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Symplectic groupoids and generalized almost subtangent manifolds

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Abstract: We obtain equivalent assertions among the integrability conditions of generalized almost subtangent manifolds, the condition of compatibility of source and target maps of symplectic groupoids with symplectic form, and generalized subtangent maps.

Key words: Lie groupoid, symplectic groupoid, generalized almost subtangent manifold

1. Introduction

The concept of groupoid was introduced by Ehresmann [5] in the 1950s, following his work on the concept of principal bundle. A groupoid G consists of 2 sets G_1 and G_0, called arrows and objects, respectively, with maps $s,t : G_1 \to G_0$ called source and target. It is equipped with a composition $m : G_2 \to G_1$ defined on the subset $G_2 = \{(g,h) \in G_1 \times G_1 | s(g) = t(h)\}$, an inclusion map of objects $e : G_0 \to G_1$, and an inversion map $i : G_1 \to G_1$. For a groupoid, the following properties are satisfied: $s(gh) = s(h)$, $t(gh) = t(g)$, $s(g^{-1}) = t(g)$, $t(g^{-1}) = s(g)$, $g(hf) = (gh)f$ whenever both sides are defined, $g^{-1}g = 1_{s(g)}$, $gg^{-1} = 1_{t(g)}$. Here we have used $gh, 1_x$ and g^{-1} instead of $m(g,h)$, $e(x)$, and $i(g)$. Generally, a groupoid G is denoted by the set of arrows G_1.

A topological groupoid is a groupoid G_1 whose set of arrows and set of objects are both topological spaces whose structure maps s,t,e,i,m are all continuous and s,t are open maps. A Lie groupoid is a groupoid G whose set of arrows and set of objects are both manifolds whose structure maps s,t,e,i,m are all smooth maps and s,t are submersions. The latter condition ensures that s and t-fibers are manifolds. One can see from the above definition the space G_2 of composable arrows is a submanifold of $G_1 \times G_1$. A Lie groupoid, as well as being a generalization of a Lie group, is an alternative formulation of the concept of principal bundle.

The concept of Lie algebroid was introduced by Pradines [13], as the first-order invariant attached to a Lie groupoid, generalizing the construction of the Lie algebra of a Lie group. Lie algebroids arise naturally as the infinitesimal parts of Lie groupoids, in complete analogy to the way that Lie algebras arise as the infinitesimal part of Lie groups and it unifies various different types of infinitesimal structures: foliated manifolds, Poisson manifolds, and infinitesimal actions of Lie algebras on manifolds. Lie's third theorem says that given a finite-dimensional Lie algebra, there is a Lie group having the given Lie algebra as its Lie algebra. A Lie algebroid is said to be integrable when it is the Lie algebroid of a Lie groupoid. In [1], the authors showed a transitive Lie algebroid that is not the Lie algebroid of any Lie groupoid. Thus the analogue of Lie's third theorem fails for

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Lie algebroids; for a complete satisfactory answer for this problem, see: [4]. Formally, a Lie algebroid structure on a real vector bundle A on a manifold M is defined by a vector bundle map $\rho_A : A \to TM$, the anchor of A, and an $\mathbb{R}$-Lie algebra bracket on $\Gamma(A), [,]_A$ satisfying the Leibnitz rule

$$[\alpha, f\beta]_A = f[\alpha,\beta]_A + L_{\rho_A(\alpha)}(f)\beta$$

for all $\alpha,\beta \in \Gamma(A), f \in C^\infty(M)$, where $L_{\rho_A(\alpha)}$ is the Lie derivative with respect to the vector field $\rho_A(\alpha)$. $\Gamma(A)$ denotes the set of sections in A.

As a unification of complex manifolds and symplectic manifolds, the notion of generalized complex manifolds was introduced by Hitchin [7]. Later the notion of generalized Kähler manifold was introduced by Gualtieri [6] and submanifolds of such manifolds have been studied in many papers. As an analogue of generalized complex structures on even dimensional manifolds, the concept of generalized almost subtangent manifolds was introduced in [15] and such manifolds have been studied in [15, 17].

Recently, Crainic [3] showed that there is a close relationship between the integrability conditions of the almost complex structure of a generalized almost complex manifold and Lie groupoids. More precisely, he obtained interesting interpretations of integrability conditions of generalized almost complex manifolds in terms of Lie groupoids and Lie algebroids.

In this paper, we study generalized almost subtangent manifolds and symplectic groupoids. We investigate relationships between the integrability conditions of the almost subtangent structure of a generalized almost subtangent manifold and Lie groupoids, and we show that such conditions can be expressed in terms of symplectic groupoids, Lie algebroids, and Poisson manifolds. We also obtain equivalent relations among an integrability condition, source map, target map of symplectic groupoids, and generalized subtangent maps. Since the source map and target map are very important in a Lie groupoid, this last result gives new tools to investigate certain properties of generalized almost subtangent manifolds and symplectic groupoids.

2. Preliminaries

In this section we recall the basic facts of Poisson geometry, Lie groupoids, and Lie algebroids. More details can be found in [10, 16]. A central idea in generalized geometry is that $TM \oplus T^*M$ should be thought of as a generalized tangent bundle to manifold M. If X and ξ denote a vector field and a dual vector field on M, respectively, then we write (X,ξ) (or $X+\xi$) as a typical element of $TM \oplus T^*M$. The Courant bracket of 2 sections $(X,\xi),(Y,\eta)$ of $TM \oplus T^*M = \mathcal{TM}$ is defined by

$$[\![(X,\xi),(Y,\eta)]\!] = ([X,Y], L_X\eta - L_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi)), \tag{2.1}$$

where d, L_X, and i_X denote exterior derivative, Lie derivative, and interior derivative with respect to X, respectively. The Courant bracket is antisymmetric but it does not satisfy the Jacobi identity. We adapt the notions $\beta(\pi^\sharp\alpha) = \pi(\alpha,\beta)$ and $\omega_\sharp(X)(Y) = \omega(X,Y)$, which are defined as $\pi^\sharp : T^*M \to TM$, $\omega_\sharp : TM \to T^*M$ for any 1-forms α and β, 2-form ω and bivector field π, and vector fields X and Y. Moreover, we denote by $[,]_\pi$, the bracket on the space of 1-forms on M defined by

$$[\alpha,\beta]_\pi = L_{\pi^\sharp\alpha}\beta - L_{\pi^\sharp\beta}\alpha - d\pi(\alpha,\beta).$$

On the other hand, a symplectic manifold is a smooth (even dimensional) manifold M with a nondegenerate closed 2-form $\omega \in \Omega^2(M)$. ω is called the symplectic form of M. Let G be a Lie groupoid on M and ω

a form on Lie groupoid G; then ω is called multiplicative if

$$m^*\omega = pr_1^*\omega + pr_2^*\omega,$$

where $pr_i : G \times G \to G$, $i = 1, 2$, are the canonical projections. If a Lie groupoid G is endowed with a symplectic form that is multiplicative, then G is called a symplectic groupoid.

A smooth manifold is a Poisson manifold if $[\pi, \pi] = 0$, where $[,]$ denotes the Schouten bracket on the space of multivector fields and π is a bivector field.

We now give a relation between Lie algebroid and Lie groupoid. The relations between Lie groupoids and Lie algebroids have been studied by many authors, see: [9, 18]. Given a Lie groupoid G on M, the associated Lie algebroid $A = Lie(G)$ has fibers $A_x = Ker(ds)_x = T_x(G(-,x))$, for any $x \in M$. Any $\alpha \in \Gamma(A)$ extends to a unique right-invariant vector field on G, which will be denoted by the same letter α. The usual Lie bracket on vector fields induces the bracket on $\Gamma(A)$, and the anchor is given by $\rho = dt : A \to TM$. As we mentioned in the introduction, not every Lie algebroid admits an integration. However, if a Lie algebroid is integrable, then there exists a canonical source simply connected integration G, and any other source simply connected integration is smoothly isomorphic to G. From now on we assume that all Lie groupoids are source-simply-connected.

In this section, finally, we recall the notion of IM form (infinitesimal multiplicative form) on a Lie algebroid [2]. An IM form on a Lie algebroid A is a bundle map

$$u : A \to T^*M$$

satisfying the following properties

(i) $\langle u(\alpha), \rho(\beta)\rangle = -\langle u(\beta), \rho(\alpha)\rangle$

(ii) $u([\alpha, \beta]) = L_\alpha(u(\beta)) - L_\beta(u(\alpha)) + d\langle u(\alpha), \rho(\beta)\rangle$

for $\alpha, \beta \in \Gamma(A)$, where $\langle,\rangle$ denotes the usual pairing between a vector space and its dual. If A is a Lie algebroid of a Lie groupoid G, then a closed multiplicative 2-form ω on G induces an IM form u_ω of A by

$$\langle u_\omega(\alpha), X\rangle = \omega(\alpha, X).$$

For the relationship between IM form and closed 2-form we have the following.

Theorem 1 *[2]If A is an integrable Lie algebroid and if G is its integration, then $\omega \mapsto u_\omega$ is a one to one correspondence between closed multiplicative 2-forms on G and IM forms of A.*

3. Symplectic groupoids and generalized subtangent structures

In this section we first recall a characterization for generalized subtangent manifolds; then we obtain certain relationships between generalized subtangent manifolds and symplectic groupoids. We recall that a generalized almost subtangent structure $\mathcal{J}$ is an endomorphism on $\mathcal{TM}$ such that $\mathcal{J}^2 = 0$ and $\mathcal{J}$ is antisymmetric with respect to the canonical symmetric bilinear operation given by

$$\langle (X,\eta), (Y,\xi)\rangle = \frac{1}{2}(\eta(Y) + \xi(X)) \text{ for all sections } (X,\eta), (Y,\xi) \in \mathcal{TM}.$$

A generalized almost subtangent structure can be represented by classical tensor fields as follows:

$$\mathcal{J} = \begin{bmatrix} a & \pi^\sharp \\ \sigma_\sharp & -a^* \end{bmatrix} \tag{3.1}$$

where π is a bivector on M, σ is a 2-form on M, $a : TM \to TM$ is a bundle map, and $a^* : T^*M \to T^*M$ is dual of a, for almost subtangent structures see:[15].

Example 1 *[15] Associated to any almost subtangent structure φ, we have a generalized almost subtangent structure by setting*

$$\mathcal{J} = \begin{bmatrix} \varphi & 0 \\ 0 & -\varphi^* \end{bmatrix},$$

where φ^* is dual of φ.

A generalized almost subtangent structure is called integrable (or just subtangent structure) if $\mathcal{J}$ satisfies the following condition

$$[\![\mathcal{J}\alpha, \mathcal{J}\beta]\!] - \mathcal{J}([\![\mathcal{J}\alpha, \beta]\!] + [\![\alpha, \mathcal{J}\beta]\!]) = 0, \tag{3.2}$$

for all sections $\alpha, \beta \in \mathcal{TM}$, such that $\alpha = (X, \xi)$, $\beta = (Y, \eta)$, where $X, Y \in TM$ and $\xi, \eta \in T^*M$.

In the sequel, we give necessary and sufficient conditions for a generalized almost subtangent structure to be integrable in terms of the above tensor fields. We note that the following result was given in [15]; here we prove one condition to give some clues how to get the other assertions.

Proposition 1 *A generalized almost subtangent manifold with $\mathcal{J}$ given by (3.1) is a generalized subtangent manifold if and only if*

(S1) π satisfies the equation

$$\pi^\sharp([\xi, \eta]_\pi) = [\pi^\sharp(\xi), \pi^\sharp(\eta)],$$

(S2) π and a are related by the following 2 formulas

$$a\pi^\sharp = \pi^\sharp a^*, \tag{3.3}$$

$$a^*([\xi, \eta]_\pi) = L_{\pi^\sharp\xi}(a^*\eta) - L_{\pi^\sharp\eta}(a^*\xi) - d\pi(a^*\xi, \eta), \tag{3.4}$$

(S3) π, a, and σ are related by the following 2 formulas

$$a^2 + \pi^\sharp\sigma_\sharp = 0, \tag{3.5}$$

$$N_a(X, Y) = \pi^\sharp(i_{X\wedge Y} d(\sigma)), \tag{3.6}$$

(S4) σ and a are related by the following 2 formulas

$$\begin{aligned} a^*\sigma_\sharp &= \sigma_\sharp a, \\ d\sigma_a(X, Y, Z) &= d\sigma(aX, Y, Z) + d\sigma(X, aY, Z) + d\sigma(X, Y, aZ) \end{aligned}$$

for all 1-forms ξ and η, and all vector fields X, Y, and Z, where $\sigma_a(X, Y) = \sigma(aX, Y)$.

Proof Since the other conditions are similar to the complex case, we will obtain only the condition (S3). The first equation of (S3) is obtained from the equation $\mathcal{J}^2 = 0$. To prove the second part of (S3), we take $\alpha = (X, 0), \beta = (Y, 0)$ in (3.2). Since $\mathcal{J}\alpha = (aX, \sigma_\sharp X)$ and $\mathcal{J}\beta = (aY, \sigma_\sharp Y)$, using (2.1) we get

$$[\![\mathcal{J}\alpha, \mathcal{J}\beta]\!] = ([aX, aY], L_{aX}\sigma_\sharp Y - L_{aY}\sigma_\sharp X + \frac{1}{2}d(\sigma_\sharp X(aY) - \sigma_\sharp Y(aX)), \tag{3.7}$$

$$\begin{aligned} -\mathcal{J}[\![\mathcal{J}\alpha, \beta]\!] &= (-a[aX, Y] - \pi^\sharp(-L_Y\sigma_\sharp X + \frac{1}{2}d(i_Y\sigma_\sharp X), -\sigma_\sharp[aX, Y] \\ &+ a^*(-L_Y\sigma_\sharp X + \frac{1}{2}d(i_Y\sigma_\sharp X)) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} -\mathcal{J}[\![\alpha, \mathcal{J}\beta]\!] &= (-a[X, aY] - \pi^\sharp(-L_X\sigma_\sharp Y - \frac{1}{2}d(i_X\sigma_\sharp Y), -\sigma_\sharp[X, aY] \\ &+ a^*(L_X\sigma_\sharp Y - \frac{1}{2}d(i_X\sigma_\sharp Y)). \end{aligned} \tag{3.9}$$

Thus from (3.7), (3.8), and (3.9) we obtain

$$\begin{aligned} [aX, aY] + [X, Y] - a[aX, Y] &- \pi^\sharp\big(-L_Y\sigma_\sharp X + \frac{1}{2}d(i_Y\sigma_\sharp X)\big) \\ -a[X, aY] - \pi^\sharp\big(-L_X\sigma_\sharp Y &- \frac{1}{2}d(i_X\sigma_\sharp Y)\big) = 0, \end{aligned}$$

where we have used $[\![\alpha, \beta]\!] = ([X, Y], 0)$. Rearranging the above expression, we arrive at

$$\begin{aligned} [aX, aY] + [X, Y] - a[aX, Y] &- a[X, aY] - \pi^\sharp\big(-L_Y(i_X\sigma) + L_X(i_Y\sigma) \\ &+ \frac{1}{2}d(i_{X\wedge Y}\sigma) + \frac{1}{2}d(i_{X\wedge Y}\sigma)\big) = 0. \end{aligned} \tag{3.10}$$

On the other hand, making use of the formula

$$i_{X\wedge Y}(d\sigma) = L_X(i_Y\sigma) - L_Y(i_X\sigma) + d(i_{X\wedge Y}\sigma) - i_{[X,Y]}\sigma, \tag{3.11}$$

we get

$$\begin{aligned} -\pi^\sharp\big(L_X(i_Y\sigma) - L_Y(i_X\sigma) + d(i_{X\wedge Y}\sigma)\big) &= -\pi^\sharp\big(i_{X\wedge Y}(d\sigma) + i_{[X,Y]}\sigma\big) \\ &= -\pi^\sharp\big(i_{X\wedge Y}(d\sigma)\big) \\ &- \pi^\sharp\big(i_{[X,Y]}\sigma\big). \end{aligned} \tag{3.12}$$

Since we have

$$\pi^\sharp\big(i_{[X,Y]}\sigma\big) = \pi^\sharp\big(\sigma_\sharp[X, Y]\big),$$

from (3.5) we obtain

$$\pi^\sharp\big(i_{[X,Y]}\sigma\big) = -a^2[X, Y] + [X, Y]. \tag{3.13}$$

Using (3.13) in (3.12), we derive

$$\begin{aligned} -\pi^{\sharp}\big(L_X(i_Y\sigma) - L_Y(i_X\sigma) + d(i_{X\wedge Y}\sigma)\big) &= -\pi^{\sharp}\big(i_{X\wedge Y}(d\sigma) + i_{[X,Y]}\sigma \\ &= -\pi^{\sharp}\big(i_{X\wedge Y}(d\sigma)\big) + a^2[X,Y] \\ &- [X,Y]. \end{aligned} \tag{3.14}$$

Thus putting (3.14) in (3.10), we obtain (3.6). □

As an analogue of a Hitchin pair on a generalized complex manifold, a Hitchin pair on a generalized almost subtangent manifold M is a pair (ω, a) consisting of a symplectic form ω and a (1,1)-tensor a with the property that ω and a commute (i.e $\omega(X, aY) = \omega(aX, Y)$) and $d\omega_a = 0$, where $\omega_a(X, Y) = \omega(aX, Y)$.

Lemma 1 *If π is a nondegenerate bivector field on a manifold M, ω is the inverse 2-form (defined by $\omega_\sharp = (\pi^\sharp)^{-1}$), and π satisfies (3.5) then $\sigma = -a^*\omega$.*

Proof For $X \in \chi(M)$, we apply $\omega_\sharp$ to (3.5) and using the dual subtangent structure a^*, we have

$$(a^*)^2\omega_\sharp(X) + \sigma_\sharp(X) = 0.$$

Now for $Y \in \chi(M)$, since ω and a are commute, we obtain

$$\omega(aX, aY) + \sigma(X, Y) = 0.$$

Thus we get

$$a^*\omega(X, Y) + \sigma(X, Y) = 0. \tag{3.15}$$

Since the equation (3.15) holds for all X and Y, we get

$$\sigma = -a^*\omega.$$

□

We say that 2-form σ is the twist of Hitchin pair (ω, a).

A symplectic+subtangent structure on a generalized almost subtangent manifold M consists of a pair (ω, J) with ω-symplectic and J-subtangent structure on M, which commute.

Lemma 2 *Let (M, ω) be a symplectic manifold. (ω, a) is a symplectic+subtangent structure if and only if $d\omega_a = 0$, $a^*\omega = 0$.*

Proof We will only prove the sufficient condition. Since (M, ω) is a symplectic manifold, then $d\omega = 0$. Since $d\omega_a = 0$, $a^*\omega = 0$, by using the following equation (see [3]),

$$i_{N_a(X,Y)}(\omega) = i_{aX\wedge Y + X\wedge aY}(d\omega_a) - i_{aX\wedge aY}(d\omega) - i_{X\wedge Y}(d(a^*\omega)), \tag{3.16}$$

we get $i_{N_a(X,Y)}(\omega) = -i_{X\wedge Y}(d(a^*\omega)) = 0$. Hence,

$$\omega(N_a(X, Y), \bullet) = 0.$$

Since ω is nondegenerate, then $N_a = 0$. Thus a is a subtangent structure. On the other hand, $a^*\omega = 0$ implies that ω and a commute. The converse is clear. □

Next we relate (S1) and the 2-form ω.

Lemma 3 *[3] If π is a nondegenerate bivector on M and ω is the inverse 2-form (defined by $\omega_\sharp = (\pi^\sharp)^{-1}$), then π satisfies (S1) if and only if ω is closed.*

Proof Taking $\xi = i_X(\omega)$ and $\eta = i_Y(\omega)$ in (S1), where X and Y are arbitrary vector fields, then applying $\omega_\sharp$ to the resulting formula, and using (3.11), we get the assertion. □

Thus, we have the following result that shows that there is a close relationship between condition (S1) and a symplectic groupoid.

Theorem 2 *Let M be a generalized subtangent manifold. Then there is a symplectic groupoid (Σ, ω) over M.*

Proof It is a well-known fact that there is a one to one correspondence between integrable Poisson structures on M and symplectic groupoids over M. In fact, the condition (S1) tells us that π is an integrable Poisson structure. □

We now give the conditions for (S2) in terms of ω and ω_a.

Lemma 4 *[3] Let ω be a symplectic form. Given a nondegenerate bivector π (i.e. $\pi^\sharp = (\omega_\sharp)^{-1}$) and a map $a: TM \to TM$, then π and a satisfy (S2) if and only if ω and a commute and ω_a is closed.*

Proof Clearly, (3.3) is equivalent to ω and a commuting. Next, to take care of (3.4), we apply $\xi = i_X(\omega) = \omega_\sharp X$ and $\eta = i_Y(\omega) = \omega_\sharp Y$ to (3.4), where X and Y are arbitrary vector fields. Using the $a^*(i_X(\omega)) = i_X(\omega_a)$, and the closed 2-form ω in formula (3.11), we see that (3.4) is equivalent to ω_a being closed. □

We now give a correspondence between generalized subtangent structures with nondegenerate π, and Hitchin pairs (ω, a).

Proposition 2 *There is a one to one correspondence between generalized subtangent structures given by (3.1) with π nondegenerate, and Hitchin pairs (ω, a). In this correspondence, π is the inverse of ω and σ is the twist of the Hitchin pair (ω, a).*

Proof Since (ω, a) is a Hitchin pair, then ω and ω_a are closed. Using (3.16), we get

$$i_{N_a(X,Y)}(\omega) = -i_{X\wedge Y}(d(a^*\omega). \tag{3.17}$$

Since $\sigma = -a^*\omega$, we derive

$$\omega_\sharp(N_a(X,Y)) \quad = \quad i_{X\wedge Y}(d(\sigma)) \tag{3.18}$$

Applying $\pi^\sharp$ to (3.18), then we get

$$N_a(X,Y) = \pi^\sharp(i_{X\wedge Y}(d(\sigma))).$$

The above equation is the second equation of (S3). Now we show that $a^*\sigma_\sharp = \sigma_\sharp a$. From (3.15), we obtain

$$a^*\sigma_\sharp \quad = \quad a^*(-a^*\omega_\sharp).$$

Hence, we have

$$a^*\sigma_\sharp \quad = \quad -a^*\omega_\sharp a.$$

From definition of twist, we get

$$a^*\sigma_\sharp = \sigma_\sharp a.$$

This equation is the first equation of (S3). Now, we will obtain

$$d\sigma_a(X,Y,Z) = d\sigma(aX,Y,Z) + d\sigma(X,aY,Z) + d\sigma(X,Y,aZ),$$

which is the second equation of (S4). Writing the equation as

$$i_{X\wedge Y}(d\sigma_a) = i_{aX\wedge Y + X\wedge aY}(d\sigma) + a^*(i_{X\wedge Y}(d\sigma))$$

and since $\sigma = -a^*\omega$, then we should find

$$i_{X\wedge Y}(d(a^*\omega_a)) = i_{aX\wedge Y + X\wedge aY}(d(a^*\omega)) + a^*(i_{X\wedge Y}(d(a^*\omega))).$$

Using (3.16), then we get

$$i_{X\wedge Y}(d(a^*\omega_a)) = i_{aX\wedge Y + X\wedge aY}(d(a^*\omega)) - i_{N_a(X,Y)}(\omega_a). \tag{3.19}$$

Since $i_{N(X,Y)}(\omega_a) = a^* i_{N(X,Y)}(\omega)$, applying $i_{X\wedge Y}(d(a^*\omega) = -i_{N_a(X,Y)}(\omega))$ to (3.19), we have

$$i_{X\wedge Y}(d(a^*\omega_a)) = i_{aX\wedge Y + X\wedge aY}(d(a^*\omega)) + a^*(i_{X\wedge Y}(d(a^*\omega))).$$

The converse is clear from Lemma 3 and Lemma 4. □

We recall that, similar to 2-forms, given a Lie groupoid G, a (1,1)-tensor $J : TG \to TG$ is called multiplicative [3] if for any $(g,h) \in G\times G$ and any $v_g \in T_gG$, $w_h \in T_hG$ such that (v_g, w_h) is tangent to $G\times G$ at (g,h), so is (Jv_g, Jw_h), and

$$(dm)_{g,h}(Jv_g, Jw_h) = J((dm)_{g,h}(v_g, w_h)).$$

Note that it is well known that there is a one to one correspondence between (1,1)-tensors a commuting with ω and 2-forms on M. On the other hand, it is easy to see that (S2) is equivalent to the fact that a^* is an IM form on the Lie algebroid T^*M associated Poisson structure π. Thus from the above discussion, Lemma 4, and Theorem 1, one can conclude the following theorem.

Theorem 3 *Let M be a generalized subtangent manifold. Let π be a Poisson structure on M and (Σ,ω) the symplectic groupoid over M. Then there exist multiplicative (1,1)-tensors J on Σ with the property that (J,ω) is a Hitchin pair.*

Proof From Theorem 3.3 of [3], we know that there is a one to one correspondence between (1,1)-tensors a on M satisfying (S2) and multiplicative (1,1)-tensors J on Σ with the property that (J,ω) is a Hitchin pair. □

We recall the notion of generalized subtangent map between generalized subtangent manifolds. This notion was given in [15] similar to the generalized complex map between generalized complex manifolds given in [3].

Let $(M_i, \mathcal{J}_i)$, $i = 1,2$, be 2 generalized subtangent manifolds, and let a_i, π_i, σ_i be the components of $\mathcal{J}_i$ in the matrix representation (3.1). A map $f : M_1 \to M_2$ is called generalized subtangent iff f maps π_1 into π_2, $f^*\sigma_2 = \sigma_1$ and $(df)\circ a_1 = a_2 \circ (df)$ [15].

A foliation on a manifold M is a decomposition of M into leaves that is locally given by the fibers of a submersion. In fact, a foliation on a manifold M can be given by a suitable foliation atlas on M, by an integrable subbundle of the tangent bundle of M, or by a locally trivial differential ideal. The equivalence of all these descriptions is a consequence of the Frobenius integrability theorem. The theory of foliations has now become a rich and exciting geometric subject by itself, see: [12]. Consider the holonomy group of a given point on a foliated manifold (M, F); this group carries information about the local structure of the foliation near the leaf passing through the given point. However, nearby points can have quite different holonomy groups, but one can see that the holonomy covering spaces of all the leaves of the foliation (M, F) can be fitted together into a smooth manifold, denoted by $Hol(M, F)$. Moreover, this manifold carries a partial multiplication operation, which incorporates all the group structures of the various holonomy groups. The resulting structure is that of a Lie groupoid, and the manifold $Hol(M, F)$ is referred to as the holonomy groupoid of the foliation. This Lie groupoid plays a central role in foliation theory, because it lies at the basis of many constructions of invariants of a foliation, such as the characteristic classes of its normal bundle, the C^*-algebra of the foliation and its K-theory, and the cyclic cohomology of the foliation. Similarly, the monodromy groupoid of a foliation has a natural Lie groupoid structure. We recall that an ètale groupoid is a Lie groupoid G with $\dim G_1 = dim G_0$. Such groupoids are related to the orbifold theory. Indeed, orbifolds can be seen as Lie groupoids that are both proper and ètale. This becomes important when one studies invariants of orbifolds and maps between orbifolds. See also [8] for relations between homotopy groups for orbifolds and Lie groupoids.

As we have seen above, Lie groupoids have relations with many other areas. One can also see that the source map and target map of a Lie groupoid are important for investigating certain properties of such groupoids. Let us recall some properties of the source map and target map to show how useful the source map and target map of a Lie groupoid are. **(a)** A Lie groupoid is weakly equivalent to a Lie group (or G is transitive) if and only if the target map $t : G(x_0, -) \longrightarrow G_0$ is a surjective submersion for any $x_0 \in G_0$. **(b)** A Lie groupoid G is weakly equivalent to a discrete group if and only if $(s, t) : G_1 \longrightarrow G_0 \times G_0$ is a covering projection that is equivalent to that $t : G(x_0, -) \longrightarrow G_0$ is a covering projection for any (or some) $x_0 \in G_0$. **(c)** A Lie groupoid G is ètale if and only if the source map of G is a local diffeomorphism. **(d)** For an ètale groupoid the fibres of the source map, the fibres of the target map, the isotropy groups, and the orbits are discrete; for more details, see:[11]. For a symplectic groupoid we also have the following result. Let (G, Q, ω) be a symplectic groupoid. Then Q has a unique Poisson structure such that s is Poisson and t is anti-Poisson [14]. Our next theorem, which is the main theorem of this paper, gives relations among source map, target map, the equation $(S3)$, and generalized subtanget map.

Theorem 4 *Assume that (π, a) satisfy (S1), (S2) with integrable π, and let (Σ, ω, J) be the induced symplectic groupoid over M and J the induced multiplicative (1, 1)-tensor. Then, for a 2-form σ on M, the following assertions are equivalent:*

(i) *(S3) is satisfied,*

(ii) $-J^*\omega = t^*\sigma - s^*\sigma$,

(iii) $(t, s) : \Sigma \to M \times \overline{M}$ *is generalized subtangent map; (condition of generalized subtangent map on M is $(dt) \circ a_1 = a_2 \circ (dt)$, this condition on $\overline{M}$ is $(ds) \circ a_1 = -a_2 \circ (ds)$).*

Proof (i)$\Leftrightarrow$ (ii). Define $\phi = \widetilde{\sigma} - t^*\sigma + s^*\sigma$, such that $\widetilde{\sigma} = -J^*\omega$ and $A = ker(ds)|_M$. We know from Theorem 1 that closed multiplicative 2-form θ on Σ vanishes if and only if IM form $u_\theta = 0$, i.e. $\theta(X,\alpha) = 0$, such that $X \in TM$, $\alpha \in A$. This case can be applied for forms with high dimension, i.e. 3-form θ vanishes if and only if $\theta(X,Y,\alpha) = 0$. Since ω and ω_J are closed, from (3.16) we get $i_{X\wedge Y}(d(J^*\omega)) = -i_{N_J(X,Y)}\omega$. Putting $\widetilde{\sigma} = -J^*\omega$, we obtain

$$i_{X\wedge Y}(d\widetilde{\sigma}) = i_{N_J(X,Y)}\omega. \tag{3.20}$$

Since $d\phi = 0 \Leftrightarrow d\phi(X,Y,\alpha) = 0$, we have

$$\begin{aligned} d\phi(X,Y,\alpha) = 0 \Leftrightarrow d\widetilde{\sigma}(X,Y,\alpha) - d(t^*\sigma)(X,Y,\alpha) \\ + d(s^*\sigma)(X,Y,\alpha) = 0 \end{aligned}$$

On the other hand, we obtain

$$d(t^*\sigma)(X,Y,\alpha) = d\sigma(dt(X),dt(Y),dt(\alpha)). \tag{3.21}$$

If we take $dt = \rho$ in (3.21) for A, we get

$$d(t^*\sigma)(X,Y,\alpha) = d\sigma(dt(X),dt(Y),\rho(\alpha)). \tag{3.22}$$

On the other hand, from [2] we know that

$$Id_\Sigma = m \circ (t, Id_\Sigma). \tag{3.23}$$

Differentiating (3.23), we obtain

$$X = dt(X). \tag{3.24}$$

Using (3.24) in (3.22), we get

$$d(t^*\sigma)(X,Y,\alpha) = d\sigma(X,Y,\rho(\alpha)).$$

In a similar way, we see that

$$d(s^*\sigma)(X,Y,\alpha) = d\sigma(ds(X),ds(Y),ds(\alpha)).$$

Since $\alpha \in kerds$, then $ds(\alpha) = 0$. Hence $d(s^*\sigma) = 0$. Thus we obtain

$$d\widetilde{\sigma}(X,Y,\alpha) = d\sigma(X,Y,\rho(\alpha)). \tag{3.25}$$

Using (3.20) in (3.25), we derive

$$\omega(N_J(X,Y),\alpha) = d\sigma(X,Y,\rho(\alpha)). \tag{3.26}$$

On the other hand, it is clear that $\phi = 0 \Leftrightarrow \widetilde{\sigma} - t^*\sigma + s^*\sigma = 0$. Thus we obtain

$$\widetilde{\sigma}(X,\alpha) = \sigma(X,\rho(\alpha)).$$

Since $\widetilde{\sigma} = -J^*\omega$, we get

$$-\omega(JX,J\alpha) = \sigma(X,\rho(\alpha)). \tag{3.27}$$

Since Poisson bivector π is integrable, it defines a Lie algebroid whose anchor map is $\rho = \pi^\sharp$. Let us use $\pi^\sharp$ instead of ρ in (3.26) and (3.27). Then we get

$$\omega(N_J(X,Y),\alpha) = d\sigma(X,Y,\pi^\sharp(\alpha)), \tag{3.28}$$

$$-\omega(JX, J\alpha) = \sigma(X, \pi^\sharp(\alpha)).$$

Since $\omega(\alpha, X) = \alpha(X)$, $\omega_J(\alpha, X) = \alpha(JX)$, from (3.28) we have

$$\begin{aligned} -\alpha(N_a(X,Y)) &= d\sigma(X, Y, \pi^\sharp(\alpha)) \\ &= i_{X\wedge Y} d\sigma(\pi^\sharp(\alpha)) \\ &= \pi(\alpha, i_{X\wedge Y} d\sigma) \\ &= -\alpha(\pi^\sharp(i_{X\wedge Y} d\sigma)). \end{aligned}$$

i.e. $\alpha(N_a(X,Y)) = \alpha(\pi^\sharp(i_{X\wedge Y} d\sigma))$. **Since** the above equation holds for all nondegenerate α, we get

$$N_a(X,Y) = \pi^\sharp(i_{X\wedge Y} d\sigma). \tag{3.29}$$

On the other hand, from (3.27) we obtain

$$\begin{aligned} \alpha(a^2 X) &= i_X \sigma(\pi^\sharp(\alpha)) \\ &= \pi(\alpha, i_X \sigma) \\ &= -\alpha(\pi^\sharp \sigma_\sharp X). \end{aligned}$$

Thus we get

$$a^2 + \pi^\sharp \sigma_\sharp = 0. \tag{3.30}$$

Then (i)$\Leftrightarrow$(ii) follows from (3.29) and (3.30).

(ii)$\Leftrightarrow$(iii) $-J^*\omega = t^*\sigma - s^*\sigma$ says (t,s) is compatiple with 2-forms. Moreover, it is clear that (t,s) and bivectors are compatible due to Σ being a symplectic groupoid. We will check the compatibility of (t,s) and (1,1)-tensors. From the compatibility condition of t and s, we will get $dt \circ J = a \circ dt$ and $ds \circ J = -a \circ ds$. **For** all $\alpha \in A$ and $V \in \chi(\Sigma)$,

$$\omega(\alpha, V) = \omega(\alpha, dtV)$$

which is equivalent to

$$\alpha(V) = \langle u_\omega(\alpha), dtV\rangle.$$

Since $u_\omega = Id$ and $u_{\omega_J} = a^*$, we get

$$\begin{aligned} \langle \alpha, a(dt(V))\rangle &= \alpha(a(dt(V))) \\ &= \omega(\alpha, dt(JV)) \\ &= \langle \alpha, dt(JV)\rangle. \end{aligned}$$

Since this equation holds for all α, then $a(dt) = dt(J)$. Using $s = t \circ i$,

$$\begin{aligned} a(ds(V)) &= ad(t\circ i)V \\ &= -ds(JV), \end{aligned}$$

which shows that $a(ds) = -ds(J)$. Thus proof is completed. □

Remark 1 *We have also seen from proof of above theorem, (S3) forces the twist to vanish cohomologically.*

References

[1] Almeida R, Molino P. Suites dAtiyah et feuilletages transversalement complets. C R Acad Sci Paris Ser I Math 1985; 300: 13–15.

[2] Bursztyn H, Crainic M, Weinstein A, Zhu C. Integration of twisted Dirac brackets. Duke Math J 2004; 123: 549–607.

[3] Crainic M. Generalized complex structures and Lie brackets. Bull Braz Math Soc, New Series 2011; 42: 559–578.

[4] Crainic M, Fernandes RL. Integrability of Lie brackets. Ann Math 2003; 157: 575–620.

[5] Ehresmann C. Catègories topologiques et catègories differentiables. (French) 1959 Colloque Gom Diff Globale (Bruxelles, 1958), Centre Belge Rech. Math., Louvain pp: 137–150.

[6] Gualtieri M. Generalized complex geometry. PhD, Univ Oxford, 2003; arXiv:math.DG/0401221.

[7] Hitchin N. Generalized Calabi-Yau manifolds. Q J Math 2003; 54: 281–308.

[8] Leida JK. Orbifolds and stable equivariant homotopy groups, PhD Thesis, University of Wisconsin-Madison, 2006.

[9] Liu ZJ, Weinstein A and Xu P. Dirac structures and Poisson homogeneous spaces. Comm Math Phys 1998; 192: 121–144.

[10] Mackenzie K. Lie groupoids and Lie algebroids in differential geometry. Cambridge, Cambridge University Press, London Mathematical Society Lecture Note Series, vol. 124, 1987.

[11] Moerdijk I, Mrcun J. Introduction to Foliations and Lie Groupoids, Cambridge University Press, 2003.

[12] Molino P. Riemannian Foliations, Birkhauser, 1988.

[13] Pradines J. Théorie de Lie pour les groupoïdes différentiables, Calcul différentiel dans la catégorie des groupoïdes infinitésimaux. C R Acad Sci Paris Ser A 1967; 264: 245–248.

[14] Racaniere S. Lie algebroids, Lie groupoids and Poisson geometry. Preprint, June, 2004.

[15] Vaisman I. Reduction and submanifolds of generalized complex manifolds. Differential Geom Appl 2007; 25: 147-166.

[16] Vaisman I. Lectures on the geometry of Poisson manifolds. Progress in Math Boston: Birkhäuser Verlag, vol. 118: 1994.

[17] Wade A. Dirac structures and paracomplex manifolds. C R Acad Sci Paris Ser I math 2004; 338: 889–894.

[18] Zhong D, He LG. On actions of groupoids and morphisms of Lie bialgebroids. Adv Math (China)2003; 32: 311–318.