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$(\mathcal{X},\mathcal{Y})$-Gorenstein projective and injective modules

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Abstract: This paper introduces and studies $(\mathcal{X},\mathcal{Y})$-*Gorenstein projective and injective modules*, which are a generalization of Enochs' Gorenstein projective and injective modules, respectively. Our main aim is to investigate the relations among various $(\mathcal{X},\mathcal{Y})$-Gorenstein projective modules.

Key words: 1-$(\mathcal{X},\mathcal{Y})$-Gorenstein projective module, n-$(\mathcal{X},\mathcal{Y})$-Gorenstein projective module, (n,m)-$(\mathcal{X},\mathcal{Y})$-Gorenstein projective module, $(\mathcal{X},\mathcal{Y})$-Gorenstein projective module

1. Introduction

Enochs and his coauthors introduced Gorenstein projective and injective modules and developed relative homological algebra. Later, many scholars further studied the classes and introduced some new classes of modules that are analogous to those of Gorenstein projective and injective modules. For example, Bennis and Mahdou [1] defined strongly Gorenstein projective and injective modules, and Ding et al. [3] defined strongly Gorenstein flat modules. A module M is called *strongly Gorenstein flat* if there is a $Hom(-,\mathcal{F})$-exact exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ with every $P_i, P^i \in \mathcal{P}$ such that $M = \ker(P_0 \to P^0)$, where $\mathcal{P}$ is the class of projective modules and $\mathcal{F}$ is the class of flat modules. In view of the contributions of Ding, Gillespie [5] called strongly Gorenstein flat modules Ding projective. In this paper, we introduce and study the classes of $(\mathcal{X},\mathcal{Y})$-Gorenstein projective and injective modules. If $\mathcal{X}=\mathcal{Y}=\mathcal{P}$, then $(\mathcal{X},\mathcal{Y})$-Gorenstein projective modules are exactly Gorenstein projectives. If $\mathcal{X}=\mathcal{P}$ and $\mathcal{Y}=\mathcal{F}$, then $(\mathcal{X},\mathcal{Y})$-Gorenstein projective modules are exactly Ding projectives.

In Section 2, we investigate how $(\mathcal{P},\mathcal{Y})$-Gorenstein projective modules behave in short exact sequences and introduce n-$(\mathcal{X},\mathcal{Y})$-Gorenstein projective and injective modules.

In Section 3, we mainly discuss the relations among $(\mathcal{X},\mathcal{Y})$-Gorenstein projective modules, 1-$(\mathcal{X},\mathcal{Y})$-Gorenstein projective modules, and n-$(\mathcal{X},\mathcal{Y})$-Gorenstein projective modules. For example, $\bigcup_{n\geq 1} n$-$(\mathcal{X},\mathcal{Y})$-$\mathcal{GP} \subseteq (\mathcal{X},\mathcal{Y})$-$\mathcal{GP}$, $\bigcap_{n\geq 2} n$-$(\mathcal{P},\mathcal{Y})$-$\mathcal{GP} = 1$-$(\mathcal{P},\mathcal{Y})$-$\mathcal{GP}$.

In Section 4, we are interested in the classes of $(\mathcal{P},\mathcal{P})$-Gorenstein projective modules, $(\mathcal{P},\mathcal{F})$-Gorenstein

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projective modules, $(\mathcal{F},\mathcal{P})$-Gorenstein projective modules, and $(\mathcal{F},\mathcal{F})$-Gorenstein projective modules. We find that $(\mathcal{P},\mathcal{F})$-$\mathcal{GP} \subsetneq (\mathcal{F},\mathcal{F})$-$\mathcal{GP}$ and $(\mathcal{P},\mathcal{P})$-$\mathcal{GP} \subsetneq (\mathcal{F},\mathcal{P})$-$\mathcal{GP}$.

Throughout this paper, R is an associative ring with identity. By an R-module, we shall mean a unitary left R-module, ${}_R\mathcal{M}$ denoting the category of left R-modules. $\mathcal{P}$, $\mathcal{I}$, and $\mathcal{F}$ denote the classes of projective, injective, and flat R-modules, respectively. $fd(M)$ stands for the flat dimensions of an R-module M, and $wD(R)$ denotes the weak dimension of a ring R.

In what follows, $\mathcal{X}$ denotes a class of R-modules. A *left $\mathcal{X}$-resolution* of M is an exact sequence $\cdots \to X_1 \to X_0 \to M \to 0$ with each $X_i \in \mathcal{X}$. A *right $\mathcal{X}$-resolution* of M is an exact sequence $0 \to M \to X^0 \to X^1 \to \cdots$ with each $X^i \in \mathcal{X}$.

The class $\mathcal{X}$ is *projectively resolving* if $\mathcal{P} \subseteq \mathcal{X}$ and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathcal{X}$ the conditions $X' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent. The class $\mathcal{X}$ is *injectively resolving* if $\mathcal{I} \subseteq \mathcal{X}$ and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathcal{X}$ the conditions $X \in \mathcal{X}$ and $X'' \in \mathcal{X}$ are equivalent.

The next *horseshoe lemma* is similar to that of [7, Lemma 6.20].

Lemma 1.1 *Let $\mathcal{Y}$ be a class of R-modules and $\mathcal{X} \subseteq \mathcal{Y}$. Assume that $\mathcal{X}$ is closed under finite direct sums, and consider a* $\mathrm{Hom}(-,\mathcal{Y})$*-exact exact sequence*

$$0 \to M' \to M \to M'' \to 0.$$

If both M' and M'' admit $\mathrm{Hom}(-,\mathcal{Y})$*-exact right $\mathcal{X}$-resolutions, then so does M.*

Remark 1.2 All the results concerning $(\mathcal{X},\mathcal{Y})$-Gorenstein projective modules have $(\mathcal{X},\mathcal{Y})$-Gorenstein injective counterparts; hence, the statements and their proofs of the dual results on $(\mathcal{X},\mathcal{Y})$-Gorenstein injective modules are left to the reader.

2. Definitions and notations

Definition 2.1 Let $\mathcal{X}$ and $\mathcal{Y}$ be 2 classes of R-modules such that $\mathcal{P} \subseteq \mathcal{X}$. An R-module M is called $(\mathcal{X},\mathcal{Y})$*-Gorenstein projective* if there is a $\mathrm{Hom}(-,\mathcal{Y})$-exact exact sequence

$$\cdots \to X_1 \to X_0 \to X^0 \to X^1 \to \cdots$$

with every $X_i, X^i \in \mathcal{X}$ such that $M = \ker(X^0 \to X^1)$.

Definition 2.2 Let $\mathcal{X}$ and $\mathcal{Y}$ be 2 classes of R-modules such that $\mathcal{I} \subseteq \mathcal{Y}$. An R-module N is called $(\mathcal{X},\mathcal{Y})$*-Gorenstein injective* if there is a $\mathrm{Hom}(\mathcal{X},-)$-exact exact sequence

$$\cdots \to Y_1 \to Y_0 \to Y^0 \to Y^1 \to \cdots$$

with every $Y_i, Y^i \in \mathcal{Y}$ such that $N = \mathrm{coker}(Y_1 \to Y_0)$.

The classes of $(\mathcal{X},\mathcal{Y})$-Gorenstein projective and injective R-modules are denoted by $(\mathcal{X},\mathcal{Y})$-$\mathcal{GP}$ and $(\mathcal{X},\mathcal{Y})$-$\mathcal{GI}$, respectively.

Remark 2.3 (1) If $\mathcal{X}$ is closed under direct sums, then $(\mathcal{X},\mathcal{Y})$-$\mathcal{GP}$ is closed under direct sums; if $\mathcal{Y}$ is closed under direct products, then $(\mathcal{X},\mathcal{Y})$-$\mathcal{GI}$ is closed under direct products.

(2) If $\mathcal{X}_1 \subseteq \mathcal{X}_2$, then $(\mathcal{X}_1,\mathcal{Y})$-$\mathcal{GP} \subseteq (\mathcal{X}_2,\mathcal{Y})$-$\mathcal{GP}$ and $(\mathcal{X}_2,\mathcal{Y})$-$\mathcal{GI} \subseteq (\mathcal{X}_1,\mathcal{Y})$-$\mathcal{GI}$; if $\mathcal{Y}_1 \subseteq \mathcal{Y}_2$, then $(\mathcal{X},\mathcal{Y}_2)$-$\mathcal{GP} \subseteq (\mathcal{X},\mathcal{Y}_1)$-$\mathcal{GP}$ and $(\mathcal{X},\mathcal{Y}_1)$-$\mathcal{GI} \subseteq (\mathcal{X},\mathcal{Y}_2)$-$\mathcal{GI}$.

Proposition 2.4 *Let M be an R-module. The following are equivalent:*

1. *M is $(\mathcal{P},\mathcal{Y})$-Gorenstein projective.*
2. *$\mathrm{Ext}_R^{\geq 1}(M,\mathcal{Y}) = 0$ and M admits a $\mathrm{Hom}(-,\mathcal{Y})$-exact right $\mathcal{P}$-resolution.*
3. *There is an exact sequence $0 \to M \to P \to K \to 0$ where P is projective and K is $(\mathcal{P},\mathcal{Y})$-Gorenstein projective.*

Proof $(1) \Leftrightarrow (2) \Rightarrow (3)$ are obvious.

$(3) \Rightarrow (2)$ Since K is $(\mathcal{P},\mathcal{Y})$-Gorenstein projective, $\mathrm{Ext}_R^{\geq 1}(K,\mathcal{Y}) = 0$ and K admits a $\mathrm{Hom}(-,\mathcal{Y})$-exact right $\mathcal{P}$-resolution

$$0 \to K \to P^0 \to P^1 \to \cdots . \qquad (\star)$$

Using the exact sequence

$$\mathrm{Ext}_R^i(P,\mathcal{Y}) \to \mathrm{Ext}_R^i(M,\mathcal{Y}) \to \mathrm{Ext}_R^{i+1}(K,\mathcal{Y}),$$

we get $\mathrm{Ext}_R^{\geq 1}(M,\mathcal{Y}) = 0$. Assembling $(\star)$ with $0 \to M \to P \to K \to 0$, we get a $\mathrm{Hom}(-,\mathcal{Y})$-exact right $\mathcal{P}$-resolution $0 \to M \to P \to P^0 \to P^1 \to \cdots$, as desired. □

Theorem 2.5 *$(\mathcal{P},\mathcal{Y})$-$\mathcal{GP}$ is projectively resolving, closed under direct summands.*

Proof By Remark 2.3(1), $(\mathcal{P},\mathcal{Y})$-$\mathcal{GP}$ is closed under direct sums. By Lemma 1.1, $(\mathcal{P},\mathcal{Y})$-$\mathcal{GP}$ is closed under extensions. Assume that $0 \to X \to Y \to Z \to 0$ is an exact sequence of R-modules, where Y and Z are $(\mathcal{P},\mathcal{Y})$-Gorenstein projective. By Proposition 2.4, there is an exact sequence $0 \to Y \to P \to K \to 0$ where P is projective and K is $(\mathcal{P},\mathcal{Y})$-Gorenstein projective. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & P & \longrightarrow & T & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Since $(\mathcal{P},\mathcal{Y})$-$\mathcal{GP}$ is closed under extensions, T is $(\mathcal{P},\mathcal{Y})$-Gorenstein projective. Again, by Proposition 2.4, X is $(\mathcal{P},\mathcal{Y})$-Gorenstein projective.

Finally, by [6, Proposition 1.4], $(\mathcal{P},\mathcal{Y})$-$\mathcal{GP}$ is closed under direct summands. □

Definition 2.6 Let $\mathcal{X}$ and $\mathcal{Y}$ be 2 classes of R-modules such that $\mathcal{P}\subseteq\mathcal{X}$. An R-module M is called n-$(\mathcal{X},\mathcal{Y})$-*Gorenstein projective* if there is a $\mathrm{Hom}(-,\mathcal{Y})$-exact exact sequence

$$0\to M\to X_n\to\cdots\to X_1\to M\to 0$$

with every $X_i\in\mathcal{X}$.

Definition 2.7 Let $\mathcal{X}$ and $\mathcal{Y}$ be 2 classes of R-modules such that $\mathcal{I}\subseteq\mathcal{Y}$. An R-module N is called n-$(\mathcal{X},\mathcal{Y})$-*Gorenstein injective* if there is a $\mathrm{Hom}(\mathcal{X},-)$-exact exact sequence

$$0\to N\to Y_n\to\cdots\to Y_1\to N\to 0$$

with every $Y_i\in\mathcal{Y}$.

The classes of n-$(\mathcal{X},\mathcal{Y})$-Gorenstein projective and injective R-modules are denoted by n-$(\mathcal{X},\mathcal{Y})$-$\mathcal{GP}$ and n-$(\mathcal{X},\mathcal{Y})$-$\mathcal{GI}$, respectively.

From the definition, we immediately get the following characterization.

Proposition 2.8 *An R-module M is n-$(\mathcal{P},\mathcal{Y})$-Gorenstein projective if and only if there is an exact sequence $0\to M\to P_n\to\cdots\to P_1\to M\to 0$ with every P_i projective and* $\mathrm{Ext}^1_R(M,\mathcal{Y})=\mathrm{Ext}^2_R(M,\mathcal{Y})=\cdots=\mathrm{Ext}^n_R(M,\mathcal{Y})=0$.

Corollary 2.9 $\mathcal{P}\subseteq 1$-$(\mathcal{P},\mathcal{Y})$-$\mathcal{GP}$.

Proof For any projective R-module P, $\mathrm{Ext}^1_R(P,\mathcal{Y})=0$ and there exists a short exact sequence $0\to P\to P\oplus P\to P\to 0$; hence, $P\in 1$-$(\mathcal{P},\mathcal{Y})$-$\mathcal{GP}$. □

3. The main results

In this section, we discuss the relations among the classes of $(\mathcal{X},\mathcal{Y})$-Gorenstein projective modules, 1-$(\mathcal{X},\mathcal{Y})$-Gorenstein projective modules, and n-$(\mathcal{X},\mathcal{Y})$-Gorenstein projective modules. First, from the definitions, we immediately get the following.

Proposition 3.1 *An R-module M is n-$(\mathcal{X},\mathcal{Y})$-Gorenstein projective if and only if there is a* $\mathrm{Hom}(-,\mathcal{Y})$*-exact exact sequence*

$$\cdots\to X_1\to\underbrace{X_n\to\cdots\to X_2\to X_1}\to X_n\to\cdots$$

with every $X_i\in\mathcal{X}$ such that $M=\ker(X_n\to X_{n-1})$.

Corollary 3.2 n-$(\mathcal{X},\mathcal{Y})$-$\mathcal{GP}\subseteq(\mathcal{X},\mathcal{Y})$-$\mathcal{GP}$, *and thus* $\bigcup_{n\geq 1} n$-$(\mathcal{X},\mathcal{Y})$-$\mathcal{GP}\subseteq(\mathcal{X},\mathcal{Y})$-$\mathcal{GP}$.

Every 1-$(\mathcal{X},\mathcal{Y})$-Gorenstein projective R-module is in particular an $(\mathcal{X},\mathcal{Y})$-Gorenstein projective module. Conversely, we have:

Proposition 3.3 *If $\mathcal{X}$ is closed under direct sums, then every $(\mathcal{X},\mathcal{Y})$-Gorenstein projective R-module is a direct summand of some 1-$(\mathcal{X},\mathcal{Y})$-Gorenstein projective R-module.*

Proof If M is an $(\mathcal{X},\mathcal{Y})$-Gorenstein projective R-module, then there is a $\mathrm{Hom}(-,\mathcal{Y})$-exact exact sequence

$$\cdots \stackrel{d_2}{\to} X_1 \stackrel{d_1}{\to} X_0 \stackrel{d_0}{\to} X_{-1} \stackrel{d_{-1}}{\to} X_{-2} \stackrel{d_{-2}}{\to} \cdots$$

with each $X_i \in \mathcal{X}$ such that $M = \ker(d_0)$. For all $z \in Z$, denote by $\mathbb{X}_z$ the following exact sequence:

$$\cdots \stackrel{d_{2-z}}{\to} X_{1-z} \stackrel{d_{1-z}}{\to} X_{-z} \stackrel{d_{-z}}{\to} X_{-1-z} \stackrel{d_{-1-z}}{\to} X_{-2-z} \stackrel{d_{-2-z}}{\to} \cdots .$$

Consider their direct sum:

$$\oplus\mathbb{X}_z = \cdots \to \oplus X_i \stackrel{\oplus d_i}{\to} \oplus X_i \stackrel{\oplus d_i}{\to} \oplus X_i \stackrel{\oplus d_i}{\to} \oplus X_i \stackrel{\oplus d_i}{\to} \cdots .$$

Since $\ker(\oplus d_i) \cong \oplus \ker d_i$, $\mathrm{Hom}(\oplus\mathbb{X}_z,\mathcal{Y}) \cong \prod \mathrm{Hom}(\mathbb{X}_z,\mathcal{Y})$, M is a direct summand of the 1-$(\mathcal{X},\mathcal{Y})$-Gorenstein projective R-module $\ker(\oplus d_i)$. □

It is easy to see that a 1-Gorenstein projective R-module is n-Gorenstein projective. Interestingly, we may also construct a 1-Gorenstein projective R-module from an n-Gorenstein projective R-module.

Lemma 3.4 *Let $M \in n$-$(\mathcal{X},\mathcal{Y})$-$\mathcal{GP}$ and $K_i = \ker(X_i \to X_{i-1})$ $(X_0 = M)$.*

1. *$K_i \in n$-$(\mathcal{X},\mathcal{Y})$-$\mathcal{GP}$ $(i = 1, 2, \cdots, n)$.*
2. *If $\mathcal{X}$ is closed under finite direct sums, then $\coprod_{i=1}^{n} K_i \in 1$-$(\mathcal{X},\mathcal{Y})$-$\mathcal{GP}$.*

Proof Since $M \in n$-$(\mathcal{X},\mathcal{Y})$-$\mathcal{GP}$, there is a $\mathrm{Hom}(-,\mathcal{Y})$-exact exact sequence $0 \to M \to X_n \to \cdots \to X_1 \to M \to 0$ with every $X_i \in \mathcal{X}$. Thus, we get 2 $\mathrm{Hom}(-,\mathcal{Y})$-exact exact sequences,

$$0 \to K_i \to X_i \to \cdots \to X_1 \to X_n \to \cdots \to X_{i+1} \to K_i \to 0$$

and

$$0 \to \coprod_{i=1}^{n} K_i \to \coprod_{i=1}^{n} X_i \to \coprod_{i=1}^{n} K_i \to 0,$$

where $K_n = M$, as desired. □

Theorem 3.5 *The following are equivalent for any R-module M:*

1. *$M \in n$-$(\mathcal{P},\mathcal{Y})$-$\mathcal{GP}$.*
2. *There is an exact sequence $0 \to M \to P_n \stackrel{f_n}{\to} \cdots \stackrel{f_2}{\to} P_1 \stackrel{f_1}{\to} M \to 0$ with every P_i projective and $\coprod_{i=1}^{n} \ker f_i \in 1$-$(\mathcal{P},\mathcal{Y})$-$\mathcal{GP}$.*

Proof (1)$\Rightarrow$(2) is obvious by Lemma 3.4(2).

(2)$\Rightarrow$(1) Since $M = \ker f_n \in (\mathcal{P},\mathcal{Y})$-$\mathcal{GP}$ by Theorem 2.5, $\mathrm{Ext}_R^{\geq 1}(M,\mathcal{Y}) = 0$ by Proposition 2.4, and thus $M \in n$-$(\mathcal{P},\mathcal{Y})$-$\mathcal{GP}$ by Proposition 2.8. □

Next we explore the relations between n-$(\mathcal{X},\mathcal{Y})$-Gorenstein projective modules and m-$(\mathcal{X},\mathcal{Y})$-Gorenstein projective modules, where $m \neq n$.

Lemma 3.6 (1) *If $n|m$, then n-$(\mathcal{X},\mathcal{Y})$-$\mathcal{GP} \subseteq m$-$(\mathcal{X},\mathcal{Y})$-$\mathcal{GP}$.*

(2) *If $m = nq + r$ $(0 < r < n)$, then m-$(\mathcal{P},\mathcal{Y})$-$\mathcal{GP} \bigcap n$-$(\mathcal{P},\mathcal{Y})$-$\mathcal{GP} \subseteq r$-$(\mathcal{P},\mathcal{Y})$-$\mathcal{GP}$.*

Proof (1) If $M \in n$-$(\mathcal{X},\mathcal{Y})$-$\mathcal{GP}$, then there is a $\mathrm{Hom}(-,\mathcal{Y})$-exact exact sequence

$$0 \to M \to X_n \to \cdots \to X_1 \to M \to 0 \qquad (\star)$$

with every $X_i \in \mathcal{X}$. If $m = nk$, then assembling $(\star)$ with itself k times, we get the desired result.

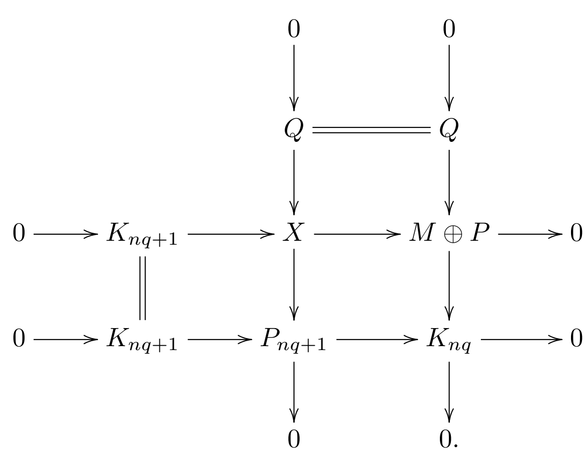
(2) By part (1), m-$(\mathcal{P},\mathcal{Y})$-$\mathcal{GP} \bigcap n$-$(\mathcal{P},\mathcal{Y})$-$\mathcal{GP} \subseteq m$-$(\mathcal{P},\mathcal{Y})$-$\mathcal{GP} \bigcap nq$-$(\mathcal{P},\mathcal{Y})$-$\mathcal{GP}$.

If $M \in m$-$(\mathcal{P},\mathcal{Y})$-$\mathcal{GP} \bigcap nq$-$(\mathcal{P},\mathcal{Y})$-$\mathcal{GP}$, then there is a $\mathrm{Hom}(-,\mathcal{Y})$-exact exact sequence

$$0 \to M \to P_m \to \cdots \to P_1 \to M \to 0 \qquad (\star\star)$$

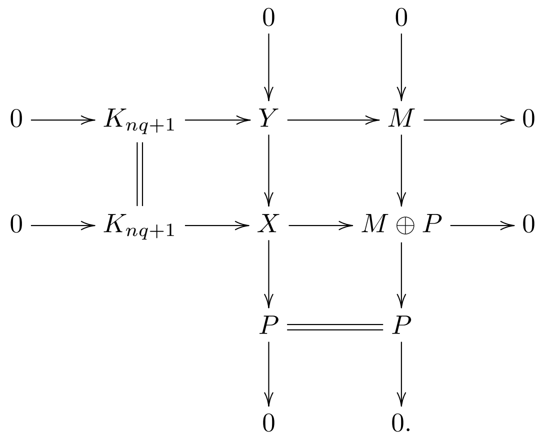
with every P_i projective. Let $K_{nq} = \ker(P_{nq} \to P_{nq-1})$. Since $M \in nq$-$(\mathcal{P},\mathcal{Y})$-$\mathcal{GP}$, there are projective R-modules P and Q such that $M \oplus P \cong K_{nq} \oplus Q$ by Schanuel's lemma [7, Exercise 3.37].

Consider the following pullback diagram:



Then X is projective.

Next, consider the following pullback diagram:



Then Y is projective. Combining this exact sequence $(\star\star)$ with the first row in the above diagram, we get a $\mathrm{Hom}(-,\mathcal{Y})$-exact exact sequence

$$0 \to M \to P_m \to \cdots \to P_{nq+2} \to Y \to M \to 0.$$

Therefore, $M \in r\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP}$. □

Theorem 3.7 *$m\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP} \bigcap n\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP} = (m,n)\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP}$, where (m,n) is the greatest common divisor of m and n.*

Proof By Lemma 3.6(1), $(m,n)\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP} \subseteq m\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP} \bigcap n\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP}$. We shall prove the converse inclusion.

If $m = nq_0 + r_0$ $(0 < r_0 < n)$, then $m\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP} \bigcap n\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP} \subseteq r_0\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP}$ by Lemma 3.6(2).

If $n = r_0q_1 + r_1$ $(0 < r_1 < r_0)$, then $r_0\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP} \bigcap n\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP} \subseteq r_1\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP}$ by Lemma 3.6(2) again, and thus $m\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP} \bigcap n\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP} \subseteq r_1\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP}$.

Continuing the above process, there is a positive integer z with $r_z = (m,n)q_{z+1}$ such that $m\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP} \bigcap n\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP} \subseteq (m,n)\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP}$. □

Corollary 3.8 *$\bigcap\limits_{n\geq 2} n\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP} = 1\text{-}(\mathcal{P},\mathcal{Y})\text{-}\mathcal{GP}$.*

4. Categories of interest

The $(\mathcal{P},\mathcal{P})$-Gorenstein projective R-modules are exactly Gorenstein projectives and the $(\mathcal{P},\mathcal{F})$-Gorenstein projective R-modules are exactly Ding projectives [5].

Example 4.1 (1) Let R be a commutative ring and $x, y \in R$ such that $Ann_R(x) = (y)$ and $Ann_R(y) = (x)$; then we have a $\mathrm{Hom}(-,\mathcal{F})$-exact exact sequence $\cdots \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} \cdots$. Thus, (x) and (y) are Ding projective.

(2) Consider the quasi-Frobenius local ring $R = k[X]/(X^2)$, where k is a field, and denote by $\bar{X}$ the residue class in R of X. Then the ideal $(\bar{X})$ is in $1\text{-}(\mathcal{P},\mathcal{F})\text{-}\mathcal{GP}$. However, it is not projective, and hence $\mathcal{P} \subsetneq 1\text{-}(\mathcal{P},\mathcal{F})\text{-}\mathcal{GP}$.

(3) Consider the commutative Noetherian local ring $R = k[[X,Y]]/(XY)$, where k is a field. By part (1), $(\bar{X})$ and $(\bar{Y})$ are Ding projective. By [1, Example 2.13], they are not in $1\text{-}(\mathcal{P},\mathcal{F})\text{-}\mathcal{GP}$, and hence $1\text{-}(\mathcal{P},\mathcal{F})\text{-}\mathcal{GP} \subsetneq (\mathcal{P},\mathcal{F})\text{-}\mathcal{GP}$.

Remark 4.2 By Proposition 3.3, an R-module is Ding projective if and only if it is a direct summand of some R-module in $1\text{-}(\mathcal{P},\mathcal{F})\text{-}\mathcal{GP}$. Hence, Example 4.1(3) shows that $1\text{-}(\mathcal{P},\mathcal{F})\text{-}\mathcal{GP}$ is not closed under direct summands.

Proposition 4.3 (1) *An $(\mathcal{F},\mathcal{F})$-Gorenstein projective R-module M is either flat or $fd(M) = \infty$.* (2) *A Ding projective R-module M is either projective or $fd(M) = \infty$.*

Proof We only prove (1). If M is an $(\mathcal{F},\mathcal{F})$-Gorenstein projective R-module, then there is a $\mathrm{Hom}(-,\mathcal{F})$-exact exact sequence $0 \to K_n \to F_n \to \cdots \to F_1 \to M \to 0$ with every F_i flat. If $fd(M) \leq n$, then K_n is flat, and thus the sequence $0 \to \mathrm{Hom}(K_{n-1},K_n) \to \mathrm{Hom}(F_n,K_n) \to \mathrm{Hom}(K_n,K_n) \to 0$ is exact. Thus, K_{n-1} is flat as a direct summand of F_n. Continuing the procedure, we get that M is flat. □

Corollary 4.4 *If $wD(R) < \infty$, then*

1. $\mathcal{F} = 1\text{-}(\mathcal{F},\mathcal{F})\text{-}\mathcal{GP} = \cdots = n\text{-}(\mathcal{F},\mathcal{F})\text{-}\mathcal{GP} = \cdots = (\mathcal{F},\mathcal{F})\text{-}\mathcal{GP}$.
2. $\mathcal{P} = 1\text{-}(\mathcal{P},\mathcal{F})\text{-}\mathcal{GP} = \cdots = n\text{-}(\mathcal{P},\mathcal{F})\text{-}\mathcal{GP} = \cdots = (\mathcal{P},\mathcal{F})\text{-}\mathcal{GP}$.

Corollary 4.5 *The following are equivalent for any R-module M:*

1. $M \in n\text{-}(\mathcal{P},\mathcal{F})\text{-}\mathcal{GP}$.
2. *There is an exact sequence* $0 \to M \to P_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} M \to 0$ *with every P_i projective and* $\coprod_{i=1}^{n} \ker f_i \in 1\text{-}(\mathcal{P},\mathcal{F})\text{-}\mathcal{GP}$.
3. *There is an exact sequence* $0 \to M \to F_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} M \to 0$ *with every F_i flat and* $\coprod_{i=1}^{n} \ker f_i \in 1\text{-}(\mathcal{P},\mathcal{F})\text{-}\mathcal{GP}$.

Proof (1)$\Leftrightarrow$(2) is straightforward by Theorem 3.5; (2)$\Rightarrow$(3) is trivial.

(3)$\Rightarrow$(2) Consider the following short exact sequences:

$$0 \to \ker f_i \to F_i \to \ker f_{i-1} \to 0.$$

By Theorem 2.5, $\ker f_{i-1}$ and $\ker f_i$ are in $(\mathcal{P},\mathcal{F})\text{-}\mathcal{GP}$, and thus every $F_i \in (\mathcal{P},\mathcal{F})\text{-}\mathcal{GP}$. Hence, F_i is projective by Proposition 4.3(2). □

Recall that a ring R is right coherent if and only if every direct product of flat R-modules is flat, a ring R is left perfect if and only if every flat R-module is projective, and a ring R is right coherent and left perfect if and only if every direct product of projective R-modules is projective (see [7, pp. 113–114]).

Proposition 4.6 *The following are equivalent for a ring R:*

1. R is left perfect.
2. $\mathcal{F} \subseteq (\mathcal{P}, \mathcal{F})$-$\mathcal{GP}$.
3. $(\mathcal{F}, \mathcal{F})$-$\mathcal{GP} = (\mathcal{P}, \mathcal{F})$-$\mathcal{GP}$.
4. $(\mathcal{F}, \mathcal{P})$-$\mathcal{GP} = (\mathcal{P}, \mathcal{F})$-$\mathcal{GP}$.

   Moreover, if R is right coherent, then they are equivalent to

5. $\mathcal{F} \subseteq (\mathcal{P}, \mathcal{P})$-$\mathcal{GP}$.
6. $(\mathcal{F}, \mathcal{P})$-$\mathcal{GP} = (\mathcal{P}, \mathcal{P})$-$\mathcal{GP}$.

Proof $(1) \Rightarrow (4)$ and $(1) \Rightarrow (6)$ are obvious since $\mathcal{F} \subseteq \mathcal{P}$.

For any flat R-module F, considering the short exact sequence $0 \to F \stackrel{id}{\to} F \to 0$, we get $\mathcal{F} \subseteq (\mathcal{F}, \mathcal{F})$-$\mathcal{GP} \subseteq (\mathcal{F}, \mathcal{P})$-$\mathcal{GP}$, and hence $(3) \Rightarrow (2)$ and $(6) \Rightarrow (5)$ are trivial.

$(4) \Rightarrow (3)$ follows from the inclusions $(\mathcal{P}, \mathcal{F})$-$\mathcal{GP} \subseteq (\mathcal{F}, \mathcal{F})$-$\mathcal{GP} \subseteq (\mathcal{F}, \mathcal{P})$-$\mathcal{GP}$.

$(2) \Rightarrow (1)$ By Proposition 4.3(2), $\mathcal{F} \subseteq \mathcal{P}$, and hence R is left perfect.

$(5) \Rightarrow (1)$ Let $\{P_i\}_{i \in I}$ be a family of projective R-modules. Since R is right coherent, $\prod P_i$ is flat. By part (5), there exists a $\mathrm{Hom}(-, \mathcal{P})$-exact exact sequence $0 \to \prod P_i \to P \to L \to 0$ with P projective. Especially, the sequence $0 \to \mathrm{Hom}(L, P_i) \to \mathrm{Hom}(P, P_i) \to \mathrm{Hom}(\prod P_i, P_i) \to 0$ is exact, since every $P_i \in \mathcal{P}$. Thus, we get an exact sequence

$$0 \to \mathrm{Hom}(L, \prod P_i) \to \mathrm{Hom}(P, \prod P_i) \to \mathrm{Hom}(\prod P_i, \prod P_i) \to 0,$$

which shows that $\prod P_i$ is projective as a direct summand of P, as desired. □

Let $\mathcal{FI}$ denote the class of FP-injective R-modules [4, Definition 6.2.3]. We give the dual result of Proposition 4.6.

Proposition 4.7 *The following are equivalent for a ring R:*

1. R is left Noetherian.
2. $\mathcal{FI} \subseteq (\mathcal{FI}, \mathcal{I})$-$\mathcal{GI}$.
3. $(\mathcal{FI}, \mathcal{FI})$-$\mathcal{GI} = (\mathcal{FI}, \mathcal{I})$-$\mathcal{GI}$.
4. $(\mathcal{I}, \mathcal{FI})$-$\mathcal{GI} = (\mathcal{FI}, \mathcal{I})$-$\mathcal{GI}$.
5. $\mathcal{FI} \subseteq (\mathcal{I}, \mathcal{I})$-$\mathcal{GI}$.
6. $(\mathcal{I}, \mathcal{FI})$-$\mathcal{GI} = (\mathcal{I}, \mathcal{I})$-$\mathcal{GI}$.

Remark 4.8 We find that $(\mathcal{P}, \mathcal{F})$-$\mathcal{GP} \subseteq (\mathcal{F}, \mathcal{F})$-$\mathcal{GP}$, $(\mathcal{P}, \mathcal{P})$-$\mathcal{GP} \subseteq (\mathcal{F}, \mathcal{P})$-$\mathcal{GP}$, $(\mathcal{FI}, \mathcal{FI})$-$\mathcal{GI} \subseteq (\mathcal{FI}, \mathcal{I})$-$\mathcal{GI}$, and $(\mathcal{I}, \mathcal{FI})$-$\mathcal{GI} \subseteq (\mathcal{I}, \mathcal{I})$-$\mathcal{GI}$; the inclusions are strict.

Corollary 4.9 *The following are equivalent for a ring R:*

1. R is left perfect.
2. $\mathcal{F} \subseteq n$-$(\mathcal{P},\mathcal{F})$-$\mathcal{GP}$.

   Moreover, if R is right coherent, then the above are equivalent to
3. $\mathcal{F} \subseteq n$-$(\mathcal{P},\mathcal{P})$-$\mathcal{GP}$ $(n \gneq 1)$.

Corollary 4.10 *The following are equivalent for a ring R:*

1. R is left perfect.
2. $\mathcal{F} \subseteq 1$-$(\mathcal{P},\mathcal{F})$-$\mathcal{GP}$.
3. $\mathcal{F} \subseteq 1$-$(\mathcal{P},\mathcal{P})$-$\mathcal{GP}$.

Proof We only have to prove $(3) \Rightarrow (1)$. Let F be any flat R-module. By (3), there is an exact sequence $0 \to F \to P \to F \to 0$ with P projective. [2, Theorem 2.5] shows that F is projective. Hence, R is left perfect. □

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References

[1] Bennis D, Mahdou N. Strongly Gorenstein projective, injective and flat modules. J Pure Appl Algebra 2007; 210: 437–445.

[2] Benson DJ, Goodearl KR. Periodic flat modules and flat modules for finite groups. Pacific J Math 2000; 196: 45–67.

[3] Ding NQ, Li YL, Mao LX. Strongly Gorenstein flat modules. J Aust Math Soc 2009; 86: 323–338.

[4] Enochs EE, Jenda OMG. Relative Homological Algebra. Berlin, Germany: Walter de Gruyter, 2000.

[5] Gillespie J. Model structures on modules over Ding-Chen rings. Homology Homotopy Appl 2010; 12: 61–73.

[6] Holm H. Gorenstein homological dimensions. J Pure Appl Algebra 2004; 189: 167–193.

[7] Rotman JJ. An Introduction to Homological Algebra. New York, NY, USA: Academic Press, 1979.