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Arithmetical rank of the edge ideals of some n-cyclic graphs with a common edge

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Abstract: In this paper, we present some lower bounds and upper bounds on the arithmetical rank of the edge ideals of some n-cyclic graphs with a common edge. For some special n-cyclic graphs with a common edge, we prove that the arithmetical rank equals the projective dimension of the corresponding quotient ring.

Key words: Arithmetical rank, edge ideal, projective dimension, set-theoretic complete intersection

1. Introduction

Let R be a Noetherian commutative ring with identity and I a proper ideal of R. The arithmetical rank (ara) of I is defined as the minimal number s of elements $a_1,\ldots,a_s$ of R such that the ideal $(a_1,\ldots,a_s)$ has the same radical as I. In this case we will say that $a_1,\ldots,a_s$ generate I up to radical. In general $\mathrm{ht}\,(I) \le \mathrm{ara}\,(I)$. If equality holds, I is called a set-theoretic complete intersection.

We consider the case where R is a polynomial ring over any field K and I is the edge ideal of a graph whose vertices are the indeterminates. The set of its generators is formed by the products of the pairs of indeterminates that form the edges of the graph. Thus, I is generated by square-free quadratic monomials and is therefore a radical ideal. The problem of the arithmetical rank of edge ideals or monomial ideals has been intensively studied by many authors over the past 3 decades (see [1, 2, 3, 5, 8, 10]).

According to a well-known result by Lyubeznik [9], if I is a square-free monomial ideal, the projective dimension of the quotient ring R/I, denoted $\mathrm{pd}_R\,(R/I)$, provides a lower bound on the arithmetical rank of I. We define the big height of I, denoted $\mathrm{bight}\,(I)$, as the maximum height of the minimal prime ideals of I. In general, we have that

$$\mathrm{ht}\,(I) \le \mathrm{bight}\,(I) \le \mathrm{pd}_R\,(R/I) \le \mathrm{ara}\,(I) \le \mu(I),$$

where the second inequality on the left is due to Morey and Villarreal [11] and $\mu(I)$ is the minimum number of generators of the ideal I. If I is not unmixed, then I is not a set-theoretic complete intersection, but it could still be true that $\mathrm{bight}\,(I) = \mathrm{pd}_R\,(R/I) = \mathrm{ara}\,(I)$. This equality has been respectively established for the edge ideals of forests by Barile [1] and Kimura and Terai [8]. A weaker condition is the equality between the arithmetical rank and the projective dimension. This is the case for all cyclic and bicyclic graphs (see [3]) and for the graphs consisting of paths and cycles with a common vertex (see [7]). In all these cases, the arithmetical rank is independent of the field K.

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The computation of the arithmetical rank of the edge ideal of a graph is still an open problem. In this paper, we consider the class of some n-cyclic graphs with a common edge and give some lower bounds and upper bounds on the arithmetical rank of the edge ideals of these graphs. For some special n-cyclic graphs with a common edge, we also prove that the arithmetical rank equals the projective dimension of the corresponding quotient ring.

2. Preliminaries

We first recall some definitions and basic facts about graphs and their edge ideals in order to make this paper self-contained. However, for more details on the notions, we refer the reader to [4, 6, 13].

Definition 2.1 *A finite graph G is an ordered pair $G = (V(G), E(G))$ where $V(G) = \{x_1, \ldots, x_n\}$ is the set of vertices of G, and $E(G)$ is a collection of 2-element subsets of $V(G)$, usually called the edges of G.*

Definition 2.2 *Let $G_i = (V(G_i), E(G_i))$ be some graphs with vertex set $V(G_i)$ and edge set $E(G_i)$, for $i = 1, \ldots, k$. The union of the graphs $G_1, G_2, \ldots, G_k$, written $G_1 \cup G_2 \cup \cdots \cup G_k$, is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$.*

Definition 2.3 *Let $G = (V(G), E(G))$ be a graph. A walk of length m in G is an alternating sequence of vertices and edges $w = \{x_1, y_1, x_2, \ldots, x_m, y_m, x_{m+1}\}$, where $y_i = \{x_i, x_{i+1}\}$ is the edge joining x_i and x_{i+1}. If $x_1 = x_{m+1}$, we call this walk closed. A walk may also be denoted $\{x_1, \ldots, x_{m+1}\}$, the edges being evident from the context.*

A cycle of length m ($m \geq 3$) is a closed walk in which the vertices $x_1, \ldots, x_m$ are distinct. We denote by C_m the graph consisting of a cycle with m vertices.

Definition 2.4 *A graph G is called an n-cyclic graph with a common edge if G is a graph consisting of n cycles $C_{3r_1}, \ldots, C_{3r_{k_1}}$, $C_{3s_1+1}, \ldots, C_{3s_{k_2}+1}$, $C_{3t_1+2}, \ldots, C_{3t_{k_3}+2}$ connected through a common edge, where $k_1 + k_2 + k_3 = n$.*

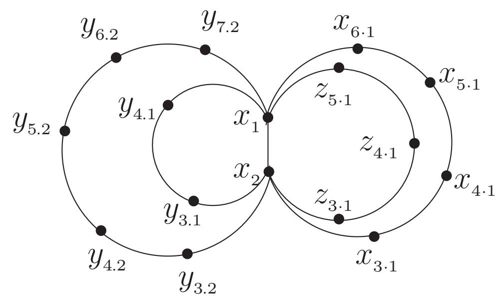
Definition 2.5 *Let G be a graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$, with $n \in \mathbb{N}$, $n \geq 1$, and whose edge set is $E(G)$. Suppose that $x_1, \ldots, x_n$ are indeterminates over the field K. The edge ideal of G in the polynomial ring $R = K[x_1, \ldots, x_n]$ is the squarefree monomial ideal*

$$I(G) = (\{x_i x_j \mid \{x_i, x_j\} \in E(G)\}).$$

For the sake of simplicity, we will use the same notation $x_i x_j$ for the monomial and for the corresponding edge.

Throughout the paper, we let G be an n-cyclic graph with a common edge consisting of the union of n cycles $C_{3r_1}, \ldots, C_{3r_{k_1}}$, $C_{3s_1+1}, \ldots, C_{3s_{k_2}+1}$, $C_{3t_1+2}, \ldots, C_{3t_{k_3}+2}$ connected through a common edge x_1x_2, where $k_1 + k_2 + k_3 = n$. We consider the following labeling for the edges of G:
$E(C_{3r_i,i}) = \{x_1x_2, x_2x_{3,i}, \ldots, x_{3r_i,i}x_1\}$ for all $1 \leq i \leq k_1$ and $r_1 \geq r_2 \geq \cdots \geq r_{k_1}$,
$E(C_{3s_i+1,i}) = \{x_1x_2, x_2y_{3,i}, \ldots, y_{3s_i+1,i}x_1\}$ for all $1 \leq i \leq k_2$ and $s_1 \geq s_2 \geq \cdots \geq s_{k_2}$,
$E(C_{3t_i+2,i}) = \{x_1x_2, x_2z_{3,i}, \ldots, z_{3t_i+2,i}x_1\}$ for all $1 \leq i \leq k_3$ and $t_1 \geq t_2 \geq \cdots \geq t_{k_3}$.
Let K be any field and $R = K[x_1, x_2, x_{3,1}, \ldots, x_{3r_1,1}, \ldots, x_{3,k_1}, \ldots, x_{3r_{k_1},k_1}, y_{3,1}, \ldots, y_{3s_1+1,1}, \ldots, y_{3,k_2}, \ldots, y_{3s_{k_2}+1,k_2}, z_{3,1}, \ldots, z_{3t_1+2,1}, \ldots, z_{3,k_3}, \ldots, z_{3t_{k_3}+2,k_3}]$ be the polynomial ring.

Example 2.6 *The following graph G is a 4-cyclic graph consisting of the union of 4 cycles C_4, C_5, C_6, and C_7 with a common edge x_1x_2.*



Figure

The edge ideal of G is $I(G) = (x_1x_2, x_2x_{3,1}, x_{3,1}x_{4,1}, x_{4,1}x_{5,1}, x_{5,1}x_{6,1}, x_{6,1}x_1, x_2y_{3,1}, y_{3,1}y_{4,1}, y_{4,1}x_1, x_2y_{3,2}, y_{3,2}y_{4,2}, y_{4,2}y_{5,2}, y_{5,2}y_{6,2}, y_{6,2}y_{7,2}, y_{7,2}x_1, x_2z_{3,1}, z_{3,1}z_{4,1}, z_{4,1}z_{5,1}, z_{5,1}x_1)$.

Definition 2.7 *Let $G = (V(G), E(G))$ be a graph. A vertex cover for G is a subset S of $V(G)$ that intersects every edge of G. If S is a minimal element (under inclusion) of the set of vertex covers of G, it is called a minimal vertex cover.*

Remark 2.8 *It is well known that the minimal primes of $I(G)$ in R are the ideals generated by the minimal vertex covers of G. Hence, for any minimal vertex cover S of G, we have*

$$|S| \leq bight(I(G)).$$

We can easily find the minimal vertex covers for the cycle graph C_m on the vertex set $\{x_1, \ldots, x_m\}$. Note that a subset S of $V(C_m)$ is a minimal vertex cover for C_m if and only if at least 1 and at most 2 of x_i, x_{i+1}, and x_{i+2} belong to S for every $i \in \{1, \ldots, m-2\}$.
If $m = 3r$, then a minimal vertex cover of C_m is

$$S = \{x_2, x_3, x_5, x_6, \ldots, x_{3r-4}, x_{3r-3}, x_{3r-1}, x_{3r}\}, \quad \text{where } |S| = 2r.$$

If $m = 3s+1$, then a minimal vertex cover of C_m is

$$S = \{x_2, x_3, x_5, x_6, \ldots, x_{3s-4}, x_{3s-3}, x_{3s-1}, x_{3s+1}\}, \quad \text{where } |S| = 2s.$$

If $m = 3t+2$, then a minimal vertex cover of C_m is

$$S = \{x_2, x_3, x_5, x_6, \ldots, x_{3t-1}, x_{3t}, x_{3t+2}\}, \quad \text{where } |S| = 2t+1.$$

By [3, Theorem 2], we obtain that the cardinalities of these minimal vertex covers are all maximum.

3. Main results

In this section, we present some lower bounds on $\text{bight}(I(G))$ and some upper bounds on $\text{ara}(I(G))$ of the edge ideals $I(G)$ of some n-cyclic graphs G with a common edge.

The following useful technique that provides an upper bound for the arithmetical rank of some ideals is a result due to Schmitt and Vogel [12].

Lemma 3.1 *[12, p. 249] Let P be a finite subset of elements of R. Let $P_0, \ldots, P_r$ be subsets of P such that*

(i) $\bigcup_{i=0}^{r} P_i = P$;

(ii) P_0 *has exactly one element;*

(iii) if p and p' are different elements of $P_i (0 < i \leq r)$, then there is an integer i' with $0 \leq i' < i$ and an element in $P_{i'}$ that divides pp'.

We set $q_i = \sum_{p \in P_i} p^{e(p)}$, where $e(p) \geq 1$ are arbitrary integers. We will write (P) for the ideal of R generated by the elements of P. Then we get

$$\sqrt{(P)} = \sqrt{(q_0, \ldots, q_r)}.$$

In particular, ara $((P)) \leq r + 1$.

In the construction given in the above lemma, if we take all exponents $e(p) = 1$, then $q_0, \ldots, q_r$ are sums of generators.

The following lemma can be used for computing the arithmetical rank of some monomial ideals.

Lemma 3.2 *Let $q_0, q_{11}, q_{12}, q_{21}, q_{22} \in R$ be such that $q_0 | q_{11}q_{22}, q_{21} | q_{11}q_{12}$, and $q_{12} | q_{21}q_{22}$. Set $q_1 = q_{11} + q_{12}$ and $q_2 = q_{21} + q_{22}$. Then*

$$\sqrt{(q_0, q_1, q_2)} = \sqrt{(q_0, q_{11}, q_{12}, q_{21}, q_{22})}.$$

Proof It suffices to show that $q_{11}, q_{12}, q_{21}, q_{22} \in \sqrt{(q_0, q_1, q_2)}$. Let $a, b \in R$ be such that $q_{11}q_{22} = aq_0$ and $q_{11}q_{12} = bq_{21}$, and then

$$\begin{aligned} q_{11}^3 &= q_{11}^2(q_{11} + q_{12}) - q_{11}^2 q_{12} = q_{11}^2(q_{11} + q_{12}) - q_{11}bq_{21} \\ &= q_{11}^2(q_{11} + q_{12}) - bq_{11}(q_{21} + q_{22}) + bq_{11}q_{22} \\ &= q_{11}^2(q_{11} + q_{12}) - bq_{11}(q_{21} + q_{22}) + baq_0 \\ &= q_{11}^2 q_1 - bq_{11}q_2 + baq_0 \in \sqrt{(q_0, q_1, q_2)}. \end{aligned}$$

This shows that $q_{11} \in \sqrt{(q_0, q_1, q_2)}$, from which $q_{12} = q_1 - q_{11} \in \sqrt{(q_0, q_1, q_2)}$. The claim for $q_{21}, q_{22} \in \sqrt{(q_0, q_1, q_2)}$ follows by symmetry. This completes the proof. □

In order to present some lower bounds on bight $(I(G))$ and some upper bounds on ara $(I(G))$ of the edge ideals $I(G)$ of some graphs G, which are the union of some cycles with a common edge, we consider 3 cases, depending on the residue modulo 3 of the lengths of the cycles. The cases $m \equiv 0, 1 \mod 3$ can be settled by a direct application of Lemma 3.1. The case $m \equiv 2 \mod 3$ is more interesting, since it needs some additional nontrivial computations on the generators. The idea of this paper is derived from Barile et al. [3]; we consider 3 cases that are treated separately in the following 3 theorems.

First, we consider the case where graph G is the union of some cycles whose lengths are all multiples of 3.

Theorem 3.3 *Let G be a k_1-cyclic graph consisting of the union of k_1 cycles $C_{3r_1}, \ldots, C_{3r_{k_1}}$ with a common edge x_1x_2. Set $q_0 = x_1x_2$, and, for any $1 \le j \le r_i - 1$, $1 \le i \le k_1$, set*

$$\begin{aligned} q_{1,i} &= x_1x_{3r_i,i} + x_2x_{3,i}, \\ q_{2j,i} &= x_{3j+1,i}x_{3j+2,i}, \\ q_{2j+1,i} &= x_{3j,i}x_{3j+1,i} + x_{3j+2,i}x_{3j+3,i}. \end{aligned}$$

Then

$$I(G) = \sqrt{(q_0, q_{1,1}, \ldots, q_{2r_1-1,1}, \ldots, q_{1,k_1}, \ldots, q_{2r_{k_1}-1,k_1})}.$$

In particular, $ara(I(G)) = ara(\sum_{i=1}^{k_1} I(C_{3r_i})) \le 1 + \sum_{i=1}^{k_1}(2r_i - 1)$.

Proof We show that the assumptions of Lemma 3.1 are fulfilled by the $1 + \sum_{i=1}^{k_1}(2r_i - 1)$ sets $P_0, P_{1,1}, \ldots, P_{2r_1-1,1}, \ldots, P_{1,k_1}, \ldots, P_{2r_{k_1}-1,k_1}$, where, for all $1 \le j \le 2r_i-1$, $1 \le i \le k_1$, $P_{j,i}$ is the set of monomials appearing in $q_{j,i}$. It is straightforward to verify that conditions (i) and (ii) are satisfied. For any $1 \le j \le r_i - 1$, $1 \le i \le k_1$, we have that $x_1x_2|(x_1x_{3r_i,i})(x_2x_{3,i})$, $x_{3j+1,i}x_{3j+2,i}|(x_{3j,i}x_{3j+1,i})(x_{3j+2,i}x_{3j+3,i})$, where $x_1x_{3r_i,i}, x_2x_{3,i}$ are the monomials in $P_{1,i}$, $x_{3j+1,i}x_{3j+2,i}$ is the only monomial in $P_{2j,i}$, and $x_{3j,i}x_{3j+1,i}, x_{3j+2,i}x_{3j+3,i}$ are the monomials in $P_{2j+1,i}$. Note that $I(G) = \sum_{i=1}^{k_1} I(C_{3r_i})$; hence, the claim is true. □

Theorem 3.4 *Let G be a k_1-cyclic graph consisting of the union of k_1 cycles $C_{3r_1}, \ldots, C_{3r_{k_1}}$ with a common edge x_1x_2. Then* $bight(I(G)) \ge 1 + \sum_{i=1}^{k_1}(2r_i - 1)$.

Proof It is easy to prove that $S = \{x_2, x_{3,1}, x_{5,1}, x_{6,1}, \ldots, x_{3r_1-4,1}, x_{3r_1-3,1}, x_{3r_1-1,1}, x_{3r_1,1}, \ldots, x_{3,k_1}, x_{5,k_1}, x_{6,k_1}, \ldots, x_{3r_{k_1}-4,k_1}, x_{3r_{k_1}-3,\,k_1}, x_{3r_{k_1}-1,\,k_1}, x_{3r_{k_1},\,k_1}\}$ is a minimal vertex cover for G. By Remark 2.8, we obtain that

$$\text{bight}\,(I(G)) \ge |S| = \sum_{i=1}^{k_1} 2r_i - (k_1 - 1) = 1 + \sum_{i=1}^{k_1}(2r_i - 1).$$

□

As a consequence of the above 2 theorems, we have:

Corollary 3.5 *Let G be a graph as in Theorem 3.4. Then*

$$bight(I(G)) = pd_R\,(R/I(G)) = ara(I(G)).$$

Now we consider the case where graph G is the union of some cycles whose lengths are all congruent to 1 modulo 3.

The following 2 theorems can be shown by arguments similar to those used for Theorem 3.3, and so we omit their proofs.

Theorem 3.6 *Let G be a k_2-cyclic graph consisting of the union of k_2 cycles $C_{3s_1+1},\dots,C_{3s_{k_2}+1}$ with a common edge x_1x_2, where $s_i=1$ for any $i\in\{1,\dots,k_2-1\}$. Set $q_0=x_1x_2$, $q_1=x_1y_{4,1}+x_2y_{3,1}$; for $2\le i\le k_2$, set $q_i=x_1y_{4,i}+x_2y_{3,i}+y_{3,i-1}y_{4,i-1}$; and, for any $1\le j\le s_{k_2}-1$, set*

$$\begin{aligned} q'_{2j} &= y_{3j+1,k_2}y_{3j+2,k_2},\\ q'_{2j+1} &= y_{3j,k_2}y_{3j+1,k_2}+y_{3j+2,k_2}y_{3j+3,k_2}, \end{aligned}$$

and, finally, $q'_{2s_{k_2}}=y_{3s_{k_2},k_2}y_{3s_{k_2}+1,k_2}$. Then

$$I(G)=\sqrt{(q_0,q_1,\dots,q_{k_2},q'_2,q'_3,\dots,q'_{2s_{k_2}})}.$$

In particular, $ara(I(G))\le 1+k_2+2(s_{k_2}-1)+1=k_2+2s_{k_2}$.

Theorem 3.7 *Let G be a k_2-cyclic graph consisting of the union of k_2 cycles $C_{3s_1+1},\dots,C_{3s_{k_2}+1}$ with a common edge x_1x_2. Set $q_0=x_1x_2$, and, for any $1\le j\le s_i-1$, $1\le i\le k_2$, set*

$$\begin{aligned} q'_{1,i} &= x_1y_{3s_i+1,i}+x_2y_{3,i},\\ q'_{2j,i} &= y_{3j+1,i}y_{3j+2,i},\\ q'_{2j+1,i} &= y_{3j,i}y_{3j+1,i}+y_{3j+2,i}y_{3j+3,i},\\ q'_{2s_i,i} &= y_{3s_i,i}y_{3s_i+1,i}. \end{aligned}$$

Then

$$I(G)=\sqrt{(q_0,q'_{1,1},\dots,q'_{2s_1,1},\dots,q'_{1,k_2},\dots,q'_{2s_{k_2},k_2})}.$$

In particular, $ara(I(G))=ara(\sum_{i=1}^{k_2}I(C_{3s_i+1}))\le 1+2\sum_{i=1}^{k_2}s_i$.

Theorem 3.8 *Let G be a k_2-cyclic graph consisting of the union of k_2 cycles $C_{3s_1+1},\dots,C_{3s_{k_2}+1}$ with a common edge x_1x_2. Then $bight(I(G))=2-k_2+2\sum_{i=1}^{k_2}s_i$.*

Proof It is obvious that $S=\{x_2,y_{3,1},y_{5,1},y_{6,1},\dots,y_{3s_1-4,1},y_{3s_1-3,1},y_{3s_1-1,1},y_{3s_1,1};x_1,y_{4,2},y_{5,2},y_{7,2},y_{8,2},\dots,y_{3s_2-2,2},y_{3s_2-1,2},y_{3s_2+1,2};\dots,y_{4,k_2},y_{5,k_2},y_{7,k_2},y_{8,k_2},\dots,y_{3s_{k_2}-2,k_2},y_{3s_{k_2}-1,k_2},y_{3s_{k_2}+1,k_2}\}$ is a minimal vertex cover for G. By Remark 2.8, we can obtain that

$$\text{bight}\,(I(G))\ge|S|=2s_1+2s_2+\sum_{i=3}^{k_2}[2(s_i-1)+1]=2-k_2+2\sum_{i=1}^{k_2}s_i.$$

Now we prove that $\text{bight}\,(I(G))\le 2-k_2+2\sum_{i=1}^{k_2}s_i$. This is equivalent to showing that the cardinality of any minimal vertex cover S of G is at most $2-k_2+2\sum_{i=1}^{k_2}s_i$. For each $i\in\{1,\dots,k_2\}$, set $S_i=S\cap V(C_{3s_i+1})$.

As x_1x_2 is the edge of C_{3s_i+1}, by Definition 2.7, we obtain that at least one of x_1 and x_2 belong to S_i, and that S_i is either a minimal vertex cover $\bar{S}_i$ of C_{3s_i+1} or becomes a minimal cover $\bar{S}_i$ after removing x_1 or x_2. The vertex to be removed certainly belongs to $\bar{S}_j$ for some j, or otherwise S would not be minimal. This shows that S is the union of the sets $\bar{S}_i$. By the characterization of minimal vertex covers given below Remark 2.8, we obtain that $\bar{S}_i$ at most contain $2s_i$ elements, one of which is x_1 or x_2. We distinguish the following cases: (1) If $x_1 \in S_i$ and $x_2 \notin S_i$ for all $1 \le i \le k_2$, or $x_2 \in S_i$ and $x_1 \notin S_i$ for all $1 \le i \le k_2$. These two cases can be shown by similar arguments, so we only consider the case $x_1 \in S_i$ and $x_2 \notin S_i$ for all $1 \le i \le k_2$. In this case, we have that

$$|S| \le 2s_1 + (2s_2 - 1) + \cdots + (2s_{k_2} - 1) = \sum_{i=1}^{k_2} 2s_i - k_2 + 1.$$

(2) If $x_1 \in S_i$ and $x_2 \in S_j$ for some $i, j \in \{1, \ldots, k_2\}$, then we get that

$$\begin{aligned} |S| \;&\le\; (2s_1 - 1) + \cdots + (2s_{i-1} - 1) + 2s_i + (2s_{i+1} - 1) + \cdots \\ &\quad +(2s_{j-1} - 1) + 2s_j + (2s_{j+1} - 1) + \cdots + (2s_{k_2} - 1) \\ &= \sum_{i=1}^{k_2} 2s_i - k_2 + 2. \end{aligned}$$

In conclusion, we have that $\operatorname{bight}(I(G)) \le \sum_{i=1}^{k_2} 2s_i - k_2 + 2$. □

As a consequence of Theorems 3.6 and 3.8, we have:

Corollary 3.9 *Let G be a k_2-cyclic graph consisting of the union of k_2 cycles $C_{3s_1+1}, \ldots, C_{3s_{k_2}+1}$ with a common edge x_1x_2, where $s_i = 1$ for any $i \in \{1, \ldots, k_2 - 1\}$. Then*

$$\mathit{bight}(I(G)) = \mathit{pd}_R(R/I(G)) = \mathit{ara}(I(G)).$$

As a consequence of Theorems 3.7 and 3.8, we have:

Corollary 3.10 *Let G be a graph as in Theorem 3.8. Then*

$$\mathit{ara}(I(G)) - \mathit{bight}(I(G)) \le k_2 - 1.$$

Then the following open question occurs.

Problem 3.11 *Is the upper bound in Corollary 3.10 sharp? In other words, can the upper bound in Corollary 3.10 be improved?*

Finally, we consider the case where graph G is the union of some cycles whose lengths are all congruent to 2 modulo 3.

Theorem 3.12 *Let G be a k_3-cyclic graph consisting of the union of k_3 cycles $C_{3t_1+2},\dots,C_{3t_{k_3}+2}$ with a common edge x_1x_2. Set $q_0=x_1x_2$, and, for any $1\le j\le t_i-1$, $1\le i\le k_3$, set*

$$\begin{aligned} q''_{1,i} &= x_2z_{3,i}+z_{4,i}z_{5,i},\\ q''_{2j,i} &= z_{3j,i}z_{3j+1,i}+z_{3j+2,i}z_{3j+3,i},\\ q''_{2j+1,i} &= z_{3j+2,i}z_{3j+3,i}+z_{3j+4,i}z_{3j+5,i},\\ q''_{2t_i,i} &= x_1z_{3t_i+2,i}+z_{3t_i,i}z_{3t_i+1,i}. \end{aligned}$$

Then

$$I(G)=\sqrt{(q_0,q''_{1,1},\dots,q''_{2t_1,1},\dots,q''_{1,k_3},\dots,q''_{2t_{k_3},k_3})}.$$

In particular, $ara\,(I(G))=ara\,(\sum_{i=1}^{k_3}I(C_{3t_i+2}))\le 1+2\sum_{i=1}^{k_3}t_i$.

Proof It suffices to show that $I(C_{3t_i+2})$ can be generated, up to radical, by $2t_i+1$ polynomials, one of which is $q_0=x_1x_2$. We consider 2 cases:

(i) If there exists some $i\in\{1,\dots,k_3\}$ such that $t_i=1$, set $q_0=x_1x_2$, $q''_{1,i}=x_2z_{3,i}+z_{4,i}z_{5,i}$, $q''_{2,i}=z_{3,i}z_{4,i}+x_1z_{5,i}$; then we get that $q_0|(x_2z_{3,i})(x_1z_{5,i})$, $z_{3,i}z_{4,i}|(x_2z_{3,i})(z_{4,i}z_{5,i})$ and $z_{4,i}z_{5,i}|(x_1z_{5,i})(z_{3,i}z_{4,i})$. Thus, by Lemma 3.2, we have that $I(C_5)=\sqrt{(q_0,q''_{1,i},q''_{2,i})}$.

(ii) If there exists some $1\le i\le k_3$ with $t_i\ge 2$, then set $J_{t_i,i}=(q_0,q''_{1,i},\dots,q''_{2t_i,i})$. It suffices to show that $I(C_{3t_i+2})\subseteq\sqrt{J_{t_i,i}}$. For all $f,g\in R$, by abuse of notation, we will write $f\equiv^{t_i}g$ whenever $f-g$ or $f+g$ belongs to $J_{t_i,i}$, and $f\equiv_{q''_{j,i}}g$ whenever $f-g$ or $f+g$ is divisible by $q''_{j,i}$. In this way, $f\equiv^{t_i}g$ or $f\equiv_{q''_{j,i}}g$ assures that $f\in J_{t_i,i}$ if and only if $g\in J_{t_i,i}$.

Set

$$\begin{aligned} u_{t_i,i} &= x_1^{2^{t_i-1}}z_{3t_i+2,i}^{2^{t_i}},\\ v_{t_i,i} &= z_{3,i}z_{4,i}z_{5,i}\prod_{j=2}^{t_i}z_{3j,i}^{3\cdot 2^{j-2}},\\ w_{t_i,i} &= (z_{3t_i,i}z_{3t_i+1,i}z_{3t_i+2,i})^{2^{t_i-1}}. \end{aligned}$$

First we prove that

$$u_{t_i,i}\equiv_{q''_{2t_i,i}}w_{t_i,i}\qquad\text{and}\qquad v_{t_i,i}\equiv^{t_i}w_{t_i,i}. \tag{1}$$

For all $t_i\ge2$, we have that $u_{t_i,i}=x_1^{2^{t_i-1}}z_{3t_i+2,i}^{2^{t_i}}=(x_1z_{3t_i+2,i})^{2^{t_i-1}}z_{3t_i+2,i}^{2^{t_i-1}}\equiv_{q''_{2t_i,i}}(z_{3t_i,i}z_{3t_i+1,i})^{2^{t_i-1}}z_{3t_i+2,i}^{2^{t_i-1}}=w_{t_i,i}$. This proves the first relation in (1).

We prove the second relation by induction on $t_i\ge2$. If $t_i=2$, then we have $q''_{2,i}=z_{3,i}z_{4,i}+z_{5,i}z_{6,i}$, $q''_{3,i}=z_{5,i}z_{6,i}+z_{7,i}z_{8,i}$, so that $v_{2,i}=z_{3,i}z_{4,i}z_{5,i}z_{6,i}^3\equiv_{q''_{2,i}}z_{5,i}^2z_{6,i}^2z_{6,i}^2\equiv_{q''_{3,i}}z_{6,i}^2z_{7,i}^2z_{8,i}^2=w_{2,i}$, which shows that

$v_{2,i} \equiv^2 w_{2,i}$, i.e. our claim is correct for $t_i = 2$. Now suppose that $t_i > 2$ and that the claim is true for $t_i - 1$. We have that

$$\begin{aligned}
v_{t_i,i} &= v_{t_i-1,i} z_{3t_i,i}^{3\cdot 2^{t_i-2}} \equiv^{t_i-1} w_{t_i-1,i} z_{3t_i,i}^{3\cdot 2^{t_i-2}} \\
&= (z_{3t_i-3,i} z_{3t_i-2,i} z_{3t_i-1,i})^{2^{t_i-2}} z_{3t_i,i}^{3\cdot 2^{t_i-2}} = (z_{3t_i-3,i} z_{3t_i-2,i})^{2^{t_i-2}} z_{3t_i-1,i}^{2^{t_i-2}} z_{3t_i,i}^{3\cdot 2^{t_i-2}} \\
&\equiv_{q''_{2t_i-2,i}} (z_{3t_i-1,i} z_{3t_i,i})^{2^{t_i-2}} z_{3t_i-1,i}^{2^{t_i-2}} z_{3t_i,i}^{3\cdot 2^{t_i-2}} = (z_{3t_i-1,i} z_{3t_i,i})^{2^{t_i-1}} z_{3t_i,i}^{2\cdot 2^{t_i-2}} \\
&\equiv_{q''_{2t_i-1,i}} (z_{3t_i+1,i} z_{3t_i+2,i})^{2^{t_i-1}} z_{3t_i,i}^{2^{t_i-1}} = (z_{3t_i,i} z_{3t_i+1,i} z_{3t_i+2,i})^{2^{t_i-1}} = w_{t_i,i}.
\end{aligned}$$

It follows that $v_{t_i,i} \equiv^{t_i} w_{t_i,i}$. Note that the symbols $q''_{2,i}, \ldots, q''_{2(t_i-1)-1,i}$ have the same meaning in $J_{t_i-1,i}$ and $J_{t_i,i}$. This completes the proof of (1).

Secondly, we show that

$$x_1^{2^{t_i}} z_{3t_i+2,i}^{2^{t_i+1}} \in J_{t_i,i}. \tag{2}$$

Noting that $x_1 z_{3t_i+2,i} z_{4,i} z_{5,i} \mid x_1 z_{3t_i+2,i} v_{t_i,i}$ and $x_1 z_{3t_i+2,i} z_{4,i} z_{5,i} \equiv_{q_0} x_1 z_{3t_i+2,i}(x_2 z_{3,i} + z_{4,i} z_{5,i}) \in J_{t_i,i}$, we deduce that $x_1 z_{3t_i+2,i} v_{t_i,i} \in J_{t_i,i}$. Thus, (1) will imply that $x_1^{2^{t_i}} z_{3t_i+2,i}^{2^{t_i+1}} = x_1^2 z_{3t_i+2,i}^2 u_{t_i,i} \in J_{t_i,i}$. Hence, we have shown that

$$x_1 z_{3t_i+2,i} \in \sqrt{J_{t_i,i}}. \tag{3}$$

Thus, we get that

$$z_{3t_i,i} z_{3t_i+1,i} = q''_{2t_i,i} - x_1 z_{3t_i+2,i} \in \sqrt{J_{t_i,i}}. \tag{4}$$

In general, whenever, for some $j \in \{2, \ldots, t_i\}$,

$$z_{3j,i} z_{3j+1,i} \in \sqrt{J_{t_i,i}}, \tag{5}$$

from the fact that $z_{3j,i} z_{3j+1,i} | (z_{3j-1,i} z_{3j,i})(z_{3j+1,i} z_{3j+2,i})$, by Lemma 3.1, one deduces that

$$z_{3j-1,i} z_{3j,i} \in \sqrt{J_{t_i,i}}. \tag{6}$$

Since $z_{3j-3,i} z_{3j-2,i} = q''_{2j-2,i} - z_{3j-1,i} z_{3j,i}$, this in turn implies that

$$z_{3j-3,i} z_{3j-2,i} \in \sqrt{J_{t_i,i}}. \tag{7}$$

Finally, since $z_{3j-3,i} z_{3j-2,i} | (z_{3j-4,i} z_{3j-3,i})(z_{3j-2,i} z_{3j-1,i})$, by Lemma 3.1, we again conclude that

$$z_{3j-2,i} z_{3j-1,i} \in \sqrt{J_{t_i,i}}. \tag{8}$$

Therefore, for all $j \in \{2, \ldots, t_i\}$, by (4) and descending induction on h, one can derive that $z_{h,i} z_{h+1,i} \in \sqrt{J_{t_i,i}}$ for all $h = 3, \ldots, 3t_i$.

In particular, we have that $z_{4,i} z_{5,i} \in \sqrt{J_{t_i,i}}$, which, together with $q''_{1,i} \in \sqrt{J_{t_i,i}}$ and $x_2 z_{3,i} = q''_{1,i} - z_{4,i} z_{5,i}$, yields $x_2 z_{3,i} \in \sqrt{J_{t_i,i}}$. Moreover, $z_{3t_i-1,i} z_{3t_i,i} \in \sqrt{J_{t_i,i}}$, $q''_{2t_i-1,i} \in \sqrt{J_{t_i,i}}$ and $z_{3t_i+1,i} z_{3t_i+2,i} = q''_{2t_i-1,i} -$

$z_{3t_i-1,i}z_{3t_i,i}$ imply that $z_{3t_i+1,i}z_{3t_i+2,i} \in \sqrt{J_{t_i,i}}$. This, together with (3) and $x_1x_2 = q_0 \in \sqrt{J_{t_i,i}}$, shows that $I(C_{3t_i+2}) \subseteq \sqrt{J_{t_i,i}}$, as claimed. □

Theorem 3.13 *Let G be a k_3-cyclic graph consisting of the union of k_3 cycles $C_{3t_1+2}, \ldots, C_{3t_{k_3}+2}$ with a common edge x_1x_2. Then bight$(I(G)) \geq 1 + 2\sum\limits_{i=1}^{k_3} t_i$.*

Proof It is obvious that $S = \{x_2, z_{3,1}, z_{5,1}, z_{6,1}, \ldots, z_{3t_1-1,1}, z_{3t_1,1}, z_{3t_1+2,1}, \ldots, z_{3,k_3}, z_{5,k_3}, z_{6,k_3}, \ldots, z_{3t_{k_3}-1,k_3}, z_{3t_{k_3},k_3}, z_{3t_{k_3}+2,k_3}\}$ is a minimal vertex cover for G. By Remark 2.8, we obtain that

$$\text{bight}\,(I(G)) \geq |S| = \sum_{i=1}^{k_3}(2t_i + 1) - (k_3 - 1) = 1 + 2\sum_{i=1}^{k_3} t_i.$$

□

As a consequence of the above 2 theorems, we have:

Corollary 3.14 *Let G be a graph as in Theorem 3.12. Then*

$$\textit{bight}\,(I(G)) = \textit{pd}_R\,(R/I(G)) = \textit{ara}\,(I(G)).$$

From Theorems 3.3, 3.7, and 3.12, we can derive an upper bound on ara$\,(I(G))$ when G is an n-cyclic graphs with a common edge. We adopt the following convention: whenever, in a sum, the index runs from 1 to 0, the sum has to be taken equal to zero.

Theorem 3.15 *Let G be an n-cyclic graph consisting of the union of n cycles $C_{3r_1}, \ldots, C_{3r_{k_1}}$, $C_{3s_1+1}, \ldots, C_{3s_{k_2}+1}$, $C_{3t_1+2}, \ldots, C_{3t_{k_3}+2}$ with a common edge x_1x_2, where $k_1 + k_2 + k_3 = n$.*

Then ara$\,(I(G)) \leq \sum\limits_{i=1}^{k_1}(2r_i - 1) + 2\sum\limits_{i=1}^{k_2} s_i + 2\sum\limits_{i=1}^{k_3} t_i + 1$.

Now we present a lower bound on bight$\,(I(G))$ of n-cyclic graphs with a common edge.

Theorem 3.16 *Let G be an n-cyclic graph consisting of the union of n cycles $C_{3r_1}, \ldots, C_{3r_{k_1}}$, $C_{3s_1+1}, \ldots, C_{3s_{k_2}+1}$, $C_{3t_1+2}, \ldots, C_{3t_{k_3}+2}$ with a common edge x_1x_2, where $k_1 + k_2 + k_3 = n$. Then*

$$\textit{bight}\,(I(G)) \geq \begin{cases} 1 + \sum\limits_{i=1}^{k_1}(2r_i - 1) + \sum\limits_{i=1}^{k_2}(2s_i - 1) + 2\sum\limits_{i=1}^{k_3} t_i & \textit{if } \; k_2 \in \{0, 1\}; \\ 2 + \sum\limits_{i=1}^{k_1}(2r_i - 1) + \sum\limits_{i=1}^{k_2}(2s_i - 1) + 2\sum\limits_{i=1}^{k_3} t_i & \textit{if } \; k_2 \geq 2. \end{cases}$$

Proof We distinguish the following cases:
(1) If $k_2 = 0$, then it is easy to check that

$$\begin{aligned} S \;=\; & \{x_2\} \cup \{x_{3,1}, x_{5,1}, x_{6,1}, \ldots, x_{3r_1-4,1}, x_{3r_1-3,1}, x_{3r_1-1,1}, x_{3r_1,1}\} \cup \cdots \\ & \cup \{x_{3,k_1}, x_{5,k_1}, x_{6,k_1}, \ldots, x_{3r_{k_1}-4,k_1}, x_{3r_{k_1}-3,k_1}, x_{3r_{k_1}-1,k_1}, x_{3r_{k_1},k_1}\} \\ & \cup \{z_{3,1}, z_{5,1}, z_{6,1}, z_{8,1}, \ldots, z_{3t_1-1,1}, z_{3t_1,1}, z_{3t_1+2,1}\} \cup \cdots \\ & \cup \{z_{3,k_3}, z_{5,k_3}, z_{6,k_3}, z_{8,k_3}, \ldots, z_{3t_{k_3}-1,k_3}, z_{3t_{k_3},k_3}, z_{3t_{k_3}+2,k_3}\} \end{aligned}$$

is a minimal vertex cover for G. Thus, by Remark 2.8, we get that

$$\text{bight}\,(I(G)) \quad \geq \quad |S| = 1 + \sum_{i=1}^{k_1}(2r_i - 1) + 2\sum_{i=1}^{k_3} t_i.$$

(2) If $k_2 = 1$, then by similar argumentation we obtain that

$$\begin{aligned} S \quad = \quad & \{x_2\} \cup \{x_{3,1}, x_{5,1}, x_{6,1}, \ldots, x_{3r_1-4,1}, x_{3r_1-3,1}, x_{3r_1-1,1}, x_{3r_1,1}\} \cup \cdots \\ & \cup \{x_{3,\,k_1}, x_{5,\,k_1}, x_{6,\,k_1}, \ldots, x_{3r_{k_1}-4,\,k_1}, x_{3r_{k_1}-3,\,k_1}, x_{3r_{k_1}-1,\,k_1}, x_{3r_{k_1},\,k_1}\} \\ & \cup \{y_{3,1}, y_{5,1}, y_{6,1}, \ldots, y_{3s_1-4,1}, y_{3s_1-3,1}, y_{3s_1-1,1}, y_{3s_1+1,1}\} \\ & \cup \{z_{3,1}, z_{5,1}, z_{6,1}, z_{8,1}, \ldots, z_{3t_1-1,1}, z_{3t_1,1}, z_{3t_1+2,1}\} \cup \cdots \\ & \cup \{z_{3,k_3}, z_{5,k_3}, z_{6,k_3}, z_{8,k_3}, \ldots, z_{3t_{k_3}-1,\,k_3}, z_{3t_{k_3},\,k_3}, z_{3t_{k_3}+2,\,k_3}\} \end{aligned}$$

is a minimal vertex cover for G. Thus, by Remark 2.8, we get that

$$\begin{aligned} \text{bight}\,(I(G)) \quad & \geq \quad |S| = 1 + \sum_{i=1}^{k_1}(2r_i - 1) + 2s_1 - 1 + \sum_{i=1}^{k_3} 2t_i \\ & = \quad 1 + \sum_{i=1}^{k_1}(2r_i - 1) + \sum_{i=1}^{1}(2s_i - 1) + 2\sum_{i=1}^{k_3} t_i. \end{aligned}$$

(3) Now suppose that $k_2 \geq 2$. It is then obvious that

$$\begin{aligned} S \quad = \quad & \{x_2\} \cup \{x_{3,1}, x_{5,1}, x_{6,1}, \ldots, x_{3r_1-4,1}, x_{3r_1-3,1}, x_{3r_1-1,1}, x_{3r_1,1}\} \cup \cdots \\ & \cup \{x_{3,\,k_1}, x_{5,\,k_1}, x_{6,\,k_1}, \ldots, x_{3r_{k_1}-4,\,k_1}, x_{3r_{k_1}-3,\,k_1}, x_{3r_{k_1}-1,\,k_1}, x_{3r_{k_1},\,k_1}\} \\ & \cup \{y_{3,1}, y_{5,1}, y_{6,1}, \ldots, y_{3s_1-4,1}, y_{3s_1-3,1}, y_{3s_1-1,1}, y_{3s_1,1}\} \\ & \cup \{x_1, y_{4,2}, y_{5,2}, y_{7,2}, y_{8,2}, \ldots, y_{3s_2-2,2}, y_{3s_2-1,2}, y_{3s_2+1,2}\} \cup \cdots \\ & \cup \{y_{4,\,k_2}, y_{5,\,k_2}, y_{7,\,k_2}, y_{8,\,k_2}, \ldots, y_{3s_{k_2}-2,\,k_2}, y_{3s_{k_2}-1,\,k_2}, y_{3s_{k_2}+1,\,k_2}\} \\ & \cup \{z_{3,1}, z_{5,1}, z_{6,1}, z_{8,1}, \ldots, z_{3t_1-1,1}, z_{3t_1,1}, z_{3t_1+2,1}\} \cup \cdots \\ & \cup \{z_{3,\,k_3}, z_{5,\,k_3}, z_{6,\,k_3}, z_{8,\,k_3}, \ldots, z_{3t_{k_3}-1,\,k_3}, z_{3t_{k_3},\,k_3}, z_{3t_{k_3}+2,\,k_3}\} \end{aligned}$$

is a minimal vertex cover for G. Thus, by Remark 2.8, we get that

$$\begin{aligned} \text{bight}\,(I(G)) \quad & \geq \quad |S| = 1 + \sum_{i=1}^{k_1}(2r_i - 1) + \sum_{i=1}^{k_2} 2s_i - k_2 + 1 + \sum_{i=1}^{k_3} 2t_i \\ & = \quad 2 + \sum_{i=1}^{k_1}(2r_i - 1) + \sum_{i=1}^{k_2}(2s_i - 1) + 2\sum_{i=1}^{k_3} t_i. \end{aligned}$$

$\square$

As a consequence of the above 2 theorems, we have:

Corollary 3.17 *Let G be an n-cyclic graph consisting of the union of n cycles $C_{3r_1}, \ldots, C_{3r_{k_1}}$ and $C_{3t_1+2}, \ldots, C_{3t_{k_3}+2}$ with a common edge x_1x_2, where $k_1 + k_3 = n$. Then*

$$bight(I(G)) = pd_R\,(R/I(G)) = ara(I(G)).$$

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