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Balanced pair algorithm for a class of cubic substitutions

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Abstract: In this article we introduce the balanced pair algorithm associated with 2 unimodular Pisot substitutions having the same incidence matrix. We are interested in beta-substitution related to the polynomial $x^3 - ax^2 - bx - 1$ for $a \geq b \geq 1$. Applying the balanced pair algorithm to these substitutions, we obtain a general formula for the associated intersection substitution.

Key words: Rauzy fractals, substitution dynamical systems, balanced pair algorithm The author was supported by project I 11 36 funded by the Austrian Science Fund

1. Introduction

In 1982 Gérard Rauzy [16] studied the symbolic dynamical system over 3 letters $\{1, 2, 3\}$,

$$1 \to 12, 2 \to 13, 3 \to 1,$$

and associated to it a set known as Rauzy fractal. The Rauzy fractal is an important object in the study of dynamical systems associated to Pisot substitutions. In particular, it plays a fundamental role in the Pisot conjecture. Geometrical and topological properties of Rauzy fractals have been studied extensively; see, among other references, [2, 7, 3, 9, 15, 16, 19, 22]. It is a compact set equal to the closure of its interior and it decomposes naturally into 3 subtiles. The interiors of the subtiles associated to a primitive unimodular Pisot substitution do not overlap provided that the substitution satisfies the so-called strong coincidence condition [2]. Many classes of substitutions are shown to satisfy this condition. For example, Barge and Diamond proved in [4] that every irreducible Pisot substitution over 2 letters satisfies it. It is conjectured that this is true for alphabets of arbitrary size but a general proof is still outstanding.

Rauzy fractals appear naturally in connection with many topics such as numeration systems, geometrical representation of symbolic dynamical systems, multidimensional continued fractions and simultaneous approximations, self-similar tilings, and Markov partitions for hyperbolic automorphisms of the torus.

The aim of this paper is to study the intersection of Rauzy fractals associated to

$$\sigma_{a,b} : \begin{cases} 1 \to 1^a 2 \\ 2 \to 1^b 3 \\ 3 \to 1 \end{cases} \qquad \text{and} \qquad \sigma^*_{a,b} : \begin{cases} 1 \to 21^a \\ 2 \to 31^b \\ 3 \to 1 \end{cases}$$

over the alphabet $\mathcal{A} = \{1, 2, 3\}$, where $1 \leq b \leq a$. The 2 substitutions $\sigma_{a,b}$ and $\sigma^*_{a,b}$ have the same incidence

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matrix. For every pair (a,b), the substitution $\sigma_{a,b}$ is an irreducible primitive unimodular Pisot substitution. Moreover, it satisfies the super coincidence condition.

Intersection of Rauzy fractals was studied first by Sing and Sirvent. In their paper [20], they studied the prefixes-suffixes automaton associated with each substitution and considered the product automaton to obtain common points for intersection of the Rauzy fractals. They studied a sequence of dynamical systems defined on sets $\mathcal{F}_k$, apart from the common dynamics of irreducible Pisot substitutions with the same incidence matrix. This common dynamics is done throughout the family of the product automata of the prefix automata associated with the power of substitution σ_1^k and σ_2^k, but these common sets have zero Lebesgue measure.

In [17, 18], under the Pisot condition, we proved that the intersection of 2 Rauzy fractals associated with 2 unimodular irreducible Pisot substitutions having the same incidence matrix have nonzero Lebesgue measure. We showed that this intersection is substitutive. This means that the intersection can be seen as a new Rauzy fractal associated to a third substitution. The new substitution for the intersection is obtained by the balanced pair algorithm.

In the present paper we continue the study of the balanced pair algorithm. We apply this algorithm to the family of substitutions $\sigma_{a,b}$ and $\sigma^*_{a,b}$. We show that the substitution for the intersection has a regular form. We discuss 3 cases where $a=b$, $b=a-1$, and $b<a-1$. In the first case, applying the balanced pair algorithm, we obtain exactly 6 minimal balanced pairs for all a, while in the second and third cases we obtain only 7 minimal balanced pairs.

In Section 2 we give the definitions of these objects. We explain the projection method to obtain the Rauzy fractal associated with an irreducible Pisot substitution. In Section 3 we introduce the balanced pair algorithm, and its application to obtain substitution for intersection of Rauzy fractals. In Section 4 we introduce the balanced pair algorithm for a class of cubic substitutions and we obtain a general result of this class of Pisot substitutions. Finally, in Section 5, we present some examples.

2. Substitutions and Rauzy fractals

Let $\mathcal{A}$ be an alphabet of k letters. We denote by $\mathcal{A}^* = \cup_{i\geq 0}\mathcal{A}^i$ the free monoid on $\mathcal{A}$, that is, the set of finite words on the alphabet $\mathcal{A}$, endowed with the concatenation map. A substitution on a finite alphabet $\mathcal{A}$ is a map σ from $\mathcal{A}$ to the set $\mathcal{A}^*$. The substitution σ is extended in a natural way to an endomorphism of the monoid $\mathcal{A}^*$ by concatenation, i.e. $\sigma(\emptyset)=\emptyset$ and $\sigma(UV)=\sigma(U)\sigma(V)$, for all U, $V\in\mathcal{A}^*$. Let $A^{\mathbb{N}}$ (respectively $A^{\mathbb{Z}}$) denote the set of 1-sided (respectively 2-sided) infinite sequences in $\mathcal{A}$. The map σ is extended to $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$ in the obvious way.

Letting $u\in\mathcal{A}^{\mathbb{N}}$ (or $u\in\mathcal{A}^{\mathbb{Z}}$), u is a *fixed point* of σ if $\sigma(u)=u$ and *periodic* if there exists $l>0$ so that it is fixed for σ^l.

The incidence matrix M_σ of σ is the square matrix of size $k\times k$ defined by $M_\sigma=(m_{ij})$, where m_{ij} is the number of occurrences of the letter i in $\sigma(j)$. We denote by $l(U)$ the vector $\mathbf{l}(U)=(l_1(U),\dots,l_k(U))^t$. A substitution σ is unimodular if $det(M_\sigma)=\pm 1$. We say that the substitution is *primitive* if its incidence matrix is primitive, i.e. all the entries of M^r are positive for some $r>0$.

Every primitive substitution has at least one periodic point. All its periodic points have the same language. Let u be fixed point of σ and (Ω_u, S) its associated dynamical system, where S is the shift map on $\mathcal{A}^{\mathbb{N}}$ (respectively on $\mathcal{A}^{\mathbb{Z}}$) defined by $S(v_0v_1\cdots)=v_1\cdots$ (respectively $S(v)=w$, where $w_i=v_{i+1}$) and Ω_u is the closure of the orbit of the fixed point u under the shift map S.

A substitution is unit of Pisot type if the dominant eigenvalue of its incidence matrix is a unit Piost number, *i.e.* the determinant is ± 1 and all the roots of the characteristic polynomial have a modulus of less than or equal to 1. In other words, the matrix M_σ has one expanding eigenvalue β and all the other eigenvalues β^i are contracting.

A substitution is *irreducible Pisot* if it is Pisot and the characteristic polynomial of the incidence matrix is irreducible. An irreducible Pisot substitution is primitive [7].

There is a long conjecture stating that the dynamical system associated to an unimodular irreducible Pisot substitution is measurably conjugate to a translation on a $(k-1)$-dimensional torus (cf. [16, 23]). This conjecture is known in the literature as the Pisot conjecture. Rauzy approached it via geometrical realization of the symbolic system. He proved it in the case of the Tribonacci substitution, $\sigma(1) = 12$, $\sigma(2) = 13$, and $\sigma(3) = 1$ (cf. [16]). In his proof, the construction of a set in $\mathbb{R}^2$, in general $\mathbb{R}^{k-1}$, plays an important role. This set is known as the Rauzy fractal associated to the substitution. For references on conditions under which the Pisot conjecture is true, we refer to, among other references, [1, 2, 3, 4, 5, 6, 7, 11, 12, 15, 16, 21, 22, 23]. Before we define Rauzy fractals, we have to introduce some constructions and notations.

Letting σ be an unimodular Pisot substitution and λ the Perron–Frobenius eigenvalue of the incidence matrix M, λ is a Pisot number. The characteristic polynomial of M might be reducible, so the algebraic degree of λ is smaller than or equal to k, the cardinality of the alphabet $\mathcal{A}$. Let E^u be the λ-expanding space of M, the eigenspace associated to the eigenvalue λ; E^s the λ-contracting space of M, the eigenspaces associated to the Galois conjugates of λ; and E^c the M-invariant space such that $\mathbb{R}^k = E^u \oplus E^s \oplus E^c$. The space E^c is trivial if and only if the substitution is irreducible. Let $\pi : \mathbb{R}^k \to E^s$ be the projection of $\mathbb{R}^k$ onto E^s along $E^u \oplus E^c$.

Definition 2.1 *A stepped line $L = (x_n)$ in $\mathbb{R}^d$ is a sequence (it could be finite or infinite) of points in $\mathbb{R}^d$ such that $x_{n+1} - x_n$ belongs to a finite set.*
A canonical stepped line is a stepped line such that $x_0 = 0$ and for all $n \geq 0$, $x_{n+1} - x_n$ belongs to the canonical basis of $\mathbb{R}^d$.

Using the abelianization map l, with any finite or infinite word W, we can associate a canonical stepped line in $\mathbb{R}^d$ as the sequence $(l(V_n))$, where V_n is the prefix of length n of W.

We introduce a suitable decomposition of the space. We denote by m the algebraic degree of the Pisot number β; one has $m \leq d$, since the characteristic polynomial of M may be reducible. We denote E_s, the beta-contracting space of the matrix M generated by the eigenspaces associated to the beta-conjugates. Let E_u be the beta-expanding line of M, i.e. the real line generated by the beta-eigenvector u_β. Let E_n be the invariant space of M that satisfies $\mathbb{R}^d = E_s \oplus E_u \oplus E_n$. It is trivial if and only if the substitution is irreducible.

Let $\pi_s : \mathbb{R}^d \to E_s$ be the linear projection on the contracting space, along $E_u \oplus E_n$, according to the natural decomposition $\mathbb{R}^d = E_s \oplus E_u \oplus E_n$.

2.1. Definition of the Rauzy fractal

An interesting property of the canonical stepped line associated with a periodic point of irreducible Pisot substitution is that it remains within a bounded distance from the expanding direction given by the right Perron–Frobenius eigenvector of M. In the reducible case, the discrete line may have other expanding directions, but the projection of the discrete line by π_s still provides a bounded set; for more details, we refer to [8].

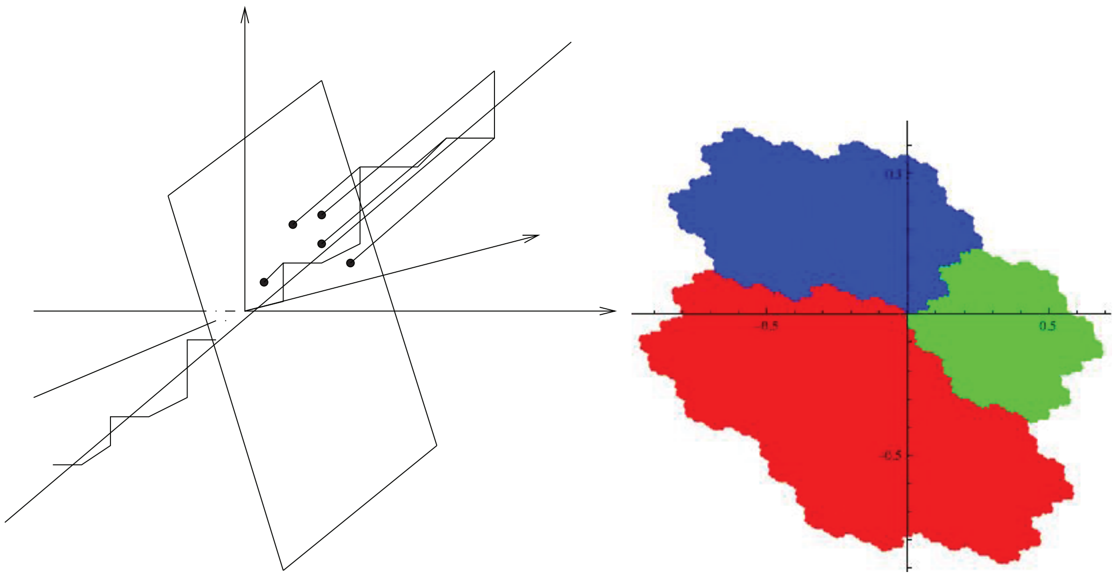

Figure 1. The projection method to get the Rauzy fractal.

Definition 2.2 *Let σ be a primitive unimodular Pisot substitution with dominant eigenvalue β. The Rauzy fractal of σ is the closure of the projection of the vertices of the canonical stepped line associated with any periodic point $u = (u_k)_{k\in\mathbb{N}}$ of σ on the beta-contracting space E_s; see Figure 1; i.e.*

$$\mathcal{R}_\sigma := \overline{\{\pi_s(l(u_0 \dots u_{k-1})), k \in \mathbb{N}\}}.$$

For each $i \in \mathcal{A}$ the subtiles of the central tile $\mathcal{R}_\sigma$ are naturally defined depending on the letter associated with the vertex of the stepped line that is projected:

$$\mathcal{R}_\sigma(i) := \overline{\{\pi_s(l(u_0 \dots u_{k-1}), k \in \mathbb{N}, u_k = i\}}.$$

Remark 1 *It follows from the primitivity of the substitution σ that the definition of $\mathcal{R}_\sigma$ and $\mathcal{R}_\sigma(i)$ ($i \in \mathcal{A}$) does not depend on the choice of the periodic point $u \in \mathcal{A}^{\mathbb{N}}$; see [2].*

We define the subgroup L of $\mathbb{Z}^d$ as :

$$L = \left\{ \sum_{i=1}^{d} n_i e_i : \sum_{i=1}^{d} n_i = 0, n_i \in \mathbb{Z} \right\}$$

Let Γ be the projection of L on the stable space, i.e. $\Gamma = \pi_s(L)$. In the irreducible case, the translation by Γ of the Rauzy fractal covers the stable space. Hence the Rauzy fractal has positive measure. The projection from the orbit of the periodic point to the Rauzy fractal extends by continuity to all of Ω_u. If we take the quotient by Γ, this projection gives a semiconjugacy between (Ω_u, S) and the translation by $\pi_s(e_i)$ on $\mathcal{R}_\sigma/\Gamma$, where e_i is any vector in the canonical base.

3. Balanced pair algorithm and fractals intersection

The balanced pair algorithm was introduced by Livshits in [12] in the study of the Pisot conjecture and was also used in [21] in the same context. A variant of this algorithm was used later in [17, 18] to study the intersection of Rauzy fractals associated with different substitutions having the same incidence matrix. This version of the balanced pair algorithm is used in the present article; we describe it in this section.

3.1. Balanced pair algorithm

We introduce the balanced pair algorithm for 2 substitutions σ_1 and σ_2 having the same incidence matrix. This algorithm was used in [17] and [18] in order to study the intersection of Rauzy fractals.

Definition 3.1 *Letting U and V be 2 finite words, we say that $\begin{pmatrix} U \\ V \end{pmatrix}$ is a* balanced pair *if $\mathbf{l}(U) = \mathbf{l}(V)$, where $\mathbf{l}(U)$ is the k-dimensional vector that gives the occurrences of the different symbols of the word U.*

Definition 3.2 *Given a word U, we denote by $\langle U \rangle_m$ the proper prefix of U of length m. A* minimal balanced pair *is a balanced pair $\begin{pmatrix} U \\ V \end{pmatrix}$, such that $\mathbf{l}(\langle U \rangle_m) \neq \mathbf{l}(\langle V \rangle_m)$, for $1 \leq m < |U|$.*

Let σ_1 and σ_2 be 2 irreducible Pisot substitutions with the same incidence matrix. Let u and v be the elements of $\mathcal{A}^{\mathbb{N}}$ that are fixed points of σ_1 and σ_2, respectively. The balanced pair algorithm gives a decomposition of the double fixed point $\begin{pmatrix} u \\ v \end{pmatrix}$ into minimal balanced pairs. The question is: when is the number of minimal balanced pairs bounded? We define the balanced pair algorithm associated to the substitutions σ_1 and σ_2 as follows:

Let $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ be the first minimal balanced pair. We iterate this first minimal balanced pair with the 2 substitutions σ_1 and σ_2, and this means that $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} \sigma_1(U) \\ \sigma_2(V) \end{pmatrix}$. Since the substitutions σ_1 and σ_2 have the same incidence matrix, the pair $\begin{pmatrix} \sigma_1(1) \\ \sigma_2(1) \end{pmatrix}$ is balanced. We decompose this new balanced pair with minimal balanced pairs.

With this decomposition, we obtain all the common points of the 2 stepped lines associated with the 2 fixed points u and v. This means that we obtain common points from the interior of the 2 associated Rauzy fractals of σ_1 and σ_2. Under the right hypotheses, considered in the next subsection, the set of minimal balanced pairs is finite, and the algorithm terminates.

3.2. Intersection of Rauzy fractals

Let σ_1 and σ_2 be 2 unimodular irreducible Pisot substitutions with the same incidence matrix. We consider their respective Rauzy fractals $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$. We suppose that 0, i.e. the origin, is an inner point of $\mathcal{R}_{\sigma_1}$. The intersection of $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$ is nonempty since it contains 0, and it is a compact set as the intersection of 2 compacts sets. Let $\mathcal{E}$ be the closure of the intersection of the interior of $\mathcal{R}_{\sigma_1}$ and the interior of $\mathcal{R}_{\sigma_2}$.

Proposition 3.1 *Let σ_1 and σ_2 be 2 unimodular irreducible Pisot substitutions with the same incidence matrix. We consider $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$ their associated Rauzy fractal. We suppose that 0 is an inner point to $\mathcal{R}_{\sigma_1}$. The set $\mathcal{E}$ then has nonempty interior and strictly positive Lebesgue measure.*

Proof Since 0 is an inner point of $\mathcal{R}_{\sigma_1}$, there exists an open set $\mathcal{U}$ such that $0 \in \mathcal{U} \subset \mathcal{R}_1$. The Rauzy fractal is the closure of its interior [22] and 0 is a point of $\mathcal{R}_{\sigma_2}$; hence, there exists a sequence of points $\{x_n\}_{n\in\mathbb{N}}$ in the interior of $\mathcal{R}_{\sigma_2}$ that converges to 0. Thus, there exist open sets $\mathcal{V}_n$ such that $x_n \in \mathcal{V}_n \subset \mathcal{R}_{\sigma_2}$. Since $\{x_n\}$ converges to 0, there exists $N \in \mathbb{N}$ such that $x_N \in \mathcal{U}$. The open set $\mathcal{U} \cap \mathcal{V}_N$ is nonempty and $\mathcal{U} \cap \mathcal{V}_N \subset \mathcal{R}_{\sigma_1} \cap \mathcal{R}_{\sigma_2}$. This implies that $\mathcal{E}$ contains a nonempty open set, and hence it has strictly positive Lebesgue measure. □

If the substitutions σ_1 and σ_2 satisfy the Pisot conjecture, then the set $\mathcal{E}$ is also a Rauzy fractal associated to the substitution defined by the balanced pair algorithm. More precisely, we will characterize this intersection of 2 Rauzy fractals associated with 2 unimodular Pisot substitutions with the same incidence matrix as follows.

Theorem 3.1 ([[18] Theorem 4.4]) *Let σ_1 and σ_2 be 2 unimodular irreducible Pisot substitutions with the same incidence matrix. Let $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$ be their 2 associated Rauzy fractals. Suppose that 0 is an inner point of $\mathcal{R}_{\sigma_1}$ and that both substitutions satisfy the Pisot conjecture. We denote by $\mathcal{E}$ the closure of the intersection of the interiors of $\mathcal{R}_{\sigma_1}$ and $\mathcal{R}_{\sigma_2}$. Then $\mathcal{E}$ has nonempty interior and is a substitutive set associated with a Pisot substitution Σ on the alphabet of minimal balanced pairs.*

Using Theorem 3.1 in the case of the substitutions $\sigma_{a,b}$ and $\sigma^*_{a,b}$, we obtain the following main result:

Theorem 3.2 *Let $\sigma_{a,b}$ and $\sigma^*_{a,b}$ be 2 substitutions defined as follows :*

$$\sigma_{a,b} : \begin{cases} 1 \to 1^a 2 \\ 2 \to 1^b 3 \\ 3 \to 1 \end{cases} \qquad \textit{and} \qquad \sigma^*_{a,b} : \begin{cases} 1 \to 21^a \\ 2 \to 31^b \\ 3 \to 1. \end{cases}$$

Let $\mathcal{R}_{a,b}$ and $\mathcal{R}^*_{a,b}$ be the respective Rauzy fractals. Then the set $\mathcal{R}_{a,b} \cap \mathcal{R}^*_{a,b}$ has nonempty interior and is a substitutive set associated to the substitution Σ obtained by the balanced pair algorithm. The substitution Σ is over 6 or 7 letters.

4. Balanced pair algorithm for a class of cubic substitutions

In this section we are interested in the 2 substitutions

$$\sigma_{a,b} : \begin{cases} 1 \to 1^a 2 \\ 2 \to 1^b 3 \\ 3 \to 1 \end{cases} \qquad \text{and} \qquad \sigma^*_{a,b} : \begin{cases} 1 \to 21^a \\ 2 \to 31^b \\ 3 \to 1 \end{cases}$$

and the intersection of their Rauzy fractals, where $1 \le b \le a$. The 2 substitutions $\sigma_{a,b}$ and $\sigma^*_{a,b}$ have the same incidence matrix defined by $M_{a,b} = \begin{pmatrix} a & b & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

For each (a,b), $\sigma_{a,b}$ and $\sigma^*_{a,b}$ are irreducible primitives unimodular substitutions. Moreover, it satisfies the super coincidence condition [5]. The substitutions $\sigma_{a,b}$ are so-called beta-substitutions (β verifies $\beta^3 = a\beta^2 + b\beta + 1$); that is, the induced dynamical system is related to beta-expansion.

The class of Rauzy fractals associated to $\sigma^*_{a,b}$ was first studied by Ito and Kimura in [10]. They showed that for $a=b=1$, the boundary of the Rauzy fractal is a Jordan curve, and they also computed its Hausdorff dimension. Later, for the same case, Messaoudi [14] constructed a finite state automaton that generates the boundary of the Rauzy fractal. This helped to show that this boundary is a quasicircle. In [14], analogous results were obtained for the car $a \geq 1$ and $b=1$. In [24] Thuswaldner gave an explicit formula for the fractal dimension of the boundary of the Rauzy fractal in the case $a \geq b \geq 1$. Recently, in [13], the authors described the boundary of the tiles by determining their neighbors in the tiling.

In this paper we are interested in the intersection of Rauzy fractals associated to $\sigma_{a,b}$ and $\sigma^*_{a,b}$. We give a general formula for substitution associated with the intersection. Applying the balanced pair algorithm, we prove that the substitution describing the intersection has a regular form and that this substitution is over 6 or 7 letters. We obtain the following proposition:

Proposition 4.1 *Let $\sigma_{a,b}$ and $\sigma^*_{a,b}$ be the 2 substitutions defined above with $a \geq b \geq 1$. Then we can characterize the set E of minimal balanced pairs as follows:*

- *If $a=b$ then* $E_1 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1^a2 \\ 21^a \end{pmatrix}, \begin{pmatrix} (1^a2)^a1^a3 \\ 31^a(21^a)^a \end{pmatrix}, \begin{pmatrix} 1^{a-1}2 \\ 21^{a-1} \end{pmatrix}, \begin{pmatrix} 1^{a-1}3 \\ 31^{a-1} \end{pmatrix}, \begin{pmatrix} (1^a2)^{a-1}1^a3 \\ 31^a(21^a)^{a-1} \end{pmatrix} \right\}.$

- *If $b=a-1$ then*

$$E_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1^a2 \\ 21^a \end{pmatrix}, \begin{pmatrix} (1^a2)^a1^b3 \\ 31^b(21^a)^a \end{pmatrix}, \begin{pmatrix} 1^{a-1}2 \\ 21^{a-1} \end{pmatrix}, \begin{pmatrix} 1^{a-1}21^{a-1}3 \\ 31^{a-1}21^{a-1} \end{pmatrix}, \begin{pmatrix} (1^a2)^{a-1}1^{a-1}3 \\ 31^{a-1}(21^a)^{a-1} \end{pmatrix}, \begin{pmatrix} 1^{a-2}3 \\ 31^{a-2} \end{pmatrix} \right\}.$$

- *If $b<a-1$ then*

$$E_3 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1^a2 \\ 21^a \end{pmatrix}, \begin{pmatrix} (1^a2)^a1^b3 \\ 31^b(21^a)^a \end{pmatrix}, \begin{pmatrix} 1^{a-1}2 \\ 21^{a-1} \end{pmatrix}, \begin{pmatrix} 1^{a-1}2(1^a2)^{a-b-1}1^b3 \\ 31^b(21^a)^{a-b-1}21^{a-1} \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} (1^a2)^{a-1}1^{a-1}3 \\ 31^{a-1}(21^a)a-1 \end{pmatrix}, \begin{pmatrix} 1^{a-1}2(1^b2)^{a-b-2}1^b3 \\ 31^b(21^a)^{a-b-2}21^{a-1} \end{pmatrix} \right\}.$$

Proof The proof of the first case was done by Sellami and Sirvent in their paper, “Symmetric intersection of Rauzy fractals (preprint). The second and the third cases can be proved similarly as the first case.

Let us prove the second case when $b=a-1$. By applying the balanced pair algorithm with $\sigma_{a,a}$ and $\sigma^*_{a,a}$ to the first minimal balanced pair $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we deduce the second new minimal balanced pair $B = \begin{pmatrix} 1^a2 \\ 21^a \end{pmatrix}$. We deduce the relation $A \longrightarrow B$. Again, we take the image of the minimal balanced pair B

$$\begin{pmatrix} 1^a2 \\ 21^a \end{pmatrix} \xrightarrow{\sigma_{a,a},\sigma^*_{a,a}} \begin{pmatrix} (1^a2)^a1^a3 \\ 31^a(21^a)^a \end{pmatrix}.$$

We obtain a new different balanced pair denoted C. This new balanced pair is minimal because the first word begins with the letter 3 and the second word finishes with 3, and the number of occurrence of this letter is only one. We thus have $B \longrightarrow C$.

By calculating the image of C, we obtain

$$\begin{pmatrix}(1^a2)^a1^{a-1}3\\31^{a-1}(21^a)^a\end{pmatrix}\overset{\sigma_{a,a},\sigma^*_{a,a}}{\longrightarrow}\begin{pmatrix}[(1^a2)^a1^{a-1}3]^a(1^a2)^{a-1}1\\1(21^a)^{a-1}[31^{a-1}(21^a)^a]^a\end{pmatrix}.$$

The last obtained balanced pair can be written as:

$$\left[\left[\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1^{a-1}2\\21^{a-1}\end{pmatrix}\right]^{a-1}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1^{a-1}21^{a-1}3\\31^{a-1}21^{a-1}\end{pmatrix}\right]^{a}\left[\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1^{a-1}2\\21^{a-1}\end{pmatrix}\right]^{a-1}\begin{pmatrix}1\\1\end{pmatrix}.$$

Two new minimal balanced pairs appear, which we denote by $D=\begin{pmatrix}1^{a-1}2\\21^{a-1}\end{pmatrix}$ and $E=\begin{pmatrix}1^{a-1}21^{a-1}3\\31^{a-1}21^{a-1}\end{pmatrix}$. The image of C is $[(AD)^{a-1}AE]^a(AD)^{a-1}A$. By applying the balanced pair algorithm to D we obtain a new balanced pair $F=\begin{pmatrix}(1^a2)^{a-1}1^{a-1}3\\31^{a-1}(21^a)^{a-1}\end{pmatrix}$. We remark that F is a minimal balanced pair, because the letter 3 is at the beginning and the end of the 2 words, and occurrence is only once.

We continue with the balanced pair algorithm; we take the image of E:

$$\begin{pmatrix}1^{a-1}21^{a-1}3\\31^{a-1}21^{a-1}\end{pmatrix}\overset{\sigma_{a,a},\sigma^*_{a,a}}{\longrightarrow}\begin{pmatrix}(1^a2)^{a-1}1^{a-1}3(1^a2)^{a-1}1\\1(21^a)^{a-1}31^{a-1}(21^a)^a]^{a-1}\end{pmatrix}.$$

The last balanced pair obtained can be decomposed with minimal balanced pairs as:

$$\left[\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1^{a-1}2\\21^{a-1}\end{pmatrix}\right]^{a-1}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1^{a-2}3\\31^{a-2}\end{pmatrix}\left[\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1^{a-1}2\\21^{a-1}\end{pmatrix}\right]^{a-1}\begin{pmatrix}1\\1\end{pmatrix}.$$

A new minimal balanced pair appears and we denote it by $G=\begin{pmatrix}1^{a-2}3\\31^{a-2}\end{pmatrix}$.

Finally, the image of G is $(AD)^{a-2}A$, and no new minimal balanced pairs appear.

In the third case we apply the same balanced pairs algorithm and we obtain the result. □

Theorem 4.1 *Let $\sigma_{a,b}$ and $\sigma^*_{a,b}$ be the 2 substitutions defined as before with $a\geq b\geq 1$. Let $\mathcal{R}_{a,b}$ and $\mathcal{R}^*_{a,b}$ be their 2 associated Rauzy fractals. Let $\Sigma_{a,b}$ be the substitution associated to the intersection of $\mathcal{R}_{a,b}$ and $\mathcal{R}^*_{a,b}$. Then $\Sigma_{a,b}$ is defined as follows:*

- *If $a=b$ then* $\Sigma^1_{a,b}:\begin{cases}A\to B\\B\to C\\C\to[(AD)^aAE]^a(AD)^aA\\D\to F\\E\to(AD)^{a-1}A\\F\to[(AD)^aAE]^{a-1}(AD)^aA.\end{cases}$

- *If* $b = a-1$ *then* $\Sigma^2_{a,b} : \begin{cases} A \to B \\ B \to C \\ C \to [(AD)^{a-1}AE]^a(AD)^{a-1}A \\ D \to F \\ E \to (AD)^{a-1}AG(AD)^{a-1}A \\ F \to [(AD)^{a-1}AE]^{a-1}(AD)^{a-1}A \\ G \to (AD)^{a-2}A. \end{cases}$

- *If* $b < a-1$ *then* $\Sigma^3_{a,b} : \begin{cases} A \to B \\ B \to C \\ C \to [(AD)^{b}AE]^a(AD)^{b}A \\ D \to F \\ E \to (AD)^{b}AG[(AD)^{b+1}AG]^{a-b-1}(AD)^{b}A \\ F \to [(AD)^{b}AE]^{a-1}(AD)^{b}A \\ G \to (AD)^{b}AG[(AD)^{a-2}AG]^{a-b-2}(AD)^{b}A. \end{cases}$

Here the capital letters A, B, C, D, E, F, G *represent minimal balanced pairs defined in Proposition 4.1; for each case, we have respectively:*

$E_1 = \{A, B, C, D, E, F\}$, $E_2 = \{A, B, C, D, E, F, G\}$, *and* $E_3 = \{A, B, C, D, E, F, G\}$.

Proof Applying $\sigma_{a,b}$ and $\sigma^*_{a,b}$ for the balanced pair algorithm defined in Proposition 4.1, we obtain for each minimal balanced pair a new balanced pair. By decomposition of this balanced pair into minimal balanced pairs we obtain the image of each one. In this case, the number of minimal balanced pairs is finite, so the image of each minimal balanced pair is a finite word on the alphabet of capital letters. Finally, we obtain the substitution associated with intersection for each case. □

Remark 2 *The characteristic polynomial of the substitutions* $\Sigma^i_{a,b}$ *where* $i = 1, 2, 3$ *respectively is:*

$P_1 = (X^3 - aX^2 - aX - 1) \times (X^3 + aX^2 + aX - 1)$.

$P_2 = (X+1) \times (X^3 - aX^2 - (a-1)X - 1) \times (X^3 + (a-1)X^2 + aX - 1)$.

$P_3 = (X+1)(X^3 - aX^2 - bX - 1) \times (X^3 + bX^2 + aX - 1)$.

We conjecture that the characteristic polynomial of the intersection substitution is a multiple of the product of the initial polynomial by its reciprocal.

5. Examples

In this section, we show examples of Rauzy fractals and their intersections for each case in Theorem 4.1.

- $a = b = 1$, $\sigma_{1,1} : \begin{cases} 1 \to 12 \\ 2 \to 13 \\ 3 \to 1 \end{cases}$, $\sigma^*_{1,1} : \begin{cases} 1 \to 21 \\ 2 \to 31 \\ 3 \to 1 \end{cases}$, and $\Sigma_{1,1} : \begin{cases} A \to B \\ B \to C \\ C \to ADAEADA \\ D \to F \\ E \to A \\ F \to ADA. \end{cases}$

The Rauzy fractals of $\sigma_{1,1}, \sigma^*_{1,1}$ and $\Sigma_{1,1}$ are shown in Figure 2.

Here, $A=\begin{pmatrix}1\\1\end{pmatrix}$, $B=\begin{pmatrix}12\\21\end{pmatrix}$, $C=\begin{pmatrix}1213\\3121\end{pmatrix}$, $D=\begin{pmatrix}2\\2\end{pmatrix}$, $E=\begin{pmatrix}3\\3\end{pmatrix}$, and, finally $F=\begin{pmatrix}13\\31\end{pmatrix}$.

$$\begin{pmatrix}1\\1\end{pmatrix}\longrightarrow\begin{pmatrix}12\\21\end{pmatrix}\longrightarrow\begin{pmatrix}1213\\3121\end{pmatrix}\longrightarrow\begin{pmatrix}1&2&1&3&1&2&1\\1&2&1&3&1&2&1\end{pmatrix}$$

$$A \longrightarrow B \longrightarrow C \longrightarrow A\ \ D\ \ A\ \ E\ \ A\ \ D\ \ A$$

$$\begin{pmatrix}2\\2\end{pmatrix}\longrightarrow\begin{pmatrix}13\\31\end{pmatrix}\longrightarrow\begin{pmatrix}1&2&1\\1&2&1\end{pmatrix}$$

$$D \longrightarrow F \longrightarrow A\ \ D\ \ A.$$

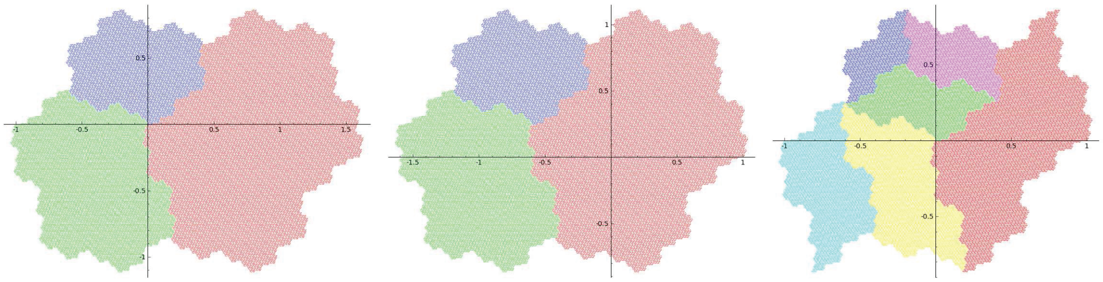

Figure 2. Rauzy fractal associated with $\sigma_{1,1}, \sigma^*_{1,1}$, and $\Sigma_{1,1}$.

- $a=3$ and $b=2$,

$$\sigma_{3,2}:\begin{cases}1\to 1112\\2\to 113\\3\to 1\end{cases},\quad \sigma^*_{3,2}:\begin{cases}1\to 2111\\2\to 311\\3\to 1\end{cases},\ \text{and}\quad \Sigma_{3,2}:\begin{cases}A\to B\\B\to C\\C\to[(AD)^2AE]^3(AD)^2A\\D\to F\\E\to(AD)^2AG(AD)^2A\\F\to[(AD)^2AE]^2(AD)^2A\\G\to ADA.\end{cases}$$

The Rauzy fractals of $\sigma_{3,2}, \sigma^*_{3,2}$, and $\Sigma_{3,2}$ are shown in Figure 3.

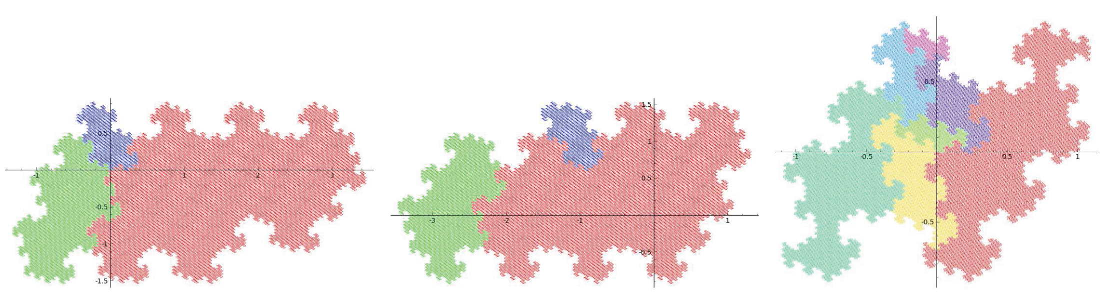

Figure 3. Rauzy fractal associated with $\sigma_{3,2}, \sigma^*_{3,2}$, and $\Sigma_{3,2}$.

- $a=4$ and $b=1$

$$\sigma_{4,1}: \begin{cases} 1 \to 11112 \\ 2 \to 13 \\ 3 \to 1 \end{cases}, \quad \sigma^*_{4,1}: \begin{cases} 1 \to 21111 \\ 2 \to 31 \\ 3 \to 1 \end{cases}, \text{ and } \quad \Sigma_{4,1}: \begin{cases} A \to B \\ B \to C \\ C \to [ADAE]^4 ADA \\ D \to F \\ E \to ADAG[(AD)^2 AG]^2 ADA \\ F \to [ADAE]^3 ADA \\ G \to ADAG(AD)^2 AGADA. \end{cases}$$

The Rauzy fractals of $\sigma_{4,1}, \sigma^*_{4,1}$, and $\Sigma_{4,1}$ are shown in Figure 4.

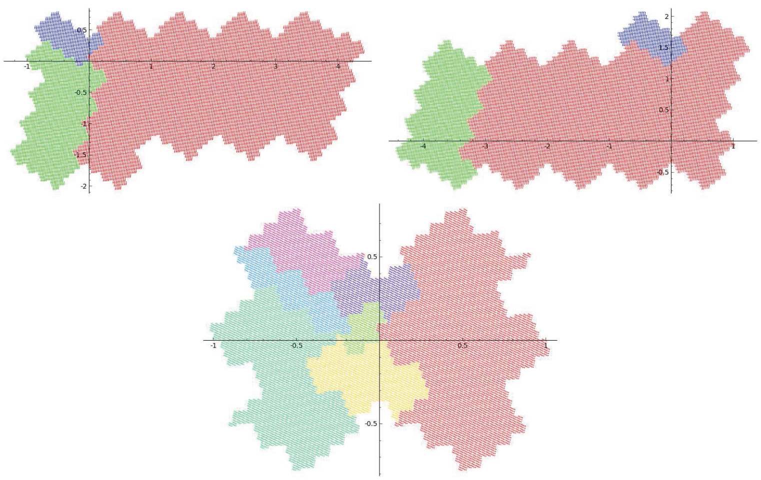

Figure 4. Rauzy fractal associated with $\sigma_{4,1}, \sigma^*_{4,1}$, and $\Sigma_{4,1}$.

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