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Generalized Heineken–Mohamed type groups

Orest ARTEMOVYCH*
Institute of Mathematics, Cracow University of Technology, Krakow, Poland

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Abstract: We prove that a torsion group G with all subgroups subnormal is a nilpotent group or $G = N(A_1 \times \cdots \times A_n)$ is a product of a normal nilpotent subgroup N and p_i-subgroups A_i, where $A_i = A_1^{(i)} \cdots A_{m_i}^{(i)} \lhd G$, $A_j^{(i)}$ is a Heineken–Mohamed type group, and $p_1, \ldots, p_n$ are pairwise distinct primes ($n \geq 1$; $i = 1, \ldots, n$; $j = 1, \ldots, m_i$ and m_i are positive integers).

Key words: Nilpotent group, indecomposable group, Heineken-Mohamed type group

1. Introduction

The concept of an HM^*-group (i.e. a p-group G with the nilpotent derived subgroup G' and the divisible Černikov quotient group G/G') was introduced by Asar [2] in the class of infinite p-groups. Quasicyclic groups, Heineken-Mohamed type groups (i.e. nonnilpotent groups with nilpotent and subnormal proper subgroups) and Čarin groups (see [5, 10] and [22, Example 1.5.1]) are HM^*-groups. Recall that a group G is called *indecomposable* if any two proper subgroups generate a proper subgroup in G and is called *decomposable* otherwise. Heineken–Mohamed type groups are indecomposable p-groups with the normalizer condition (see examples in [15, 18, 24]). With indecomposable groups are connected barely transitive groups (see [21, Lemma 2.10]).

Casolo [11] proved that a torsion group G with all subgroups subnormal contains a normal nilpotent subgroup N such that G/N is an abelian divisible group of finite rank (see also [12]). We make this result more precise in the following:

Theorem 1.1 *Let G be a torsion group with all subgroups subnormal. Then G is a nilpotent group or $G = NA$ is a product of a normal nilpotent subgroup N and a nonnilpotent subgroup*

$$A = A_1 \times \cdots \times A_n \ (n \geq 1),$$

where A_i is a p_i-group, $A_i = A_1^{(i)} \cdots A_{m_i}^{(i)} \lhd G$, $A_j^{(i)}$ is a Heineken–Mohamed type group, and $p_1, \ldots, p_n$ are pairwise distinct primes ($i = 1, \ldots, n$; $j = 1, \ldots, m_i$ and m_i are positive integers).

Recently Smith [28] established that a group G with all subgroups subnormal such that its torsion part is a π-group, where π is a finite set of primes, is nilpotent-by-(divisible Černikov).

*Correspondence: artemo@usk.pk.edu.pl
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Throughout, p will always denote a prime, $\mathbb{C}_{p^\infty}$ the quasicyclic p-group, and $\mathbb{Z}$ the ring of integers. For a group G, $G', G'', \dots, G^{(n)}, \dots$ will indicate the terms of derived series of G and G^n the subgroup of G generated by the nth powers of all elements in G.

For notations see [19, 23, 27].

2. Preliminary results

In this part we collect some results concerning groups G with the locally nilpotent derived subgroup G' and the divisible Černikov quotient p-group G/G'.

Lemma 2.1 *Let G be a group with the locally nilpotent derived subgroup G' and the quotient group G/G' be a divisible Černikov p-group. Then the following hold:*

(i) $G' = [G', G]$*;*

(ii) each nontrivial homomorphic image of G has infinite exponent and, in particular, $G = G^m$ for each positive integer m;

(iii) G' contains no proper G-invariant subgroup of finite index;

(iv) if G is a hypercentral group, then it is a divisible Černikov p-group.

Proof For the proof see, e.g., [2, 3]. □

Lemma 2.2 *[20, Chapter XV, § 63] Let G be a group with the normalizer condition. If H is a normal hypercentral subgroup of G and the quotient group G/H is cyclic, then G is a hypercentral group.*

Lemma 2.3 *[13, Lemma 1.18] If a hypercentral group $G = AB$ is a product of an abelian normal subgroup A and a divisible abelian subgroup B, then G is abelian.*

Lemma 2.4 *[13, Proposition 1.7] If a p-group G contains an abelian subgroup of finite index, then it is hypercentral.*

Lemma 2.5 *Let G be a nonhypercentral group with the normalizer condition. If G contains a normal hypercentral soluble subgroup A with the quasicyclic quotient group G/A, then every nonhypercentral subgroup is subnormal in G.*

Proof If H is a proper nonhypercentral subgroup of G, then, in view of Lemma 2.2, $G = AH$ and, by [17, Lemma],

$$G/A'(A \cap H)$$

is a hypercentral group. Therefore, it is abelian by Lemma 2.3 and $A'H \lhd G$. Since $A'H/A''(A' \cap H)$ is hypercentral, by the same argument we obtain that $A''H \lhd A'H$. By induction, H is subnormal in G. □

Lemma 2.6 *The following properties hold:*

(i) *Any indecomposable soluble torsion group G is a locally finite p-group.*

(ii) *Let G be a group with the hypercentral derived subgroup G' and the quotient group G/G' be a divisible Černikov p-group. If G is indecomposable, then it is a locally finite p-group.*

(iii) *Any indecomposable locally nilpotent group is a p-group.*

Proof (i) If G is abelian and indecomposable, then it is a quasicyclic or cyclic p-group. Therefore, we suppose that G is nonabelian. Then G/G' is a divisible Černikov p-group. Assume that there is an integer k such that $G/G^{(k)}$ is a p-group and $G^{(k)}$ is not a p-group, where $1 \leq k \leq n-1$ and n is the derived length of G.

The group G is a soluble torsion group and so it is a locally finite group. Since G is soluble and indecomposable, it is countable. Then G has a local system consisting of its finite subgroups. Since the group G contains an element of order p and, by Hall's Theorem, it has a Hall p-subgroup and a Hall p'-subgroup, G is generated by them. Then G being indecomposable implies that G is a p-group.

(ii) We denote the derived subgroup G' by H. If H is not torsion, then the torsion part τH is normal in H and the quotient group $H_1 = H/\tau H$ is torsion-free hypercentral. Then H_1/H_1' is not torsion (see, e.g., [27, Exercises 12.2.6]) and therefore without loss of generality we can assume that H is a torsion-free abelian group. Let q be a prime different from p. By [9, Lemma 2.3], there exists a proper submodule N of a $\mathbb{Z}[G/H]$-module H such that H/N is torsion as a group and contains an element of order q. Then G/N is a decomposable group, a contradiction. Hence, G is a torsion group and this yields that G is a locally finite p-group.

(iii) It follows by the same argument as in (ii). □

If G is a barely transitive group with the locally nilpotent derived subgroup G' and the quotient group G/G' is a divisible Černikov p-group, then G is an indecomposable locally finite p-group by Lemma 2.3 and [7, Proposition 2.4]. In view of Lemma 2.6, it is a very interesting open question whether nontorsion (and therefore non-"barely transitive") indecomposable soluble groups exist.

Lemma 2.7 *Let G be a group with the hypercentral derived subgroup G' and the divisible Černikov quotient p-group G/G'. Then the following hold:*

(1) *if the derived subgroup G' has no proper supplements in G, then G is a p-group;*

(2) *if G is an indecomposable group, then every proper subgroup of it is hypercentral;*

(3) *if G is a soluble group with the derived subgroup G' of finite exponent, then every finite subgroup is subnormal in G;*

(4) *if G is a soluble group with the normalizer condition, then*

- (i) *G' is not supplemented nontrivially in G;*
- (ii) *$G/G' \cong \mathbb{C}_{p^\infty}$ if and only if G is an indecomposable group;*
- (iii) *if G' is nilpotent of finite exponent, then all subgroups are subnormal in G.*

Proof (1) We can prove by the same argument as in the proof of Lemma 2.6(ii).

(2) Since G/G' is a quasicyclic group and $G'H$ is proper in G for any proper subgroup of G, we conclude that $G'H = G'\langle a\rangle$ for some element $a \in G$. Hence, $G'H$ is a hypercentral group.

(3) If F is a finite subgroup of G, then $G'F$ is a proper subgroup of G and we deduce that $\overline{G}'\overline{F}$ is a nilpotent subgroup of $\overline{G} = G/G''$ by [4, Lemma 3.8]. As a consequence, $G''F$ is subnormal in G. Repeating this process, we obtain that $G^{(n)}F$ is subnormal in G for any positive integer n.

(4) Let G be a soluble group with the normalizer condition. Since G' is hypercentral and G/G' is a p-group, G is a p-group. By [17, Lemma], $G = G'S$ for some proper subgroup S and it follows that the quotient group

$$\overline{G} = G/G''(G' \cap S)$$

is hypercentral, which leads to a contradiction. If, moreover, G' is of finite exponent, then every subgroup is subnormal in G by condition (3) and [2, Lemma 3.6]. □

Lemma 2.8 *Let G be a nonhypercentral group with the hypercentral derived subgroup G' and the divisible Černikov quotient p-group G/G'. Then the following hold:*

(1) *if all subgroups of G are subnormal, then*

$$G = G_1 \cdots G_n \cdot D \ (n \geq 1),$$

where D is a nilpotent group, G_i is a Heineken–Mohamed type group, $D' \leq G' = G'_1 \cdots G'_n \leq D$ and $G'_i = [G'_i, G_i] = [G', G_i] = [G'_i, G]$;

(2) *if G/G' is a quasicyclic p-group, then*

(*a*) *G satisfies the normalizer condition if and only if it is a minimal nonhypercentral group;*

(*b*) *every proper subgroup is subnormal in G if and only if G is a Heineken–Mohamed type group.*

Proof (1) By definition,

$$\overline{G} = G/G' = \overline{T}_1 \times \cdots \times \overline{T}_s$$

is a direct product of quasicyclic p-subgroups $\overline{T}_i$ $(i = 1, \dots, s)$. If T_i is an inverse image of $\overline{T}_i$ in G, then

$$G' \leq T_i \lhd G.$$

Since G is nonhypercentral, there exists an integer l $(1 \leq l \leq s)$ such that $T_1, \dots, T_l$ are nonhypercentral, while $T_{l+1}, \dots, T_s$ are hypercentral groups. We denote the product $T_{l+1} \cdots T_s$ by D. Then D is hypercentral and, by [26, Theorem 2.7], it is a nilpotent group.

Let $T = T_i$ $(i = 1, \dots, l)$. Obviously,

$$[G', T] \leq T' \leq G'.$$

Moreover, the quotient group $T/[G', T]$ (and respectively $T/[T', T]$) is nilpotent (and consequently abelian), and therefore we obtain that

$$T' = [T', T] = [T', G] = [G', T].$$

Since $G/(T'_1 \cdots T'_l)$ is a hypercentral group, we conclude that the derived subgroup $G' = T'_1 \cdots T'_l$.

(2) Let G/G' be a quasicyclic p-group.

(a) Suppose that G satisfies the normalizer condition. If it is decomposable, then $G = G'S$ for some $S < G$, and so the quotient group

$$G/G''(G' \cap S)$$

is abelian by [17, Lemma], a contradiction. Hence, G is an indecomposable group. By Lemma 2.7(2), G is a minimal nonhypercentral group. The converse is an immediate consequence of the fact that $G'K$ is proper in G for every proper subgroup K of G and therefore K is not equal to its normalizer $N_G(K)$.

(b) Assume that every proper subgroup is subnormal in G. Then G satisfies the normalizer condition. By [11, Theorem 1], the derived subgroup G' is nilpotent. If G is a hypercentral group, then, by [26, Theorem 2.7], it is nilpotent and, consequently, $G \cong \mathbb{C}_{p^\infty}$. Therefore, we assume that G is nonhypercentral and, by (a), it is a minimal nonhypercentral group. If we assume that G is decomposable (and so $G = G'S$ for some $S < G$), then, by [17, Lemma],

$$\overline{G} = G/(G' \cap S) = \overline{G}' \rtimes \overline{S}$$

is a hypercentral group. However, then it is abelian, a contradiction. Hence, G is an indecomposable group and for any $K < G$ there exists $a \in G$ such that $G'K = G'\langle a \rangle < G$ and, by [1, Lemma 2.4], $G'K$ is a nilpotent group. Thus, G is a Heineken–Mohamed type group.

The converse is clear. □

Corollary 2.9 *Let G be a nonabelian indecomposable soluble group. If G' is a nilpotent group of finite (respectively infinite) exponent, then G is a Heineken–Mohamed type group (respectively a minimal nonhypercentral group).*

Proof By Lemma 2.6, G is a p-group for some prime p. Let B be a proper subgroup of G/G' and $E(G', B)$ be a subgroup of G, which is an extension of G' by B. If G' is a nilpotent group of finite (respectively infinite) exponent, then [4, Lemma 3.8] (respectively Lemma 2.4) yields that the quotient group $E(G', B)/G''$ is nilpotent (respectively hypercentral). By the Hall Theorem [14, Theorem 7] (respectively the Betten Theorem [8]), $E(G', B)$ is a nilpotent (respectively hypercentral) group. Hence, G is a minimal nonnilpotent (respectively nonhypercentral) group. Moreover, if the derived subgroup G' is of finite exponent, then, by Lemma 2.7, G is a Heineken–Mohamed type group. □

Belyaev and Kuzucuoğlu [6, Corollary 1] proved that a Heineken–Mohamed type group has a faithful barely transitive representation. Then we prove the following:

Corollary 2.10 *A locally finite barely transitive group G with almost locally p-soluble point stabilizer is a Heineken–Mohamed type group or $G \cong \mathbb{C}_{p^\infty}$.*

Proof By [21, Lemma 2.10], G is indecomposable and, by Proposition 3 of [16], the derived subgroup G' is nilpotent of finite exponent. If G is nonabelian, then it is a Heineken–Mohamed type group by Corollary 2.9. □

3. Proof of Theorem 1.1

Suppose that G is a nonnilpotent p-group. As it was proved in [11], a torsion group G with subnormal proper subgroups contains a nilpotent normal subgroup N such that G/N is a divisible abelian Černikov p-group. If

$N^p = N$, then, by [25, Lemma 1], a subgroup $N\langle a\rangle$ is nilpotent (and so abelian) for any $a \in G$, a contradiction. Hence, $N^p \neq N$.

Let k, l be nonnegative integers. By [2, Lemma 3.3],

$$\overline{G} = G/N^{p^k} = \overline{T}\ \overline{N}$$

is a product of a maximal normal HM^*-subgroup $\overline{T}$ and a normal nilpotent subgroup $\overline{N}$. If we denote the inverse image of $\overline{T}$ by T and $M = T \cap N$, then, by [2, Lemma 3.3],

$$T/M^{p^k} = \overline{T}_0\ \overline{M},$$

where $\overline{T}_0$ is a maximal normal HM^*-subgroup of T/M^{p^k}. Let T_0 be an inverse image of $\overline{T}_0$ in a group T. If $l < k$, then

$$T_0/M^{p^k}T_0' \text{ and } T_0/M^{p^l}T_0'$$

are divisible Černikov groups, $M^{p^k}T_0' \leq M^{p^l}T_0'$, and so the section

$$M^{p^l}T_0'/M^{p^k}T_0'$$

is a finite group. There exists a subgroup D of T_0 such that D/T_0' is the divisible part of T_0/T_0'. Then $T = T_0M = DM$ and $D' = T_0'$. This yields that D is an HM^*-group. The conclusion follows from Lemma 2.8.

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