A note on infinite groups whose subgroups are close to be normal-by-finite

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Abstract: A group G is said to have the CF-property if the index $|X : X_G|$ is finite for every subgroup X of G. Extending previous results by Buckley, Lennox, Neumann, Smith, and Wiegold, it is proven here that if G is a locally graded group whose proper subgroups have the CF-property, then G is abelian-by-finite, provided that all its periodic sections are locally finite. Groups in which all proper subgroups of infinite rank have the CF-property are also studied.

Key words: Normal-by-finite subgroup, CF-group, group of infinite rank

1. Introduction

A subgroup X of a group G is said to be *normal-by-finite* if the core X_G of X in G has finite index in X. Finite subgroups and normal subgroups of an arbitrary group are obvious examples of normal-by-finite subgroups. A group G is called a *CF-group* if all its subgroups are normal-by-finite. Clearly, Tarski groups (i.e. infinite simple groups whose proper nontrivial subgroups have prime order) have the CF-property, but Buckley et al. [3] proved that any locally finite CF-group contains an abelian subgroup of finite index. A generalization of this result can be found in [7].

Recall that a group G is *locally graded* if every finitely generated nontrivial subgroup of G contains a proper subgroup of finite index. Although it is an open question whether an arbitrary locally graded CF-group G is abelian-by-finite, it was shown in [11] that this is actually the case if all periodic homomorphic images of G are locally finite. Groups with this property form a large class, containing in particular every group whose finitely generated subgroups are generalized radical (a group is called *generalized radical* if it has an ascending series whose factors are either locally nilpotent or locally finite).

The class of CF-groups is obviously subgroup-closed, and it is easy to show that there exists a periodic metabelian group that is not a CF-group but in which all proper subgroups have the CF-property. On the other hand, the aim of this paper is to show that the above quoted result can be extended to the case of groups whose proper subgroups have the CF-property. Of course, a group G satisfies this condition if and only if the index $|X : X_Y|$ is finite, whenever $X \leq Y < G$, which means that the subgroup X is close to be normal-by-finite in G. Our first main result is the following.

Theorem A *Let G be a locally graded group whose periodic sections are locally finite. If every proper subgroup of G has the CF-property, then G is abelian-by-finite.*

It was also proved in [11] that if G is a locally graded group for which there exists a positive integer k such that $|X : X_G| \leq k$ for every subgroup X, then G is abelian-by-finite. Groups with such property are called BCF-*groups*. In particular, all periodic sections of a locally graded BCF-group are locally finite, and hence our main theorem has the following consequence.

Corollary *Let G be a locally graded group whose proper subgroups have the BCF-property. Then G is abelian-by-finite.*

Recall that a group G is said to have *finite rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with this property. In recent years a series of papers have been published on the behavior of groups of infinite rank in which every proper subgroup of infinite rank satisfies a suitable property (see, for instance, [4, 5, 6, 8]). Our second main result provides a further contribution to this topic.

Theorem B *Let G be a locally (soluble-by-finite) group of infinite rank whose proper subgroups of infinite rank have the CF-property. Then G is abelian-by-finite.*

It is an open question whether an arbitrary locally graded group of infinite rank must contain a proper subgroup of infinite rank, and this problem seems to be the main obstacle in order to extend Theorem B to the case of locally graded groups.

Most of our notation is standard and can be found in [10].

2. Proofs

If $\mathfrak{X}$ is a class of groups, recall that a group G is *minimal non*-$\mathfrak{X}$ if it is not an $\mathfrak{X}$-group but all proper subgroups of G belong to $\mathfrak{X}$. Let A be a group of type p^∞ for some prime number p, and consider the standard wreath product $G = A \wr \langle x \rangle$ of A by a group $\langle x \rangle$ of order 2. Since A is not normal in G, the group G does not have the CF-property. On the other hand, every proper subgroup of G is either finite or abelian or a finite extension of a group of type p^∞, and hence it is a CF-group. Therefore the periodic metabelian group G is minimal non-CF.

Our first lemma describes a general property of CF-groups, which is of independent interest.

Lemma 1 *Let G be a CF-group whose periodic homomorphic images are locally finite. Then the subgroup G'' is locally finite.*

Proof Let a be any element of infinite order of G. As the subgroup $\langle a \rangle$ is normal-by-finite, there exists a positive integer $n(a)$ (depending of course on a) such that the cyclic subgroup $\langle a^{n(a)} \rangle$ is normal in G, and hence

$$[G', a^{n(a)}] = \{1\}.$$

Let A be the subgroup of G generated by the powers $a^{n(a)}$, where a runs over all elements of infinite order of G. Clearly, A is an abelian normal subgroup of G, and the factor group G/A is periodic, and so even locally finite. Moreover, $A \cap G'$ is contained in the center of G', so that $G'/Z(G')$ is locally finite, and it follows from a classical result of Schur that also the subgroup G'' is locally finite (see, for instance, [10] Part 1, Theorem 4.12). □

The structure of groups whose proper subgroups are abelian-by-finite was studied by Bruno and Phillips (see [1, 2]); in particular, they proved that every imperfect minimal non-(abelian-by-finite) group is periodic. For our purposes, we collect the main results of Bruno and Phillips in the next 2 lemmas.

Lemma 2 *Let G be a group whose proper subgroups are abelian-by-finite. If $G \neq G'$, then either G is abelian-by-finite or G/G' is a group of type p^∞ and G' is an abelian q-group, where p and q are prime numbers. Moreover, in the latter case, if $p \neq q$ there exists in G a subgroup P of type p^∞ such that $G = PG'$, and G' is a minimal normal subgroup of G.*

Lemma 3 *Let G be a locally finite p-group whose proper subgroups are abelian-by-finite. Then either G is abelian-by-finite or it is minimal nonhypercentral.*

Proof of Theorem A   Assume for a contradiction that the group G is not abelian-by-finite. It follows from the result of Smith and Wiegold [11] that every proper subgroup of G is abelian-by-finite, so that G cannot contain proper subgroups of finite index, and in particular it is not finitely generated. Then every finitely generated subgroup E of G has the CF-property, and so E'' is locally finite by Lemma 1. Therefore, the subgroup G'' is likewise locally finite. On the other hand, an application of Lemma 2 yields that G is either perfect or locally finite, and hence G must be locally finite. It follows that the commutator subgroup G' is properly contained in G (see [1], Proposition 3). By Lemma 2, we have that G/G' is a group of type p^∞ and G' is a q-group, where p and q are prime numbers.

Suppose first that $p \neq q$, so that again Lemma 2 yields that G' is an (infinite abelian) minimal normal subgroup of G, and there exists in G a subgroup P of type p^∞ such that $G = PG'$. Then $Z(G) = C_P(G')$ is finite, and the factor group $G/Z(G)$ has a trivial center. Clearly, $G/Z(G)$ is likewise a counterexample, and hence replacing G by $G/Z(G)$ it can be assumed without loss of generality that $Z(G) = \{1\}$. Since G' is a minimal normal subgroup of G, there exist elements a of G' and y of P such that $\langle a \rangle^y \neq \langle a \rangle$. Let b be any element of G' such that $\langle b \rangle^y = \langle b \rangle$, so that $b^y = b^i$ for some positive integer $i < q$. The set

$$K = \{c \in G' \mid c^y = c^i\}$$

is a proper G-invariant subgroup of G', so that $K = \{1\}$. Thus, $b = 1$, and hence y moves all cyclic nontrivial subgroups of G', which means that

$$\langle u \rangle \cap \langle u \rangle^y = \{1\}$$

for every element u of G'. Application of Dietzmann's lemma and Maschke's theorem yields that there exists in G' an infinite collection $(E_n)_{n \in \mathbb{N}}$ of finite nontrivial $\langle y \rangle$-invariant subgroups such that

$$E = \langle E_n \mid n \in \mathbb{N} \rangle = \operatorname*{Dr}_{n \in \mathbb{N}} E_n.$$

For each positive integer n, let x_n be a nontrivial element of E_n, and consider the infinite subgroup

$$X = \langle x_n \mid n \in \mathbb{N} \rangle = \operatorname*{Dr}_{n \in \mathbb{N}} \langle x_n \rangle.$$

Then $X \cap X^y = \{1\}$, and so in particular the core of X in $\langle y, E \rangle$ is trivial. Therefore, the CF-property does not hold for the proper subgroup $\langle y, E \rangle$ of G, and this contradiction shows that $p = q$, so that G is a p-group.

References

[1] Bruno B. On groups with abelian-by-finite proper subgroups. Boll Un Mat Ital 1984; 3: 797–807.

[2] Bruno B, Phillips RE. A note on groups with nilpotent-by-finite proper subgroups. Arch Math (Basel) 1995; 65: 369–374.

[3] Buckley J, Lennox JC, Neumann BH, Smith H, Wiegold J. Groups with all subgroups normal-by-finite. J Austral Math Soc Ser A 1995; 59: 384–398.

[4] De Falco M, de Giovanni F, Musella C. Groups whose proper subgroups of infinite rank have a transitive normality relation. Mediterranean J Math 2013; 10: 1999–2006.

[5] De Falco M, de Giovanni F, Musella C, Trabelsi N. Groups with restrictions on subgroups of infinite rank. Rev Mat Iberoamericana 2014; 30: 535–548.

[6] De Falco M, de Giovanni F, Musella C, Trabelsi N. Groups whose proper subgroups of infinite rank have finite conjugacy classes. Bull Austral Math Soc 2014; 89: 41–48.

[7] de Giovanni F, Martusciello M, Rainone C. Locally finite groups whose subgroups have finite normal oscillation. Bull Austral Math Soc 2014; 89: 479–487.

[8] de Giovanni F, Saccomanno F. A note on groups of infinite rank whose proper subgroups are abelian-by-finite. Colloquium Math (in press).

[9] Dixon MR, Evans MJ, Smith H. Groups with all proper subgroups nilpotent-by-finite rank. Arch Math (Basel) 2000; 75: 81–91.

[10] Robinson DJS. Finiteness Conditions and Generalized Soluble Groups. Berlin, Germany: Springer, 1972.

[11] Smith H, Wiegold J. Locally graded groups with all subgroups normal-by-finite. J Austral Math Soc Ser A 1996; 60: 222-227.