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# Coextended weak entwining structures

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**Abstract:** In this paper, we formulate the definition of coextended weak entwining structure in a strict monoidal category with equalizers. For a coextended weak entwining structure $(A, D, \psi, \alpha)$, we introduce the notions of weak $(D, \alpha)$-cleft extension and weak $(D, \alpha)$-Galois extension (with normal basis), proving that weak $(D, \alpha)$-Galois extensions with normal basis are equivalent to weak $(D, \alpha)$-cleft extensions.

**Key words:** Monoidal category, coextended weak entwining structure, weak cleft extension, weak Galois extension, normal basis

## 1. Introduction

The definition of the normal basis for extensions associated to a Hopf algebra $H$ in a category of modules over a commutative ring was introduced by Kreimer and Takeuchi in [14]. Using this notion, Doi and Takeuchi characterized in [10] $H$-Galois extensions with normal basis in terms of $H$-cleft extensions. This result can be extended for Hopf algebras living in symmetric closed categories [13] and, in [2, 3, 5], we can find a more general formulation in the context of entwining structures, weak entwining structures, and lax entwining structures, respectively.

The objective of the present paper is to prove similar results for the same kind of extensions associated to an idempotent comonoid morphism $\alpha$ in a strict monoidal category $\mathcal{C}$ with equalizers. These extensions will be called coextended weak entwining structures and, if $\alpha$ is the identity, they coincide with weak entwining structures. The typical example of coextended weak entwining structure and cleft extensions in this setting can be obtained by working with comonoid projections of weak Hopf algebras. If $H$, $D$ are weak Hopf algebras in $\mathcal{C}$ and $f : H \to D$, $g : D \to H$ are comonoid morphisms such that $g \circ f = id_H$, we can define a quadruple $(H, D, \psi, \alpha)$, where $\psi = (H \otimes (f \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (g \otimes \delta_H)$ and $\alpha = f \circ g$ is a comonoid idempotent morphism. There also exists an extension $H_D \hookrightarrow D$, with $H_D$ the equalizer of $\varrho_D = (D \otimes g) \circ \delta_D$ and $\zeta_D = (\mu_D \otimes g) \circ (D \otimes (\delta_D \circ \eta_D))$. The quadruple $(H, D, \psi, \alpha)$ is a coextended weak entwining structure, and $H_D \hookrightarrow D$ is an example of a cleft extension associated to this type of entwining structure. Note that $(H, D, \psi)$ is not a weak entwining structure, because $\psi \circ (D \otimes \eta_H) = ((\Pi_H^R \circ g) \otimes \alpha) \circ \delta_D$ with $\Pi_H^R$ the source morphism of $H$. Actually, we have that $(H, D, \psi)$ is a weak entwining structure iff $\alpha = id_D$, but in this case $f$, $g$ are isomorphisms.

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The organization of the paper is the following. In the second section, we introduce the notion of a coextended weak entwining structure, and we obtain the main properties of these algebra structures. In particular, we find a categorical isomorphism between the category of entwining modules associated to a coextended weak entwining structure $(A, D, \psi, \alpha)$ and the category of entwining modules for a certain weak entwining structure obtained from $(A, D, \psi, \alpha)$. In Section 3, we define the notion of cleft extension for a coextended weak entwining structure, and we prove that this extension induces an example of weak crossed product in the sense of [11]. This crossed product characterizes completely the cleft extension and is the motivation for the definition of Galois extension with normal basis in this setting. Finally, in the last section, we formulate the definition of weak $(D, \alpha)$-Galois extension with normal basis for a coextended weak entwining structure $(A, D, \psi, \alpha)$, and in Theorem 4.5 we characterize these extensions using the notion of cleftness introduced in Section 3. If the morphism $\alpha$ is the identity, we recover the results proved in [2].

## 2. Coextended weak entwining structures

In what follows, $(\mathcal{C}, \otimes, K)$ denotes an strict monoidal category with equalizers where $\otimes$ is the tensor product and $K$ the unit object. It is easy to prove that, if $\mathcal{C}$ admits equalizers, then every idempotent morphism splits, i.e. for every morphism $q : Y \to Y$ such that $q = q \circ q$, there exist an object $Z$ (called the image of $q$) and morphisms $i : Z \to Y$ and $p : Y \to Z$ satisfying $q = i \circ p$ and $p \circ i = id_Z$.

A monoid in $\mathcal{C}$ is a triple $A = (A, \eta_A, \mu_A)$, where $A$ is an object in $\mathcal{C}$ and $\eta_A : K \to A$ (unit), $\mu_A : A \otimes A \to A$ (product) are morphisms in $\mathcal{C}$ such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. Given 2 monoids $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, $f : A \to B$ is called a monoid morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$, $f \circ \eta_A = \eta_B$. Also, if $\mathcal{C}$ is a braided monoidal category with braiding $c$ and $A$, $B$ are monoids, so is $A \otimes B$, where $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$.

A comonoid in $\mathcal{C}$ is a triple $D = (D, \varepsilon_D, \delta_D)$, where $D$ is an object in $\mathcal{C}$ and $\varepsilon_D : D \to K$ (counit), $\delta_D : D \to D \otimes D$ (coproduct) are morphisms in $\mathcal{C}$ such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are comonoids, $f : D \to E$ is called a comonoid morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$, $\varepsilon_E \circ f = \varepsilon_D$. If $\mathcal{C}$ is a braided monoidal category with braiding $c$ and $D$, $E$ are comonoids, $D \otimes E$ is a comonoid with counit $\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E$ and coproduct $\delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$.

Finally, if $A$ is a monoid, $D$ is a comonoid, and $f, g : D \to A$ are morphisms, the convolution product of $f$ and $g$, denoted by $f * g$, is defined by

$$f * g = \mu_A \circ (f \otimes g) \circ \delta_D.$$

**Definition 2.1** *Let $(A, D, \psi, \alpha)$ be a quadruple, where $A$ is a monoid, $D$ a comonoid, $\psi : D \otimes A \to A \otimes D$ a morphism, and $\alpha : D \to D$ an idempotent comonoid morphism. We say that $(A, D, \psi, \alpha)$ is a coextended weak entwining structure on $\mathcal{C}$ if the following identities hold:*

*(a1)* $\psi \circ (D \otimes \mu_A) = (\mu_A \otimes D) \circ (A \otimes \psi) \circ (\psi \otimes A)$,

*(a2)* $(A \otimes \delta_D) \circ \psi = (\psi \otimes D) \circ (D \otimes \psi) \circ (\delta_D \otimes A)$,

*(a3)* $\psi \circ (D \otimes \eta_A) = (e \otimes D) \circ \delta_D \circ \alpha$,

*(a4)* $(A \otimes \varepsilon_D) \circ \psi = \mu_A \circ (e \otimes A)$,

*where* $e : D \to A$ *is the morphism defined by* $e = (A \otimes \varepsilon_D) \circ \psi \circ (D \otimes \eta_A)$. *The morphism* $\psi$ *is called the intertwining.*

If the idempotent morphism is the identity, we obtain the notion of weak entwining structure introduced by Caenepeel and De Groot [8] as a generalization of entwining structures defined by Brzezinski and Majid [7]. Entwining structures are coextended weak entwining structures, where $e = \eta_A \otimes \varepsilon_D$ and $\alpha = id_D$. If $e = \eta_A \otimes \varepsilon_D$ and $\alpha \neq id_D$, we will say that $(A, D, \psi, \alpha)$ is a co-extended entwining structure. In this case,

$$\psi \circ (D \otimes \eta_A) = \eta_A \otimes \alpha,$$

and, as a consequence, the morphism

$$\Delta_{A\otimes D} = (\mu_A \otimes D) \circ (A \otimes (\psi \circ (D \otimes \eta_A))) : A \otimes D \to A \otimes D$$

is equal to $A \otimes \alpha$.

**Proposition 2.2** *Let* $(A, D, \psi, \alpha)$ *be a quadruple as in Definition 2.1. Then (a3) holds if and only if*

$$\psi \circ (D \otimes \eta_A) = (e \otimes \alpha) \circ \delta_D \tag{1}$$

*and*

$$e \circ \alpha = e \tag{2}$$

*hold.*

**Proof** Assume that (1) and (2) hold. Then

$$(e \otimes D) \circ \delta_D \circ \alpha = ((e \circ \alpha) \otimes \alpha) \circ \delta_D = (e \otimes \alpha) \circ \delta_D = \psi \circ (D \otimes \eta_A)$$

and (a3) holds. Conversely, by (a3),

$$e = (A \otimes \varepsilon_D) \circ \psi \circ (D \otimes \eta_A) = (e \otimes \varepsilon_D) \circ \delta_D \circ \alpha = e \circ \alpha,$$

and (2) holds. On the other hand, using (2), we obtain (1) because

$$\psi \circ (D \otimes \eta_A) = (e \otimes D) \circ \delta_D \circ \alpha = ((e \circ \alpha) \otimes \alpha) \circ \delta_D = (e \otimes \alpha) \circ \delta_D.$$

□

**Proposition 2.3** *Let* $(A, D, \psi, \alpha)$ *be a coextended weak entwining structure. Then the equalities*

$$\mu_A \circ (A \otimes e) \circ \psi = (A \otimes \varepsilon_D) \circ \psi, \tag{3}$$

$$\psi = (A \otimes \alpha) \circ \psi, \tag{4}$$

$$\psi = \psi \circ (\alpha \otimes A) \tag{5}$$

*hold.*

**Proof** First, note that, by (a1), we obtain (3) because

$$\mu_A \circ (A \otimes e) \circ \psi = (\mu_A \otimes \varepsilon_D) \circ (A \otimes \psi) \circ (\psi \otimes \eta_A) = (A \otimes \varepsilon_D) \circ \psi.$$

Moreover, (4) holds because

$$\begin{aligned} &\psi \\ &= (\mu_A \otimes D) \circ (A \otimes \psi) \circ (\psi \otimes \eta_A) \\ &= (\mu_A \otimes D) \circ (A \otimes ((e \otimes \alpha) \circ \delta_D)) \circ \psi \\ &= ((\mu_A \circ (A \otimes e) \circ \psi) \otimes \alpha) \circ (A \otimes \psi) \circ (\delta_D \otimes A) \\ &= (((A \otimes \varepsilon_D) \circ \psi) \otimes \alpha) \circ (A \otimes \psi) \circ (\delta_D \otimes A) \\ &= (A \otimes ((\varepsilon_D \otimes \alpha) \circ \delta_D)) \circ \psi \\ &= (A \otimes \alpha) \circ \psi, \end{aligned}$$

where the first equality follows by (a1), the second by (a3), and the third and fifth by (a2); the fourth is a consequence of (3), and the last one follows because $D$ is a comonoid.

The equality (5) follows because

$$\begin{aligned} &\psi \\ &= \psi \circ (D \otimes (\mu_A \circ (\eta_A \otimes A)) \\ &= (\mu_A \otimes D) \circ (A \otimes \psi) \circ ((\psi \circ (D \otimes \eta_A)) \otimes A) \\ &= (\mu_A \otimes D) \circ (A \otimes \psi) \circ (((e \otimes D) \circ \delta_D \circ \alpha) \otimes A) \\ &= (((A \otimes \varepsilon_D) \circ \psi) \otimes D) \circ (D \otimes \psi) \circ ((\delta_D \circ \alpha) \otimes A) \\ &= (A \otimes ((\varepsilon_D \otimes \alpha) \circ \delta_D)) \circ \psi \circ (\alpha \otimes A) \\ &= \psi \circ (\alpha \otimes A), \end{aligned}$$

where the first identity is a consequence of the unit properties, the second follows by (a1), the third relies on (a3), and the fourth relies on (a4). Finally, in the fifth equality, we used (a2). □

Note that by (4) and (5) we obtain that

$$(A \otimes ((D \otimes \alpha) \circ \delta_D)) \circ \psi = (\psi \otimes D) \circ (D \otimes \psi) \circ (((D \otimes \alpha) \circ \delta_D) \otimes A) \tag{6}$$

and

$$(A \otimes ((D \otimes \alpha) \circ \delta_D)) \circ \psi = (\psi \otimes D) \circ (D \otimes \psi) \circ (\delta_D \otimes A). \tag{7}$$

If $\alpha : D \to D$ is an idempotent morphism of comonoids, there exist an object $D_\alpha$ and 2 morphisms $i_\alpha : D_\alpha \to D$, $p_\alpha : D \to D_\alpha$ such that $i_\alpha \circ p_\alpha = \alpha$ and $p_\alpha \circ i_\alpha = id_{D_\alpha}$. Therefore, $D_\alpha$ is a comonoid with counit and coproduct defined by

$$\varepsilon_{D_\alpha} = \varepsilon_D \circ i_\alpha, \;\; \delta_{D_\alpha} = (p_\alpha \otimes p_\alpha) \circ \delta_D \circ i_\alpha.$$

As a consequence, the quadruple $(D_\alpha, D, i_\alpha, p_\alpha)$ is a comonoid projection. That is, $i_\alpha$ and $p_\alpha$ are comonoid morphisms such that $p_\alpha \circ i_\alpha = id_{D_\alpha}$. Under these conditions, we have that the triple $(A, D_\alpha, \psi^\alpha)$, where

$$\psi^\alpha = (A \otimes p_\alpha) \circ \psi \circ (i_\alpha \otimes A) : D_\alpha \otimes A \to A \otimes D_\alpha, \tag{8}$$

is a weak entwining structure. Indeed, first note that by (5) and (a1) we obtain

$$\begin{aligned}
&(\mu_A \otimes D_\alpha) \circ (A \otimes \psi^\alpha) \circ (\psi^\alpha \otimes A) \\
&= (\mu_A \otimes p_\alpha) \circ (A \otimes (\psi \circ (\alpha \otimes A))) \circ ((\psi \circ (i_\alpha \otimes A)) \otimes A) \\
&= (\mu_A \otimes p_\alpha) \circ (A \otimes \psi) \circ ((\psi \circ (i_\alpha \otimes A)) \otimes A) \\
&= \psi^\alpha \circ (D_\alpha \otimes \mu_A).
\end{aligned}$$

On the other hand, by (a2) and (5), we have

$$\begin{aligned}
&(\psi^\alpha \otimes D_\alpha) \circ (D_\alpha \otimes \psi^\alpha) \circ (\delta_{D_\alpha} \otimes A) \\
&= (A \otimes p_\alpha \otimes p_\alpha) \circ (\psi \otimes D) \circ (D \otimes \psi) \circ (((\alpha \otimes \alpha) \circ \delta_D \circ i_\alpha) \otimes A) \\
&= (A \otimes p_\alpha \otimes p_\alpha) \circ (\psi \otimes D) \circ (D \otimes \psi) \circ ((\delta_D \circ i_\alpha) \otimes A) \\
&= (A \otimes ((p_\alpha \otimes p_\alpha) \circ \delta_D)) \circ \psi \circ (i_\alpha \otimes A) \\
&= (A \otimes ((p_\alpha \otimes p_\alpha) \circ \delta_D \circ \alpha)) \circ \psi \circ (i_\alpha \otimes A) \\
&= (A \otimes \delta_{D_\alpha}) \circ \psi^\alpha.
\end{aligned}$$

By (1) and the equality (2),

$$\psi^\alpha \circ (D_\alpha \otimes \eta_A) = (A \otimes p_\alpha) \circ \psi \circ (i_\alpha \otimes \eta_A) = (e \otimes p_\alpha) \circ \delta_D \circ i_\alpha = (e_\alpha \otimes D_\alpha) \circ \delta_{D_\alpha},$$

where $e_\alpha = (A \otimes D_\alpha) \circ \psi^\alpha \circ (D_\alpha \otimes A) = e \circ i_\alpha$. Finally, by (a4),

$$(A \otimes \varepsilon_{D_\alpha}) \circ \psi^\alpha = (A \otimes \varepsilon_D) \circ \psi \circ (i_\alpha \otimes A) = \mu_A \circ ((e \circ i_\alpha) \otimes A) = \mu_A \circ (e_\alpha \otimes A).$$

Conversely, if $(A, D_\alpha, \Gamma)$ is a weak entwining structure, the quadruple $(A, D, {}^\alpha\Gamma, \alpha)$, where

$${}^\alpha\Gamma = (A \otimes i_\alpha) \circ \Gamma \circ (p_\alpha \otimes A) : D \otimes A \to A \otimes D, \tag{9}$$

is a coextended weak entwining structure and, trivially, $({}^\alpha\Gamma)^\alpha = \Gamma$. Indeed, first note that

$$\begin{aligned}
&(\mu_A \otimes D) \circ (A \otimes {}^\alpha\Gamma) \circ ({}^\alpha\Gamma \otimes A) \\
&= (\mu_A \otimes i_\alpha) \circ (A \otimes \Gamma) \circ ((\Gamma \circ (p_\alpha \otimes A)) \otimes A) \\
&= {}^\alpha\Gamma \circ (D \otimes \mu_A).
\end{aligned}$$

On the other hand,

$$({}^\alpha\Gamma \otimes D) \circ (D \otimes {}^\alpha\Gamma) \circ (\delta_D \otimes A)$$

$= (A \otimes i_\alpha \otimes D) \circ (\Gamma \otimes i_\alpha) \circ (D_\alpha \otimes \Gamma) \circ (((p_\alpha \otimes p_\alpha) \circ \delta_D) \otimes A)$

$= (A \otimes i_\alpha \otimes D) \circ (\Gamma \otimes i_\alpha) \circ (D_\alpha \otimes \Gamma) \circ ((\delta_{D_\alpha} \circ p_\alpha) \otimes A)$

$= (A \otimes ((i_\alpha \otimes i_\alpha) \circ \delta_{D_\alpha})) \circ \Gamma \circ (p_\alpha \otimes A)$

$= (A \otimes \delta_D) \circ {}^{\alpha}\Gamma.$

Finally,

$$^{\alpha}\Gamma \circ (D \otimes \eta_A) = (A \otimes i_\alpha) \circ \Gamma \circ (p_\alpha \otimes \eta_A) = (u \otimes i_\alpha) \circ \delta_{D_\alpha} \circ p_\alpha = ({}^{\alpha}u \otimes \alpha) \circ \delta_D,$$

where ${}^{\alpha}u = u \circ p_\alpha = (A \otimes \varepsilon_D) \circ {}^{\alpha}\Gamma \circ (D \otimes \eta_A)$, and

$$(A \otimes \varepsilon_D) \circ^{\alpha} \Gamma = (A \otimes \varepsilon_{D_\alpha}) \circ \Gamma \circ (p_\alpha \otimes A) = \mu_A \circ ((u \circ p_\alpha) \otimes A) = \mu_A \circ ({}^{\alpha}u \otimes A).$$

With $\mathcal{E}nt^w_{co}$ we will denote the category of coextended weak entwining structures, defined by the following.

- Objects: coextended weak entwining structures.
- Morphisms from the object $(A, D, \psi, \alpha)$ to the object $(A', D', \psi', \alpha')$: pairs $(f, g)$, where $f : A \to A'$ is a morphism, $g : D \to D'$ is a comonoid morphism, and the equalities

$$(f \otimes g) \circ \psi = \psi' \circ (g \otimes f), \tag{10}$$

$$\alpha' \circ g = g' \circ \alpha \tag{11}$$

hold.

In a similar way, we define the category of weak entwining structures, denoted by $\mathcal{E}nt^w$. In this case:

- Objects: weak entwining structures.
- Morphisms from the object $(A, D, \psi)$ to the object $(A', D', \psi')$: pairs $(f, g)$, where $f : A \to A'$ is a morphism, $g : D \to D'$ is a comonoid morphism, and the equality (10) holds.

Obviously there exists an inclusion functor $i : \mathcal{E}nt^w \to \mathcal{E}nt^w_{co}$, where $i((B, C, \Gamma)) = (B, C, \Gamma, id_C)$ for the objects, and $i((f, g)) = (f, g)$ for the morphisms. There also exists a functor

$$F : \mathcal{E}nt^w_{co} \to \mathcal{E}nt^w$$

defined by

$$F((A, D, \psi, \alpha)) = (A, D_\alpha, \psi^\alpha)$$

on objects, and by

$$F((f, g)) = (f, p_{\alpha'} \circ g \circ i_\alpha)$$

on morphisms.

It is easy to show that $i$ is left adjoint of $F$ with unit defined by $u_{(A,D,\psi,\alpha)} = (id_A, p_\alpha)$ and counit $v = id_{\mathcal{E}nt^w}$. Moreover, $i$ is also right adjoint of $F$, with unit $u' = id_{\mathcal{E}nt^w}$ and counit $v'_{(A,D,\psi,\alpha)} = (id_A, i_\alpha)$.

If $T = i \circ F$, the pair $(T,u)$ is an idempotent coaugmented functor because

$$u_{T((A,D,\psi,\alpha))} = u_{(A,D_\alpha,\psi^\alpha,id_\alpha)} = (id_A, id_{D_\alpha}) = T((id_A, p_\alpha)) = T(u_{(A,D,\psi,\alpha)}).$$

Then, by Proposition 1.2 of [9], for every object $(A,D,\psi,\alpha)$, the morphism $u_{(A,D,\psi,\alpha)}$ is initial among all morphisms from $(A,D,\psi,\alpha)$ to objects isomorphic to $T((A',D',\psi',\alpha'))$, for some $(A',D',\psi',\alpha')$.

**Definition 2.4** *Let $(A,D,\psi,\alpha)$ be a coextended weak entwining structure. We denote by $\mathcal{M}_A^D(\psi,\alpha)$ the category whose objects are triples $(M,\phi_M,\rho_M)$, where $(M,\phi_M)$ is a right $A$-module (i.e. $\phi_M \circ (\phi_M \otimes A) = \phi_M \circ (M \otimes \mu_A)$, $id_M = \phi_M \circ (M \otimes \eta_A)$), $(M,\rho_M)$ is a right $D$-comodule (i.e. $(\rho_M \otimes D) \circ \rho_M = (M \otimes \delta_D) \circ \rho_M$, $(M \otimes \varepsilon_D) \circ \rho_M = id_M$), and*

$$\rho_M \circ \phi_M = (\phi_M \otimes D) \circ (M \otimes \psi) \circ (\rho_M \otimes A). \tag{12}$$

*The objects of $\mathcal{M}_A^D(\psi,\alpha)$ will be called coextended weak entwined modules, and a morphism in $\mathcal{M}_A^D(\psi,\alpha)$ is a morphism of $A$-modules and $D$-comodules. If $\alpha = id_D$, $\mathcal{M}_A^D(\psi, id_D)$ is the category of weak entwined modules introduced in [8]. In this case, $\mathcal{M}_A^D(\psi, id_D)$ will be denoted by $\mathcal{M}_A^D(\psi)$.*

*If $(M,\phi_M,\rho_M)$ is a coextended weak entwined module, by (a1), we obtain that*

$$\Delta_{M\otimes D} = (\phi_M \otimes D) \circ (M \otimes (\psi \circ (D \otimes \eta_A))) : M \otimes D \to M \otimes D \tag{13}$$

*is an idempotent morphism, and by (a3) we have*

$$\Delta_{M\otimes D} = (\phi_M \otimes D) \circ (M \otimes ((e \otimes D) \circ \delta_D \circ \alpha)) \tag{14}$$

*and by (2)*

$$\Delta_{M\otimes D} = ((\phi_M \circ (M \otimes e)) \otimes \alpha) \circ (M \otimes \delta_D). \tag{15}$$

*Using (a2), it is also easy to show that*

$$\Delta_{M\otimes A} = (M \otimes ((A \otimes \varepsilon_D) \circ \psi)) \circ (\rho_M \otimes A) : M \otimes A \to M \otimes A \tag{16}$$

*is an idempotent morphism and, by (a4), we have the equality*

$$\Delta_{M\otimes A} = (M \otimes (\mu_A \circ (e \otimes A))) \circ (\rho_M \otimes A). \tag{17}$$

**Proposition 2.5** *Let $(A,D,\psi,\alpha)$ be a coextended weak entwining structure. For any $(M,\phi_M,\rho_M)$ in $\mathcal{M}_A^D(\psi,\alpha)$ the following identities hold:*

$$\phi_M \circ (M \otimes e) \circ \rho_M = id_M, \tag{18}$$

$$\rho_M = (M \otimes \alpha) \circ \rho_M. \tag{19}$$

**Proof** The equality (18) follows by (12), and (19) holds because

$$\begin{aligned} &\rho_M \\ &= \rho_M \circ \phi_M \circ (M \otimes \eta_A) \\ &= (\phi_M \otimes D) \circ (M \otimes \psi) \circ (\rho_M \otimes \eta_A) \end{aligned}$$

$$= (\phi_M \otimes \alpha) \circ (M \otimes ((e \otimes D) \circ \delta_D)) \circ \rho_M$$

$$= ((\phi_M \circ (M \otimes e) \circ \rho_M) \otimes \alpha) \circ \rho_M$$

$$= (M \otimes \alpha) \circ \rho_M,$$

where the first equality follows by the unit properties, the second by (12), the third by (1), and the fourth by the comodule condition, while the last one relies on (18). □

**Proposition 2.6** *Let $(A, D, \psi, \alpha)$ be a coextended weak entwining structure. The categories $\mathcal{M}_A^D(\psi, \alpha)$ and $\mathcal{M}_A^{D_\alpha}(\psi^\alpha)$ are isomorphic.*

**Proof** Define the functors

$$F_\alpha : \mathcal{M}_A^D(\psi, \alpha) \to \mathcal{M}_A^{D_\alpha}(\psi^\alpha)$$

and

$$G_\alpha : \mathcal{M}_A^{D_\alpha}(\psi^\alpha) \to \mathcal{M}_A^D(\psi, \alpha)$$

by

$$F_\alpha((M, \phi_M, \rho_M)) = (M, \phi_M, \rho_M^\alpha = (M \otimes p_\alpha) \circ \rho_M),$$

$$G_\alpha((N, \varphi_N, \varrho_N)) = (N, \varphi_N, \ {}^\alpha\varrho_N = (N \otimes i_\alpha) \circ \rho_N)$$

on objects, and by the identity, on morphisms. Then, by (19), we obtain that $G_\alpha \circ F_\alpha = id_{\mathcal{M}_A^D(\psi,\alpha)}$, and by the properties of $i_\alpha$, $p_\alpha$, the identity $F_\alpha \circ G_\alpha = id_{\mathcal{M}_A^{D_\alpha}(\psi^\alpha)}$ holds. □

**Example 2.7** *Weak Hopf algebras (monoids) are generalizations of Hopf algebras and were introduced by Böhm et al. in [4]. The definition is as follows:*

*A weak Hopf algebra $H$, in a symmetric monoidal category $\mathcal{C}$ with symmetry isomorphism $c$, is a monoid $(H, \eta_H, \mu_H)$ and comonoid $(H, \varepsilon_H, \delta_H)$, such that the following axioms hold:*

*(b1)* $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H\otimes H}$.

*(b2)* $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H)$

$$= (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H).$$

*(b3)* $(\delta_H \otimes H) \circ \delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$

$$= (H \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H).$$

*(b4) There exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of $H$) verifiying:*

- *(b4-1)* $id_H * \lambda_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H)$.
- *(b4-2)* $\lambda_H * id_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$.
- *(b4-3)* $\lambda_H * id_H * \lambda_H = \lambda_H$.

*As a consequence of this definition, a weak Hopf algebra is a Hopf algebra if and only if the morphism* $\delta_H$ *(coproduct) is unit-preserving (i.e.* $\eta_H \otimes \eta_H = \delta_H \circ \eta_H$*), or if and only if the counit is a monoid morphism (i.e.* $\varepsilon_H \circ \mu_H = \varepsilon_H \otimes \varepsilon_H$*).*

*If* $H$ *is a weak Hopf algebra, the antipode* $\lambda_H$ *is unique, antimultiplicative, and anticomultiplicative and leaves the unit and the counit invariant, i.e.* $\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}$, $\delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H$, $\lambda_H \circ \eta_H = \eta_H$, $\varepsilon_H \circ \lambda_H = \varepsilon_H$.

*If we define the morphisms* $\Pi_H^L$ *(target),* $\Pi_H^R$ *(source),* $\overline{\Pi}_H^L$*, and* $\overline{\Pi}_H^R$ *by*

$$\Pi_H^L = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_{H,H}) \circ ((\delta_H \circ \eta_H) \otimes H) : H \to H,$$

$$\Pi_H^R = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_{H,H} \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)) : H \to H,$$

$$\overline{\Pi}_H^L = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H) : H \to H,$$

$$\overline{\Pi}_H^R = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)) : H \to H,$$

*it is straightforward to show that they are idempotent (see [4]).*

*Let* $(H, H, \Gamma)$ *be the triple where* $\Gamma = (H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H)$*. Then* $(H, H, \Gamma)$ *is a weak entwining structure with* $u = \Pi_H^R$*. This entwining structure is a particular instance of the following: Let* $H$ *be a weak Hopf algebra and let* $(A, \rho_A)$ *be a monoid, which is also a right* $H$*-comodule, such that* $\mu_{A\otimes H} \circ (\rho_A \otimes \rho_A) = \rho_A \circ \mu_A$*. We call* $A$ *a right* $H$*-comodule monoid if any of the following equivalent conditions hold:*

*(c1)* $(\rho_A \otimes H) \circ \rho_A \circ \eta_A = (A \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\rho_A \otimes \delta_H) \circ (\eta_A \otimes \eta_H)$,

*(c2)* $(\rho_A \otimes H) \circ \rho_A \circ \eta_A = (A \otimes \mu_H \otimes H) \circ (\rho_A \otimes \delta_H) \circ (\eta_A \otimes \eta_H)$,

*(c3)* $(A \otimes \overline{\Pi}_H^R) \circ \rho_A = (\mu_A \otimes H) \circ (A \otimes \rho_A) \circ (A \otimes \eta_A)$,

*(c4)* $(A \otimes \Pi_H^L) \circ \rho_A = ((\mu_A \circ c_{A,A}) \otimes H) \circ (A \otimes \rho_A) \circ (A \otimes \eta_A)$,

*(c5)* $(A \otimes \overline{\Pi}_H^R) \circ \rho_A \circ \eta_A = \rho_A \circ \eta_A$,

*(c6)* $(A \otimes \Pi_H^L) \circ \rho_A \circ \eta_A = \rho_A \circ \eta_A$.

*Under these conditions,* $(A, H, \Gamma = (A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes \rho_A))$ *is a weak entwining structure, and* $(A, \mu_A, \varrho_A) \in \mathcal{M}_A^H(\Gamma)$*. Then, if* $(H, D, f, g)$ *is a comonoid projection, that is,* $D$ *is a comonoid and* $f : H \to D$*,* $g : D \to H$ *are comonoid morphisms such that* $g \circ f = id_H$*, we have that* $\alpha = f \circ g : D \to D$ *is an idempotent comonoid morphism such that* $D_\alpha = H$*,* $p_\alpha = g$ *and* $i_\alpha = f$*. As a consequence,* $(A, D, {}^{\alpha}\Gamma = (A \otimes f) \circ \Gamma \circ (g \otimes A), \alpha)$ *is a coextended weak entwining structure, and* $(A, \mu_A, \rho_A = (A \otimes f) \circ \varrho_A)$ *is an object in* $\mathcal{M}_A^D({}^{\alpha}\Gamma, \alpha)$*. By Proposition 2.6, the categories* $\mathcal{M}_A^D({}^{\alpha}\Gamma, \alpha)$ *and* $\mathcal{M}_A^H(\Gamma)$ *are isomorphic.*

*Interesting examples of comonoid projections between weak Hopf algebras appear associated to exact factorizations of groupoids. First, note that, as group algebras are the natural examples of Hopf algebras, groupoid algebras provide examples of weak Hopf algebras. Recall that a groupoid* $G$ *is simply a small category where all morphisms are isomorphisms. In this example, we consider finite groupoids, i.e. groupoids with a finite number of objects. The set of objects of* $G$*, called also the base of* $G$*, will be denoted by* $G_0$*, and the set* *of morphisms by $G_1$. The identity morphism on $x \in G_0$ will be denoted by $id_x$, and for a morphism $\sigma : x \to y$ in $G_1$, we write $s(\sigma)$ and $t(\sigma)$, respectively, for the source and the target of $\sigma$.*

*Let $G$ be a groupoid and let $R$ be a commutative ring. The groupoid algebra is the direct product in $R$-Mod*

$$RG = \bigoplus_{\sigma \in G_1} R\sigma,$$

*with the product of 2 morphisms being equal to their composition, if the latter is defined and 0 otherwise, i.e. $\mu_{RG}(\tau \otimes \sigma) = \tau \circ \sigma$ if $s(\tau) = t(\sigma)$ and $\mu_{RG}(\tau \otimes \sigma) = 0$ if $s(\tau) \neq t(\sigma)$. The unit element is $1_{RG} = \sum_{x \in G_0} id_x$. The algebra $RG$ is a cocommutative weak Hopf algebra, with coproduct $\delta_{RG}$, counit $\varepsilon_{RG}$, and antipode $\lambda_{RG}$, given by the following formulas: $\delta_{RG}(\sigma) = \sigma \otimes \sigma$, $\varepsilon_{RG}(\sigma) = 1$, $\lambda_{RG}(\sigma) = \sigma^{-1}$. The target and source morphisms are $\Pi^L_{RG}(\sigma) = id_{t(\sigma)}$, $\Pi^R_{RG}(\sigma) = id_{s(\sigma)}$, and $\lambda_{RG} \circ \lambda_{RG} = id_{RG}$, i.e. the antipode is involutory.*

*A wide subgroupoid $U$ of a groupoid $G$ is a a subcategory of $G$, provided with a functor $F : U \to G$ that is the identity on the objects, and induces inclusions $hom_U(x, y) \subset hom_G(x, y)$, i.e. it has the same base, and (perhaps) fewer arrows.*

*Let $G$ be a groupoid. An exact factorization of $G$ is a pair of wide subgroupoids of $G$, $U$, and $V$, such that for any $\sigma \in G_1$, there exist unique $\sigma_V \in V_1$, $\sigma_U \in U_1$, such that $\sigma = \sigma_U \circ \sigma_V$. Following the notation of [15], we denote $G$ by $U \bowtie V$, because in Theorems 2.10 and 2.15 of [15] it was proven that the notion of a groupoid with exact factorization is equivalent to the notion of a matched pair of groupoids and to the notion of a vacant double groupoid. Any groupoid $G$ with an exact factorization $U \bowtie V$ induces a nontrivial example of a comonoid projection between weak Hopf algebras. Put $H = RV$ and $D = RG$ and define $f : H \to D$ by $f(\sigma) = \sigma$ and $g : D \to H$ by $g(\tau) = \tau_V$. It is then easy to show that $f$ is a monoid-comonoid morphism and $g \circ f = id_H$. Moreover, $g$ is a comonoid morphism, and it does not satisfy the condition of monoid morphism (see Example 3.3 of [12]).*

**Proposition 2.8** *Let $(A, D, \psi, \alpha)$ be a coextended weak entwining structure. Let $\rho_A : A \to A \otimes D$ be a morphism such that $(A, \mu_A, \rho_A)$ belongs to $\mathcal{M}^D_A(\psi, \alpha)$. If for all $(M, \phi_M, \rho_M) \in \mathcal{M}^D_A(\psi, \alpha)$ we denote by $M_D$ the equalizer of $\rho_M$ and $\zeta_M = (\phi_M \otimes D) \circ (M \otimes (\rho_A \circ \eta_A))$ and by $i^D_M$ the injection of $M_D$ in $M$, we have the following:*

*i) The triple $(A_D, \eta_{A_D}, \mu_{A_D})$ is a monoid, where $\eta_{A_D} : K \to A_D$ and $\mu_{A_D} : A_D \otimes A_D \to A_D$ are the factorizations of $\eta_A$ and $\mu_A \circ (i^D_A \otimes i^D_A)$, respectively, through the equalizer $i^D_A$.*

*ii) The pair $(M_D, \phi_{M_D})$ is a right $A_D$-module, where $\phi_{M_D} : M_D \otimes A_D \to M_D$ is the factorization of $\phi_M \circ (i^D_M \otimes i^D_A)$ through the equalizer $i^D_M$.*

**Proof** The proof for this proposition is the one used in the weak entwining setting to get a similar result (see Proposition 1.5 of [1]). □

**Proposition 2.9** *Let $(A, D, \psi, \alpha)$ be a coextended weak entwining structure in the conditions of Proposition 2.8. Then, for all $(M, \phi_M, \rho_M) \in \mathcal{M}^D_A(\psi, \alpha)$, the following identity holds:*

$$\rho_M \circ \phi_M \circ (i^D_M \otimes A) = (\phi_M \otimes D) \circ (i^D_M \otimes \rho_A). \quad (20)$$

**Proof** By (12) and the module condition,

$$\begin{aligned}
&\rho_M \circ \phi_M \circ (i_M^D \otimes A)\\
&= (\phi_M \otimes D) \circ (M \otimes \psi) \circ ((\rho_M \circ i_M^D) \otimes A)\\
&= (\phi_M \otimes D) \circ (\phi_M \otimes \psi) \circ (i_M^D \otimes (\rho_A \circ \eta_A) \otimes A)\\
&= (\phi_M \otimes D) \circ (i_M^D \otimes ((\mu_A \otimes D) \circ (A \otimes \psi) \circ ((\rho_A \circ \eta_A) \otimes A)))\\
&= (\phi_M \otimes D) \circ (i_M^D \otimes (\rho_A \circ \mu_A \circ (\eta_A \otimes A)))\\
&= (\phi_M \otimes D) \circ (i_M^D \otimes \rho_A),
\end{aligned}$$

and the proof is complete. □

## 3. Cleft extensions for coextended weak entwining structures

The aim of this section is to introduce the notion of cleft extension for coextended weak entwining structures. As a particular instance, we will obtain the definition of weak cleft extension as defined in [1].

**Proposition 3.1** *Let $(A, D, \psi, \alpha)$ be a coextended weak entwining structure in the conditions of Proposition 2.8. Then, if $h : D \to A$ is a right $D$-comodule morphism for $\rho_D = \delta_D$, the following identity holds:*

$$h * e = h. \tag{21}$$

*Moreover, if the coaction for $D$ is $\varrho_D = (D \otimes \alpha) \circ \delta_D$, (21) holds. Also, if $h$ is a morphism of right $D$-comodules for $\rho_D = \delta_D$ and $\rho_A$, it is a morphism of right $D$-comodules for $\varrho_D = (D \otimes \alpha) \circ \delta_D$ and $\rho_A$.*

**Proof** The equality follows by (18). If we change the coaction of $D$, by (2), we obtain the same equality and the last assertion follows by (19), composing with $A \otimes \alpha$ in the equality $\rho_A \circ h = (h \otimes D) \circ \delta_D$. □

**Definition 3.2** *Let $(A, D, \psi, \alpha)$ be a coextended weak entwining structure in the conditions of Proposition 2.8. By $Reg_\alpha^{WR}(D, A)$ we denote the set of morphisms $h \in Hom_{\mathcal{C}}(D, A)$ such that there exists a morphism $h^{-1} \in Hom_{\mathcal{C}}(D, A)$ (the left weak $\alpha$-inverse of $h$) satisfying*

$$(h^{-1} * h) \circ \alpha = e. \tag{22}$$

*First, note that (22) is equivalent to*

$$(h^{-1} \circ \alpha) * (h \circ \alpha) = e, \tag{23}$$

*and if $\alpha = id_D$ we recover the set $Reg^{WR}(D, A)$ introduced in [1].*

*On the other hand, by $Reg_\alpha(D, A)$ we denote the set of morphisms $h : D \to A$ such that there exists a morphism $h^{-1} : D \to A$ (the left $\alpha$-inverse of $h$) satisfying $(h^{-1} * h) \circ \alpha = (h * h^{-1}) \circ \alpha = \varepsilon_D \otimes \eta_A$. Of course, if $(A, D, \psi, \alpha)$ is a coextended entwining structure, $Reg_\alpha(D, A) \subset Reg_\alpha^{WR}(D, A)$. In this setting, if $\alpha = id_D$ we recover the classical set of regular morphisms $Reg(D, A)$.*

*As a consequence of this definition, if* $h \in Reg_{\alpha}^{WR}(D,A)$*, then* $h' = h \circ \alpha \in Reg_{\alpha}^{WR}(D,A)$ *with* $h'^{-1} = h^{-1} \circ \alpha$ *because, by (2),*

$$(h^{-1} \circ \alpha) * (h \circ \alpha) = (h^{-1} * h) \circ \alpha = e \circ \alpha = e$$

*and* $h' \circ \alpha = h'$*,* $h'^{-1} \circ \alpha = h'^{-1}$*. We can then assume without loss of generality that when we choose an element* $h \in Reg_{\alpha}^{WR}(D,A)$*, it satisfies*

$$h \circ \alpha = h, \quad h^{-1} \circ \alpha = h^{-1}. \tag{24}$$

*Finally, we have that if* $h \in Reg_{\alpha}^{WR}(D,A)$*, with left weak* $\alpha$*-inverse* $h^{-1}$*, then* $h^{\alpha} = h \circ i_{\alpha} \in Reg^{WR}(D_{\alpha},A)$ *with left weak inverse* $(h^{\alpha})^{-1} = h^{-1} \circ i_{\alpha}$*. Conversely, if* $l \in Reg^{WR}(D_{\alpha},A)$ *with left weak inverse* $l^{-1}$*,* ${}^{\alpha}l = l \circ p_{\alpha} \in Reg_{\alpha}^{WR}(D,A)$ *with left weak* $\alpha$*-inverse* $({}^{\alpha}l)^{-1} = l^{-1} \circ p_{\alpha}$*. Finally, note that* ${}^{\alpha}(h^{\alpha}) = h$ *and* $({}^{\alpha}l)^{\alpha} = l$*.*

**Proposition 3.3** *Let* $(A, D, \psi, \alpha)$ *be a coextended weak entwining structure in the conditions of Proposition 2.8. Let* $h \in Reg_{\alpha}^{WR}(D,A)$*. Then, if* $h$ *is a morphism of right* $D$*-comodules for* $\varrho_D = (D \otimes \alpha) \circ \delta_D$*, the interwining* $\psi$ *is completely determined in the following form:*

$$\psi = (\mu_A \otimes D) \circ (A \otimes (\rho_A \circ \mu_A)) \circ (((h^{-1} \otimes h) \circ \delta_D) \otimes A), \tag{25}$$

*and equivalently*

$$\psi = (\mu_A \otimes D) \circ (A \otimes (\rho_A \circ \mu_A)) \circ (((h^{-1} \otimes h) \circ \delta_D \circ \alpha) \otimes A). \tag{26}$$

**Proof** Indeed:

$$\begin{aligned}
&(\mu_A \otimes D) \circ (A \otimes (\rho_A \circ \mu_A)) \circ (((h^{-1} \otimes h) \circ \delta_D) \otimes A) \\
&= (\mu_A \otimes D) \circ (A \otimes ((\mu_A \otimes D) \circ (A \otimes \psi) \circ (\rho_A \otimes A))) \circ (((h^{-1} \otimes h) \circ \delta_D) \otimes A) \\
&= (\mu_A \otimes D) \circ (h^{-1} \otimes ((\mu_A \circ (h \otimes A)) \otimes D) \circ (D \otimes D \otimes (\psi \circ (\alpha \otimes A))) \circ (D \otimes \delta_D \otimes A) \circ (\delta_D \otimes A) \\
&= ((\mu_A \circ ((h^{-1} * h) \otimes A)) \otimes D) \circ (D \otimes (\psi \circ (\alpha \otimes A))) \circ (\delta_D \otimes A) \\
&= ((\mu_A \circ (e \otimes A)) \otimes D) \circ (D \otimes (\psi \circ (\alpha \otimes A))) \circ (\delta_D \otimes A) \\
&= ((\mu_A \circ (e \otimes A)) \otimes D) \circ (D \otimes \psi) \circ (\delta_D \otimes A) \\
&= (((A \otimes \varepsilon_D) \circ \psi) \otimes D) \circ (D \otimes \psi) \circ (\delta_D \otimes A) \\
&= (A \otimes ((\varepsilon_D \otimes D) \circ \delta_D)) \circ \psi \\
&= \psi.
\end{aligned}$$

In the last equalities, the first one follows for the entwining module condition for $A$, the second one by the comodule morphism condition for $h$, and the third one by the coassociativity of $\delta_D$. The fourth one follows because $h \in Reg_{\alpha}^{WR}(D,A)$, and the fifth one follows by (5). The sixth equality relies on (a4), and the seventh follows by (a2). Finally, the last one follows by the counit properties.

The equality (26) follows from (25), using (24). □

**Definition 3.4** *Let $(A, D, \psi, \alpha)$ be a coextended weak entwining structure in the conditions of Proposition 2.8. We say that $A_D \hookrightarrow A$ is a weak $(D, \alpha)$-cleft extension if there exists a morphism $h \in Reg_{\alpha}^{WR}(D, A)$ of right $D$-comodules for $\varrho_D = (D \otimes \alpha) \circ \delta_D$, such that the equality*

$$\psi \circ (D \otimes h^{-1}) \circ \delta_D \circ \alpha = \zeta_A \circ (e * h^{-1}) \circ \alpha \tag{27}$$

*holds. Note that by (2), (5), and (24), the equality (27) can be rewritten as*

$$\psi \circ (D \otimes h^{-1}) \circ \delta_D = \zeta_A \circ (e * h^{-1}). \tag{28}$$

*Then, if $\alpha = id_D$, we have the notion of weak $D$-cleft extension introduced in [1].*

*Furthermore, if $g = e * h^{-1}$ we have*

$$g * h = (e * h^{-1}) * h = e * (h^{-1} * h) = e * e = e,$$

$$e * g = e * (e * h^{-1}) = (e * e) * h^{-1} = e * h^{-1} = g$$

*and*

$$\begin{aligned}
&\psi \circ (D \otimes g) \circ \delta_D \\
&= (\mu_A \circ D) \circ (A \otimes \psi) \circ ((\psi \circ (D \otimes e) \circ \delta_D) \otimes h^{-1}) \circ \delta_D \\
&= (\mu_A \circ D) \circ (A \otimes \psi) \circ (((\psi \otimes \varepsilon_D) \circ (D \otimes \psi) \circ (\delta_D \otimes \eta_A)) \otimes h^{-1}) \circ \delta_D \\
&= (\mu_A \circ D) \circ (A \otimes \psi) \circ ((\psi \circ (D \otimes \eta_A)) \otimes h^{-1}) \circ \delta_D \\
&= \psi \circ (D \otimes h^{-1}) \circ \delta_D \\
&= \zeta_A \circ g,
\end{aligned}$$

*where the first equality follows by the coassociativity of $\delta_D$, the second by the definition of $e$, the third by (a2) and the counit properties, the fourth by (a1) and the unit properties, and, finally, the last by (28).*

*Therefore, we can also assume without loss of generality that*

$$e * h^{-1} = h^{-1}, \tag{29}$$

*and then (27) is equivalent to*

$$\psi \circ (D \otimes h^{-1}) \circ \delta_D = \zeta_A \circ h^{-1}. \tag{30}$$

*The morphism $h$ will be called a cleaving morphism for the extension $A_D \hookrightarrow A$.*

**Proposition 3.5** *Let $(A, D, \psi, \alpha)$ be a coextended weak entwining structure in the conditions of Proposition 2.8. If $A_D \hookrightarrow A$ is a weak $(D, \alpha)$-cleft extension with cleaving morphism $h$ and $(M, \phi_M, \rho_M) \in \mathcal{M}_A^D(\psi, \alpha)$, the morphism*

$$q_M^D = \phi_M \circ (A \otimes h^{-1}) \circ \rho_M : M \to M$$

*satisfies the equality*

$$\phi_M \circ (q_M^D \otimes (q_A^D \circ \eta_A)) = q_M^D \circ q_M^D. \tag{31}$$

*As a consequence, if* $M=A$*, we have*

$$\mu_A\circ(q_A^D\otimes(q_A^D\circ\eta_A))=q_A^D\circ q_A^D \tag{32}$$

*and*

$$q_A^D\circ h=h*h^{-1}. \tag{33}$$

**Proof** The equality (31) follows by (30) because, if $(M,\phi_M,\rho_M)\in\mathcal{M}_A^D(\psi,\alpha)$, we have

$$\begin{aligned}
&q_M^D\circ q_M^D\\
&=\phi_M\circ(M\otimes h^{-1})\circ(\phi_M\otimes D)\circ(M\otimes\psi)\circ(\rho_M\otimes A)\circ(M\otimes h^{-1})\circ\rho_M\\
&=\phi_M\circ(M\otimes(\mu_A\circ(A\otimes h^{-1})\circ\psi\circ(D\otimes h^{-1})\circ\delta_D))\circ\rho_M\\
&=\phi_M\circ(M\otimes(\mu_A\circ(A\otimes h^{-1})\circ\zeta_A\circ h^{-1})\circ\rho_M\\
&=\phi_M\circ(M\otimes(\mu_A\circ(\mu_A\otimes h^{-1})\circ(h^{-1}\otimes(\rho_A\circ\eta_A))))\circ\rho_M\\
&=\phi_M\circ(q_M^D\otimes(q_A^D\circ\eta_A)).
\end{aligned}$$

On the other hand, by the comodule morphism condition for $h$ and (24), we obtain

$$q_A^D\circ h=\mu_A\circ(h\otimes(h^{-1}\circ\alpha))\circ\delta_D=h*h^{-1}.$$

□

**Proposition 3.6** *Let* $(A,D,\psi,\alpha)$ *be a coextended weak entwining structure in the conditions of Proposition 2.8. Let* $h\in Reg_\alpha^{WR}(D,A)$ *be a morphism of right* $D$*-comodules for* $\varrho_D=(D\otimes\alpha)\circ\delta_D$*. Let* $q_M^D$ *be the morphism introduced in the previous proposition. The following assertions are equivalent:*

*(i) For every* $(M,\phi_M,\rho_M)\in\mathcal{M}_A^D(\psi,\alpha)$ *the morphism* $q_M^D$ *factorizes through the equalizer* $i_M^D$*; that is, there exists a unique morphism* $p_M^D:M\to M_D$ *such that* $p_M^D\circ i_M^D=q_M^D$*.*

*(ii) The morphism* $q_A^D$ *factorizes through the equalizer* $i_A^D$*; that is, there exists a unique morphism* $p_A^D:A\to A_D$ *such that* $p_A^D\circ i_A^D=q_A^D$*.*

*(iii)* $A_D\hookrightarrow A$ *is a weak* $(D,\alpha)$*-cleft extension with cleaving morphism* $h$*.*

**Proof** Trivially (i) ⇒ (ii). If (ii) holds, we have that $A_D\hookrightarrow A$ is a weak $(D,\alpha)$-cleft extension with cleaving morphism $h$, because

$$\begin{aligned}
&\psi\circ(D\otimes h^{-1})\circ\delta_D\\
&=(\mu_A\otimes D)\circ(A\otimes(\rho_A\circ\mu_A))\circ(((h^{-1}\otimes h)\circ\delta_D)\otimes A)\circ(D\otimes h^{-1})\circ\delta_D\\
&=(\mu_A\otimes D)\circ(A\otimes(\rho_A\circ\mu_A))\circ(h^{-1}\otimes h\otimes h^{-1}))\circ(D\otimes\delta_D)\circ\delta_D\\
&=(\mu_A\otimes D)\circ(A\otimes(\rho_A\circ\mu_A))\circ(h^{-1}\otimes h\otimes(h^{-1}\circ\alpha)))\circ(D\otimes\delta_D)\circ\delta_D
\end{aligned}$$

$= (\mu_A \otimes D) \circ (h^{-1} \otimes (\rho_A \circ q_A^D \circ h)) \circ \delta_D$

$= (\mu_A \otimes D) \circ (h^{-1} \otimes (\zeta_A \circ q_A^D \circ h)) \circ \delta_D$

$= \zeta_A \circ (h^{-1} * (q_A^D \circ h))$

$= \zeta_A \circ (h^{-1} * (h * h^{-1}))$

$= \zeta_A \circ ((h^{-1} * h) * h^{-1})$

$= \zeta_A \circ (e * h^{-1})$

$= \zeta_A \circ h^{-1},$

where the first equality follows by (25), the second and the eighth by the coassociativity of $\delta_D$, and the third by (24). In the fourth equality we used the comodule morphism condition for $h$, and the fifth one relies on (ii). The sixth equality is a consequence of the associativity of $\mu_A$, and the seventh follows by (33). Finally, the ninth and the tenth equalities follow by the properties of $h^{-1}$.

If (iii) holds, we obtain (i), because using that $(M, \phi_M, \rho_M) \in \mathcal{M}_A^D(\psi, \alpha)$, we have that

$\rho_M \circ q_M^D$

$= (\phi_M \otimes D) \circ (M \otimes \psi) \circ (\rho_M \otimes h^{-1}) \circ \rho_M$

$= (\phi_M \otimes D) \circ (M \otimes (\psi \circ (D \otimes h^{-1}) \circ \delta_D)) \circ \rho_M$

$= (\phi_M \otimes D) \circ (M \otimes (\zeta_A \circ h^{-1})) \circ \rho_M$

$= \zeta_M \circ q_M^D,$

and the proof is complete. □

Let $(A, D, \psi, \alpha)$ be a coextended weak entwining structure in the conditions of Proposition 2.8, and assume that $A_D \hookrightarrow A$ is a weak $(D, \alpha)$-cleft extension with cleaving morphism $h$. By Proposition 2.6, the triple $(A, \mu_A, \rho_A^\alpha = (A \otimes p_\alpha) \circ \rho_A)$ is an object in $\mathcal{M}_A^{D_\alpha}(\psi^\alpha)$. Let $A_{D_\alpha}$ be the equalizer of $\rho_A^\alpha$ and $\zeta_A^\alpha = (\mu_A \otimes D_\alpha) \circ (A \otimes (\rho_A^\alpha \circ \eta_A))$. Then there exists a morphism $\beta_A : A_D \to A_{D_\alpha}$ such that

$$i_A^{D_\alpha} \circ \beta_A = i_A^D. \tag{34}$$

On the other hand, by (19), we know that $(A \otimes i_\alpha) \circ \rho_A^\alpha = \rho_A$, $(A \otimes i_\alpha) \circ \zeta_A^\alpha = \zeta_A$ and, as a consequence, if $A \otimes -$ preserves equalizers, there exists a morphism $\beta'_A : A_{D_\alpha} \to A_D$ such that $i_A^D \circ \beta'_A = i_A^{D_\alpha}$. Moreover,

$$i_A^D \circ \beta'_A \circ \beta_A = i_A^{D_\alpha} \circ \beta_A = i_A^D, \;\; i_A^{D_\alpha} \circ \beta_A \circ \beta'_A = i_A^D \circ \beta'_A = i_A^{D_\alpha},$$

and this implies that $\beta'_A$ is the inverse of $\beta_A$. Therefore, $A_{D_\alpha}$ and $A_D$ are isomorphic as monoids.

If $h$ is the cleaving morphism for $A_D \hookrightarrow A$, then $h^\alpha = h \circ i_\alpha \in Reg^{WR}(D_\alpha, A)$ with left weak inverse $(h^\alpha)^{-1} = h^{-1} \circ i_\alpha$ (see Definition (3.2)), and $h^\alpha$ is a morphism of right $D_\alpha$-comodules because

$$(A \otimes p_\alpha) \circ \rho_A \circ h^\alpha = (h \otimes p_\alpha) \circ \varrho_D \circ i_\alpha = (h^\alpha \otimes D_\alpha) \circ \delta_{D_\alpha}.$$

Finally, for $(h^\alpha)^{-1}$ we have

$$\psi^\alpha\circ(D_\alpha\otimes(h^\alpha)^{-1})\circ\delta_{D_\alpha}=(A\otimes p_\alpha)\circ\psi\circ(D\otimes h^{-1})\circ\delta_D\circ i_\alpha=(A\otimes p_\alpha)\circ\zeta_A\circ h^{-1}\circ i_\alpha=\zeta_A^\alpha\circ(h^\alpha)^{-1},$$

and then $A_{D_\alpha}\hookrightarrow A$ is a weak $D$-cleft extension for $(A,D_\alpha,\psi^\alpha)$ with cleaving morphism $h^\alpha$.

Conversely, assume that $(A,\varrho_A)$ is a right $D_\alpha$-comodule such that $(A,\mu_A,\varrho_A)$ is an object in $\mathcal{M}_A^{D_\alpha}(\psi^\alpha)$. By Proposition 2.6, the triple $(A,\mu_A,\ {}^\alpha\varrho_A=(A\otimes i_\alpha)\circ\varrho_A)$ is an object in $\mathcal{M}_A^D(\psi,\alpha)$. Let $A_{D_\alpha}$ be the equalizer of $\varrho_A$ and $\zeta_A=(\mu_A\otimes D_\alpha)\circ(A\otimes(\varrho_A\circ\eta_A))$. Then there exists a morphism $\pi_A:A_{D_\alpha}\to A_D$ such that

$$i_A^{D_\alpha}=i_A^D\circ\pi_A, \tag{35}$$

where $i_A^D$ is the equalizer morphism of ${}^\alpha\varrho_A$ and ${}^\alpha\zeta_A=(\mu_A\otimes D)\circ(A\otimes({}^\alpha\varrho_A\circ\eta_A))$. On the other hand, if $A\otimes -$ preserves equalizers, we have $\varrho_A\circ i_A^D=\zeta_A\circ i_A^D$, because $(A\otimes i_\alpha)\circ\varrho_A\circ i_A^D=(A\otimes i_\alpha)\circ\zeta_A\circ i_A^D$, and then there exists a morphism $\pi'_A:A_D\to A_{D_\alpha}$ such that $i_A^{D_\alpha}\circ\pi'_A=i_A^D$. Moreover, $\pi'_A\circ\pi_A=id_{A_{D_\alpha}}$ and $\pi_A\circ\pi'_A=id_{A_D}$. Therefore, $A_{D_\alpha}$ and $A_D$ are isomorphic as monoids.

By Definition 3.2, if $l\in Reg^{WR}(D_\alpha,A)$ with left weak inverse $l^{-1}$, ${}^\alpha l=l\circ p_\alpha\in Reg_\alpha^{WR}(D,A)$ with left weak $\alpha$-inverse $({}^\alpha l)^{-1}=l^{-1}\circ p_\alpha$. Moreover, if $l$ is a morphism of right $D_\alpha$ comodules, we have that ${}^\alpha l$ is a morphism of right $D$-comodules for $\varrho_D=(D\otimes\alpha)\circ\delta_D$. Indeed:

$${}^\alpha\varrho_A\circ\ {}^\alpha l=(\ {}^\alpha l\otimes\alpha)\circ\delta_D\circ\alpha=(\ {}^\alpha l\otimes\alpha)\circ\delta_D=(\ {}^\alpha l\otimes D)\circ\varrho_D.$$

Finally,

$$\psi\circ(D\otimes(\ {}^\alpha l)^{-1})\circ\delta_D=(A\otimes\alpha)\circ\psi\circ(\alpha\otimes(\ {}^\alpha l)^{-1})\circ\delta_D=(A\otimes i_\alpha)\circ\psi^\alpha\circ(D_\alpha\otimes l^{-1})\circ\delta_{D_\alpha}\circ p_\alpha$$
$$=(\mu_A\otimes i_\alpha)\circ((\ {}^\alpha l)^{-1}\otimes(\varrho_A\circ\eta_A))=\ {}^\alpha\zeta_A\circ(\ {}^\alpha l)^{-1}),$$

and then, if $A_{D_\alpha}\hookrightarrow A$ is a weak $D_\alpha$-cleft extension for $(A,D_\alpha,\psi^\alpha)$ with cleaving morphism $l$, we have that $A_D\hookrightarrow A$ is a weak $(D,\alpha)$-cleft extension for $(A,D,\psi,\alpha)$ with cleaving morphism ${}^\alpha l$.

**Example 3.7** Let $H$ be a weak Hopf algebra and let $(A,\rho_A)$ be a right $H$-comodule monoid. By example (2.7), we know that $(A,H,\Gamma=(A\otimes\mu_H)\circ(c_{H,A}\otimes H)\circ(H\otimes\rho_A))$ is a weak entwining structure. Let $D$ be a weak Hopf algebra and let $(H,D,f,g)$ be a comonoid projection. Then for $\alpha=f\circ g$ we have that $(A,D,{}^\alpha\Gamma,\alpha)$ is a coextended weak entwining structure and $({}^\alpha\Gamma)^\alpha=\Gamma$. If $A_H\hookrightarrow A$ is a weak $H$-cleft extension for $(A,H,\Gamma)$ with cleaving morphism $l:H\to A$, we have that $A_D\hookrightarrow A$ is a weak $(D,\alpha)$-cleft extension for $(A,D,{}^\alpha\Gamma,\alpha)$ with cleaving morphism ${}^\alpha l=l\circ g$. Also, if $A\otimes -$ preserves equalizers, $A_H\backsimeq A_D$ as monoids. In particular, if $(H,H,\Gamma=(H\otimes\mu_H)\circ(c_{H,H}\otimes H)\circ(H\otimes\delta_H))$ is the weak entwining estructure associated to $H$, we have that $H_H=H_L=Im(\Pi_H^L)$ and $H_L\hookrightarrow H$ is a weak $H$-cleft extension with cleaving morphism $l=id_H$ and $l^{-1}=\lambda_H$. Then $H_D\hookrightarrow D$ is a $(D,\alpha)$-cleft extension for $(H,D,{}^\alpha\Gamma,\alpha)$ with cleaving morphism ${}^\alpha l=g$ and $({}^\alpha l)^{-1}=\lambda_H\circ g$. Finally, if $H\otimes -$ preserves equalizers, $H_L\backsimeq H_D$ as monoids.

**Proposition 3.8** *Let $(A,D,\psi,\alpha)$ be a coextended weak entwining structure in the conditions of Proposition 2.8, and assume that $A_D\hookrightarrow A$ is a weak $(D,\alpha)$-cleft extension with cleaving morphism $h$. Then the following equality holds:*

$$\phi_{M_D}\circ(M_D\otimes p_A^D)=p_M^D\circ\phi_M\circ(i_M^D\otimes A), \tag{36}$$

*for all* $(M,\phi_M,\rho_M)\in \mathcal{M}_A^D(\psi,\alpha)$.

**Proof** Using that $i_M^D$ is an equalizer morphism, we obtain (36), because:

$$\begin{aligned}
& i_M^D\circ p_M^D\circ\phi_M\circ(i_M^D\otimes A)\\
&=\phi_M\circ(\phi_M\otimes h^{-1})\circ(M\otimes\psi)\circ((\rho_M\circ i_M^D)\otimes A)\\
&=\phi_M\circ(\phi_M\otimes h^{-1})\circ(M\otimes\psi)\circ((\zeta_M\circ i_M^D)\otimes A)\\
&=\phi_M\circ(i_M^D\otimes(\mu_A\circ(\mu_A\otimes h^{-1})\circ(A\otimes\psi)\circ((\rho_A\circ\eta_A)\otimes A)))\\
&=\phi_M\circ(i_M^D\otimes q_A^D)\\
&=i_M^D\circ\phi_{M_D}\circ(M_D\otimes p_A^D).
\end{aligned}$$

The first equality follows by (12), the second by the properties of the equalizer morphism $i_M^D$, and the third by the module structure of $M$ and the associativity of $\mu_A$. In the fourth, we apply (12) for $A$, and the last one follows by the definition of $\phi_{M_D}$. □

**Proposition 3.9** *Let $(A,D,\psi,\alpha)$ be a coextended weak entwining structure in the conditions of Proposition 2.8. Suppose that $A_D\hookrightarrow A$ is a weak $(D,\alpha)$-cleft extension with cleaving morphism $h$. Define, for $M\in\mathcal{M}_A^D(\psi,\alpha)$:*

$$\omega_M=\phi_M\circ(i_M^D\otimes h):M_D\otimes D\to M$$

*and*

$$\omega_M'=(p_M^D\otimes D)\circ\rho_M:M\to M_D\otimes D.$$

*The following assertions hold:*

*(i) The morphisms $\omega_M$ and $\omega_M'$ are morphisms of right $D$-comodules for the coaction $\varrho_{M_D\otimes D}=M_D\otimes\varrho_D$. Moreover, they satisfy $\omega_M\circ\omega_M'=id_M$ and then $\Omega_M=\omega_M'\circ\omega_M:M_D\otimes D\to M_D\otimes D$ is an idempotent morphism.*

*(ii) In particular, if we consider $M=A$, we have that $\omega_A$ and $\omega_A'$ are also morphisms of left $A_D$-modules for $\varphi_{A_D\otimes D}=\mu_{A_D}\otimes D$ and $\varphi_A=\mu_A\circ(i_A^D\otimes A)$.*

**Proof** (i) By (20) the morphism $\omega_M$ satisfies

$$\rho_M\circ\omega_M=(\phi_M\otimes D)\circ(i_M^D\otimes(\rho_A\circ h))=(\phi_M\otimes D)\circ(i_M^D\otimes((h\otimes D)\circ\varrho_D))=(\omega_M\otimes D)\circ\varrho_{M_D\otimes D},$$

and then it is a morphism of right $D$-comodules. Also, $\omega_M'$ is a morphism of right $D$-comodules by (19). Indeed:

$$\begin{aligned}
\varrho_{M_D\otimes D}\circ\omega_M'&=(p_M^D\otimes D\otimes\alpha)\circ(M\otimes\delta_D)\circ\rho_M=(((p_M^D\otimes D)\circ\rho_M)\otimes\alpha)\circ\rho_M\\
&=(((p_M^D\otimes D)\circ\rho_M)\otimes D)\circ\rho_M=(\omega_M'\otimes D)\circ\rho_M.
\end{aligned}$$

On the other hand, by (18) we have

$$\omega_M\circ\omega_M'=\phi_M\circ(M\otimes e)\circ\rho_M=id_M,$$

and then $\Omega_M$ is idempotent.

(ii) If $M = A$ we have

$$\varphi_A \circ (A_D \otimes \omega_A) = \mu_A \circ ((\mu_A \circ (i_A^D \otimes i_A^D)) \otimes h) = \omega_A \circ \varphi_{A_D \otimes D},$$

and by (20) and (36), the identities

$$\omega'_A \circ \varphi_A = ((p_A^D \circ \mu_A) \otimes D) \circ (i_A^D \otimes \rho_A) = \mu_{A_D} \circ (A_D \otimes \omega'_A)$$

hold. Therefore, $\omega_A$ and $\omega'_A$ are morphisms of left $A_D$-modules. □

**Proposition 3.10** *In the conditions of Proposition 3.9, denote by $M_D \times D$ the image of the idempotent morphism $\Omega_M$ and consider the right $D$-comodule structure $\varrho_{M_D \times D} = (r_M \otimes D) \circ (M_D \otimes \varrho_D) \circ s_M$, where $r_M$, $s_M$ are the projection and the injection related to $\Omega_M$. Then there exists an isomorphism $b_M : M \to M_D \times D$ of right $D$-comodules. Finally, if $M = A$, $b_A$ is also a morphism of left $A_D$-modules for the left $A_D$-module structure of $A_D \times D$ given by $\varphi_{A_D \times D} = r_A \circ (\mu_{A_D} \otimes D) \circ (A_D \otimes s_A)$.*

**Proof** Let $b_M : M \to M_D \times D$ be the morphism defined by

$$b_M = r_M \circ \omega'_M.$$

Then $b_M$ is an isomorphism with inverse $b_M^{-1} = \omega_M \circ s_M$. Indeed, by Proposition 3.9

$$b_M \circ b_M^{-1} = r_M \circ \Omega_M \circ s_M = id_{M_D \times D},$$

$$b_M^{-1} \circ b_M = \omega_M \circ \omega'_M \circ \omega_M \circ \omega'_M = id_M.$$

On the other hand, $b_M$ is a morphism of right $D$-comodules because:

$$\varrho_{M_D \times D} \circ b_M = (r_M \otimes D) \circ (M_D \otimes ((D \otimes \alpha) \circ \delta_D)) \circ \omega'_M = (b_M \otimes \alpha) \circ \rho_M = (b_M \otimes D) \circ \rho_M.$$

Finally, by (36) and (20) we obtain that $b_A$ is a morphism of left $A_D$-modules because:

$$\varphi_{A_D \times D} \circ (A_D \otimes b_A) = r_A \circ ((\mu_{A_D} \circ (A_D \otimes p_A^D)) \otimes D) \circ (A_D \otimes \rho_A)$$

$$= r_A \circ ((p_A^D \circ \mu_A \circ (i_A^D \otimes A)) \otimes D) \circ (A_D \otimes \rho_A) = r_A \circ (p_A^D \otimes D) \circ \rho_A \circ \mu_A \circ (i_A^D \otimes A) = b_A \circ \varphi_A.$$

□

**Remark 3.11** Note that, in the conditions of Proposition 3.9, the morphism $\omega'_M$ is a morphism of right $D$-comodules for the coaction $\rho_{M_D \otimes D} = M_D \otimes \delta_D$. As a consequence, $b_M$ is a morphism of right $D$-comodules for $\rho_{M_D \times D} = (r_M \otimes D) \circ (M_D \otimes \delta_D) \circ s_M$.

**Proposition 3.12** *In the conditions of Proposition 3.9, the morphism*

$$\phi_A = \mu_A \circ (\mu_A \otimes h^{-1}) \circ (h \otimes \psi) \circ (\delta_D \otimes A)$$

*factorizes through the equalizer* $i_D^A$*. Moreover, if* $\phi'_A$ *is this factorization, we have the following equalities:*

$$\phi'_A = p_A^D \circ \mu_A \circ (h \otimes A), \tag{37}$$

$$\mu_{A_D} \circ (\phi'_A \otimes \phi'_A) \circ (D \otimes \psi \otimes A) \circ (\delta_D \otimes A \otimes A) = \phi'_A \circ (D \otimes \mu_A). \tag{38}$$

*Finally, if we define the morphism* $\phi_{A_D} : D \otimes A_D \to A_D$ *by* $\phi_{A_D} = \phi'_A \circ (D \otimes i_A^D)$*, we obtain:*

$$\mu_{A_D} \circ (\phi'_A \otimes \phi_{A_D}) \circ (D \otimes \psi \otimes A_D) \circ (\delta_D \otimes i_A^D \otimes A_D) = \phi_{A_D} \circ (D \otimes \mu_{A_D}). \tag{39}$$

**Proof** By (5) and (a3), the proof is similar to the one used to prove Proposition 1.15 of [1]. □

**Proposition 3.13** *In the conditions of Proposition 3.9, the morphism* $\sigma_A : D \otimes D \to A$ *defined by*

$$\sigma_A = \phi_A \circ (D \otimes h),$$

*where* $\phi_A$ *is the morphism introduced in the previous proposition, factorizes through the equalizer* $i_D^A$*. Moreover, if* $\sigma_{A_D}$ *is this factorization, then*

$$\sigma_{A_D} = p_D^A \circ \mu_A \circ (h \otimes h). \tag{40}$$

**Proof** By (5), the proof is similar to the one used to prove Proposition 1.17 of [1]. □

In the conditions of Proposition 3.9, we have that $b_A$ is an isomorphism of algebras, where the algebra structure is the one induced by $b_A$:

$$\eta_{A_D \times D} = b_A \circ \eta_A, \quad \mu_{A_D \times D} = b_A \circ \mu_A \circ (b_A^{-1} \otimes b_A^{-1}). \tag{41}$$

In the next proposition we obtain that $\mu_{A_D \times D}$ can be identified in another way as a weak crossed product (see [11] for the definition and main properties of weak crossed products).

**Proposition 3.14** *In the conditions of Proposition 3.9,* $(A_D, D, \psi_D^{A_D}, \sigma_D^{A_D})$ *with*

$$\psi_D^{A_D} = (\phi'_A \otimes D) \circ (D \otimes \psi) \circ (\delta_D \otimes i_D^A), \quad \sigma_D^{A_D} = (\phi'_A \otimes D) \circ (D \otimes \psi) \circ (\delta_D \otimes h),$$

*is a quadruple such that the associated idempotent morphism*

$$\nabla_{A_D \otimes D} = (\mu_{A_D} \otimes D) \circ (A \otimes (\psi_D^{A_D} \circ (D \otimes \eta_{A_D}))) : A_D \otimes D \to A_D \otimes D$$

*is* $\Omega_A$*, and satisfies the twisted and cocycle condition (see Definitions 3.5 and 3.6 of [11]).*

*Moreover, if* $m_{A_D \times D}$ *denotes the associative product induced by the quadruple*

$$(A_D, D, \psi_D^{A_D}, \sigma_D^{A_D}),$$

*we have that* $m_{A_D \times D} = \mu_{A_D \times D}$*, where* $\mu_{A_D \times D}$ *is the product defined in (41).*

**Proof** First note that, by (37) and (5),

$$\psi_D^{A_D} = ((p_A^D \circ \mu_A) \otimes D) \circ (h \otimes (\psi \circ (\alpha \otimes A))) \circ (\delta_D \otimes i_A^D) = \omega'_A \circ \mu_A \circ (h \otimes i_A^D),$$

and similarly

$$\sigma_D^{A_D} = \omega'_A \circ \mu_A \circ (h \otimes h).$$

Then, using that $\omega'_A$ is a morphism of left $A_D$-modules,

$$\nabla_{A_D \otimes D} = \varphi_{A_D \otimes D} \circ (A_D \otimes (\omega'_A \circ h)) = \omega'_A \circ \varphi_A \circ (A_D \otimes h) = \Omega_A,$$

and, as a consequence, $\sigma_D^{A_D} \circ \nabla_{A_D \otimes D} = \sigma_D^{A_D} \circ \Omega_A = \sigma_D^{A_D}$.

The quadruple $(A_D, D, \psi_D^{A_D}, \sigma_D^{A_D})$ satisfies the twisted condition because, by the left $A_D$-module condition for $\omega'_A$ and the associativity of $\mu_A$, we have:

$$\begin{aligned}
&(\mu_{A_D} \otimes D) \circ (A_D \otimes \sigma_D^{A_D}) \circ (\psi_D^{A_D} \otimes D) \circ (D \otimes \psi_D^{A_D}) \\
&= \omega'_A \circ \mu_A \circ ((\omega_A \circ \omega'_A \circ \mu_A) \otimes A) \circ (h \otimes ((i_A^D \otimes h) \circ \omega'_A \circ \mu_A \circ (h \otimes i_A^D))) \\
&= \omega'_A \circ \mu_A \circ (h \otimes (\omega_A \circ \omega'_A \circ \mu_A \circ (h \otimes i_A^D))) \\
&= \omega'_A \circ \mu_A \circ ((\mu_A \circ (h \otimes h)) \otimes i_A^D) \\
&= \omega'_A \circ \mu_A \circ ((\omega_A \circ \omega'_A \circ \mu_A \circ (h \otimes h)) \otimes i_A^D) \\
&= (\mu_{A_D} \otimes D) \circ (A_D \otimes \psi_D^{A_D}) \circ (\sigma_D^{A_D} \otimes A_D).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&(\mu_{A_D} \otimes D) \circ (A_D \otimes \sigma_D^{A_D}) \circ (\psi_D^{A_D} \otimes D) \circ (D \otimes \sigma_D^{A_D}) \\
&= \omega'_A \circ \mu_A \circ ((\mu_A \circ (h \otimes h)) \otimes h) \\
&= (\mu_{A_D} \otimes D) \circ (A_D \otimes \sigma_D^{A_D}) \circ (\sigma_D^{A_D} \otimes D),
\end{aligned}$$

and $(A_D, D, \psi_D^{A_D}, \sigma_D^{A_D})$ satisfies the cocycle condition. Therefore, by Proposition 3.8 of [11], the product

$$m_{A_D \times D} = r_A \circ (\mu_{A_D} \otimes D) \circ (\mu_{A_D} \otimes \sigma_D^{A_D}) \circ (A_D \otimes \psi_D^{A_D} \otimes D) \circ (s_A \otimes s_A)$$

is associative and $m_{A_D \times D} = \mu_{A_D \times D}$ because, by the left $A_D$-module condition for $\omega'_A$ and the associativity of $\mu_A$,

$$(\mu_{A_D} \otimes D) \circ (\mu_{A_D} \otimes \sigma_D^{A_D}) \circ (A_D \otimes \psi_D^{A_D} \otimes D) = \omega'_A \circ \mu_A \circ (\omega_A \otimes \omega_A).$$

□

## 4. Galois extensions for coextended weak entwining structures

Let $(A, D, \psi, \alpha)$ be a coextended weak entwining structure in the conditions of Proposition 2.8. Let $A \square D$ be the image of the idempotent morphism $\Delta_{A \otimes D}$ defined in (13), and let $i_{A \otimes D} : A \square D \to A \otimes D$ and

$p_{A\otimes D} : A\otimes D \to A\square D$ be the morphisms satisfying $\Delta_{A\otimes D} = i_{A\otimes D}\circ p_{A\otimes D}$ and $p_{A\otimes D}\circ i_{A\otimes D} = id_{A\square D}$. Under these conditions,

$$A\square D \xrightarrow{i_{A\otimes D}} A\otimes D \overset{\Delta_{A\otimes D}}{\underset{id_{A\otimes D}}{\rightrightarrows}} A\otimes D$$

is an equalizer diagram. Let $t_A : A\otimes A \to A\otimes D$ be the morphism defined by $t_A = (\mu_A\otimes D)\circ(A\otimes\rho_A)$. By the associativity of $\mu_A$, (12), and the properties of $\eta_A$, we have that

$$\Delta_{A\otimes D}\circ t_A = (\mu_A\otimes D)\circ(\mu_A\otimes\psi)\circ(A\otimes\rho_A\otimes\eta_A)$$

$$= (\mu_A\otimes D)\circ(A\otimes((\mu_A\otimes D)\circ(A\otimes\psi)\circ(\rho_A\otimes\eta_A))) = t_A.$$

Therefore, there exists a unique morphism (called the lifted canonical morphism) $r_{A\otimes D} : A\otimes A \to A\square D$, such that $i_{A\otimes D}\circ r_{A\otimes D} = t_A$, and equivalently, $r_{A\otimes D} = p_{A\otimes D}\circ t_A$.

On the other hand, it is obvious that $(A, \varphi_A = \mu_A\circ(i_A^D\otimes A))$ is a left $A_D$-module and $(A, \varphi'_A = \mu_A\circ(A\otimes i_A^D))$ is a right $A_D$-module. With $n_A$ we denote the coequalizer morphism of $A\otimes\varphi_A$ and $\varphi'_A\otimes A$.

$$A\otimes A_D\otimes A \overset{A\otimes\varphi_A}{\underset{\varphi'_A\otimes A}{\rightrightarrows}} A\otimes A \xrightarrow{n_A} A\otimes_{A_D} A$$

As in 1.5 of [2], we can prove that the morphism $r_{A\otimes D}$ satisfies

$$i_{A\otimes D}\circ r_{A\otimes D}\circ(A\otimes\varphi_A) = i_{A\otimes D}\circ r_{A\otimes D}\circ(\varphi'_A\otimes A) \tag{42}$$

and, as a consequence, there exists a unique morphism (called the canonical morphism)

$$\gamma_A : A\otimes_{A_D} A \to A\square D,$$

such that $\gamma_A\circ n_A = r_{A\otimes D}$.

On the other hand, $\gamma_A$ is a morphism of right $D$-comodules, where $\rho_{A\otimes_{A_D}A} : A\otimes_{A_D} A \to (A\otimes_{A_D} A)\otimes C$ is the factorization of $(n_A\otimes D)\circ(A\otimes\rho_A)$ through the coequalizer $n_A$, i.e. $\rho_{A\otimes_{A_D}A}$ is the unique morphism such that $\rho_{A\otimes_{A_D}A}\circ n_A = (n_A\otimes D)\circ(A\otimes\rho_A)$, and $\rho_{A\square D} : A\square D \to A\square D\otimes D$ is defined by $\varrho_{A\square D} = (p_{A\otimes D}\otimes D)\circ(A\otimes\varrho_D)\circ i_{A\otimes D}$. Moreover, $\gamma_A$ is a morphism of right $D$-comodules with $\rho_{A\square D} = (p_{A\otimes D}\otimes D)\circ(A\otimes\delta_D)\circ i_{A\otimes D}$. Finally, if $A\otimes -$ preserves coequalizers, $\gamma_A$ is a morphism of left $A$-modules where $\varphi_{A\otimes_{A_D}A} : A\otimes(A\otimes_{A_D} A) \to A\otimes_{A_D} A$ is the factorization of $n_A\circ(\mu_A\otimes A)$ through the coequalizer $A\otimes n_A$, i.e. $\varphi_{A\otimes_{A_D}A}$ is the unique morphism such that $\varphi_{A\otimes_{A_D}A}\circ(A\otimes n_A) = n_A\circ(\mu_A\otimes A)$, and $\varphi_{A\square D} : A\otimes A\square D \to A\square D$ is defined by $\varphi_{A\square D} = p_{A\otimes D}\circ(\mu_A\otimes D)\circ(A\otimes i_{A\otimes D})$.

**Definition 4.1** *Let $(A, D, \psi, \alpha)$ be a coextended weak entwining structure in the conditions of Proposition 2.8. We say that $A_D \hookrightarrow A$ is a weak $(D,\alpha)$-Galois extension if the canonical morphism $\gamma_A$ is an isomorphism.*

*If $\alpha = id_D$, the notion of weak $(D,\alpha)$-Galois extension is the one defined for weak entwined structures in [2] with the name of weak $D$-Galois extension (in this last definition the condition of $A\otimes -$ preserving coequalizers was required, but it is only necessary if we want $\gamma_A$ to be a morphism of left $A$-modules). This kind of extension was introduced by Brzeziński in [6] for weak entwining structures in a category of modules over a commutative ring.*

**Proposition 4.2** *Let* $(A, D, \psi, \alpha)$ *be a coextended weak entwining structure in the conditions of Proposition 2.8. Let* $(A, D_\alpha, \psi^\alpha)$ *be the weak entwining structure, where* $\psi^\alpha$ *is the morphism defined in (8). If* $A_D \hookrightarrow A$ *is a weak* $(D,\alpha)$*-Galois extension, then* $A_{D_\alpha} \hookrightarrow A$ *is a weak* $D_\alpha$*-Galois extension, where* $\rho_A^\alpha = (A \otimes p_\alpha) \circ \rho_A$*. Conversely, let* $(A, D_\alpha, \Gamma)$ *be a weak entwining structure and let* $(A, \varrho_A)$ *be a right* $D_\alpha$*-comodule such that* $(A, \mu_A, \varrho_A)$ *is an object in* $\mathcal{M}_A^{D_\alpha}(\Gamma)$*. If* $A_{D_\alpha} \hookrightarrow A$ *is a weak* $D_\alpha$*-Galois extension, we have that* $A_D \hookrightarrow A$ *is a weak* $(D,\alpha)$*-Galois extension for* $(A, D, {}^\alpha\Gamma, \alpha)$*, where* ${}^\alpha\Gamma$ *is the morphism defined in (9).*

**Proof** Let $\Delta^\alpha_{A\otimes D_\alpha} : A \otimes D_\alpha \to A \otimes D_\alpha$ be the morphism defined by $\Delta^\alpha_{A\otimes D_\alpha} = (A \otimes p_\alpha) \circ \Delta_{A\otimes D} \circ (A \otimes i_\alpha)$. Then the equality

$$\Delta^\alpha_{A\otimes D_\alpha} = (\mu_A \otimes D_\alpha) \circ (A \otimes (\psi^\alpha \circ (D_\alpha \otimes \eta_A)) \tag{43}$$

holds. Let $A\square D_\alpha$ be the image of $\Delta^\alpha_{A\otimes D_\alpha}$, and let $p^\alpha_{A\otimes D_\alpha}$, $i^\alpha_{A\otimes D_\alpha}$ be the projection and the injection associated to $\Delta^\alpha_{A\otimes D_\alpha}$. By (4) and (5) it is easy to show that

$$v^\alpha_A = p_{A\otimes D} \circ (A \otimes i_\alpha) \circ i^\alpha_{A\otimes D_\alpha} : A\square D_\alpha \to A\square D$$

is an isomorphism, with inverse

$$(v^\alpha_A)^{-1} = p^\alpha_{A\otimes D_\alpha} \circ (A \otimes p_\alpha) \circ i_{A\otimes D} : A\square D \to A\square D_\alpha,$$

such that

$$v^\alpha_A \circ r^\alpha_{A\otimes D_\alpha} = p_{A\otimes D} \circ (A \otimes i_\alpha) \circ \Delta^\alpha_{A\otimes D_\alpha} \circ (A \otimes p_\alpha) \circ (\mu_A \otimes D) \circ (A \otimes \rho_A) = r_{A\otimes D},$$

where $r^\alpha_{A\otimes D_\alpha}$ is the lifted canonical morphism associated to $t^\alpha_A = (\mu_A \otimes D_\alpha) \circ (A \otimes \rho^\alpha_A)$.

Consider $(A, \varphi^\alpha_A = \mu_A \circ (i^{D_\alpha}_A \otimes A))$, $(A, \varphi'^\alpha_A = \mu_A \circ (A \otimes i^{D_\alpha}_A))$, and let $n^\alpha_A$ be the coequalizer morphism of $A \otimes \varphi^\alpha_A$ and $\varphi'^\alpha_A \otimes A$, i.e.

$$A \otimes A_{D_\alpha} \otimes A \underset{\varphi'^\alpha_A \otimes A}{\overset{A\otimes\varphi^\alpha_A}{\rightrightarrows}} A \otimes A \xrightarrow{n^\alpha_A} A \otimes_{A_{D_\alpha}} A$$

is a coequalizer diagram. Then the existence of the isomorphism $\beta_A : A_D \to A_{D_\alpha}$ satisfying (34) implies that there exists a unique isomorphism $d^\alpha_A : A \otimes_{A_D} A \to A \otimes_{A_{D_\alpha}} A$ such that $d^\alpha_A \circ n_A = n^\alpha_A$, where $n_A$ is the coequalizer morphism of the morphisms $\varphi_A$ and $\varphi'_A$ defined in the previous page. Then, if $\gamma^\alpha_A : A \otimes_{A_{D_\alpha}} A \to A\square D_\alpha$ is the canonical morphism for the extension associated to $(A, D_\alpha, \psi^\alpha)$, we have

$$\gamma^\alpha_A \circ d^\alpha_A \circ n_A = \gamma^\alpha_A \circ n^\alpha_A = r^\alpha_{A\otimes D_\alpha} = (v^\alpha_A)^{-1} \circ r_{A\otimes D} = (v^\alpha_A)^{-1} \circ \gamma_A \circ n_A,$$

and this implies that $\gamma^\alpha_A \circ d^\alpha_A = (v^\alpha_A)^{-1} \circ \gamma_A$, where $\gamma_A$ is the canonical morphism of the extension associated to $(A, D, \psi, \alpha)$. Thus, if $\gamma_A$ is an isomorphism, $\gamma^\alpha_A$ is an isomorphism.

Conversely, let ${}^\alpha\Delta_{A\otimes D} : A\otimes D \to A\otimes D$ be the morphism defined by ${}^\alpha\Delta_{A\otimes D} = (A\otimes i_\alpha)\circ\Delta_{A\otimes D_\alpha}\circ(A\otimes p_\alpha)$. Then ${}^\alpha\Delta_{A\otimes D} = (\mu_A \otimes D) \circ (A \otimes ({}^\alpha\Gamma \circ (D \otimes \eta_A))$. Let $A\square D$ be the image of ${}^\alpha\Delta_{A\otimes D}$, and let ${}^\alpha p_{A\otimes D}$, ${}^\alpha i_{A\otimes D}$ be the projection and the injection associated to ${}^\alpha\Delta_{A\otimes D}$. The morphism

$${}^\alpha v_A = p_{A\otimes D_\alpha} \circ (A \otimes p_\alpha) \circ {}^\alpha i_{A\otimes D} : A\square D \to A\square D_\alpha$$

is an isomorphism, with inverse

$$({}^{\alpha}v_A)^{-1} = {}^{\alpha}p_{A\otimes D} \circ (A \otimes i_\alpha) \circ i_{A\otimes D_\alpha} : A\Box D_\alpha \to A\Box D,$$

such that ${}^{\alpha}r_{A\otimes D} \circ {}^{\alpha}v_A = r_{A\otimes D_\alpha}$. Consider $(A, {}^{\alpha}\varphi_A = \mu_A \circ (i_A^D \otimes A))$, $(A, {}^{\alpha}\varphi'_A = \mu_A \circ (A \otimes i_A^D))$, and let ${}^{\alpha}n_A$ be the coequalizer morphism of $A \otimes {}^{\alpha}\varphi_A$ and ${}^{\alpha}\varphi'_A \otimes A$, i.e.

$$\begin{array}{ccccc} & A \otimes^{\alpha} \varphi_A & & {}^{\alpha}n_A & \\ A \otimes A_D \otimes A & \rightrightarrows & A \otimes A & \longrightarrow & A \otimes_{A_D} A \\ & {}^{\alpha}\varphi'_A \otimes A & & & \end{array}$$

is a coequalizer diagram. Then the existence of the isomorphism $\pi_A : A_{D_\alpha} \to A_D$ satisfying (35) implies that there exists a unique isomorphism ${}^{\alpha}d_A : A \otimes_{A_{D_\alpha}} A \to A \otimes_{A_D} A$ such that ${}^{\alpha}d_A \circ n_A = {}^{\alpha}n_A$, where $n_A$ is the coequalizer of $\varphi_A = \mu_A \circ (i_A^{D_\alpha} \otimes A)$ and $\varphi'_A = \mu_A \circ (A \otimes i_A^{D_\alpha})$. Then, if ${}^{\alpha}\gamma_A : A \otimes_{A_D} A \to A\Box D$ is the canonical morphism for the extension associated to $(A, D, {}^{\alpha}\Gamma, \alpha)$, we have that $\gamma_A \circ ({}^{\alpha}d_A)^{-1} = {}^{\alpha}v_A \circ {}^{\alpha}\gamma_A$, where $\gamma_A$ is the canonical morphism of the extension associated to $(A, D_\alpha, \Gamma)$. Thus, if $\gamma_A$ is an isomorphism, ${}^{\alpha}\gamma_A$ is an isomorphism. □

**Definition 4.3** *Let $(A, D, \psi, \alpha)$ be a coextended weak entwining structure in the conditions of Proposition 2.8. Let $A_D \hookrightarrow A$ be a weak $(D, \alpha)$-Galois extension. We will say that $A_D \hookrightarrow A$ satisfies the normal basis property (or $A_D \hookrightarrow A$ is a coextended weak $(D, \alpha)$-Galois extension with normal basis) if there exists an idempotent morphism of left $A_D$-modules and right $D$-comodules $\Pi_A : A_D \otimes D \to A_D \otimes D$, for $\varphi_{A_D\otimes D} = \mu_{A_D} \otimes D$ and $\rho_{A_D\otimes D} = A_D \otimes \varrho_D$, such that*

$$\Pi_A = (A_D \otimes \alpha) \circ \Pi_A = \Pi_A \circ (A_D \otimes \alpha), \tag{44}$$

*and an isomorphism of left $A_D$-modules and right $D$-comodules $g_A : A \to A_D \boxtimes D$, where $A_D \boxtimes D$ is the image of $\Pi_A$ and*

$$\varphi_{A_D\boxtimes D} = r_A \circ \varphi_{A_D\otimes D} \circ (A_D \otimes s_A), \qquad \rho_{A_D\boxtimes D} = (r_A \otimes D) \circ \rho_{A_D\otimes D} \circ s_A,$$

*being $s_A : A_D \boxtimes D \to A_D \otimes D$ and $r_A : A_D \otimes D \to A_D \times D$ the morphisms such that $s_A \circ r_A = \Pi_A$ and $r_A \circ s_A = id_{A_D\boxtimes D}$.*

*Note that, if $\alpha = id_D$, we can recall the definition of weak $D$-Galois extension with normal basis introduced in [2] for weak Galois extensions associated to a weak entwining structure $(A, D, \psi)$.*

**Proposition 4.4** *Let $(A, D, \psi, \alpha)$ be a coextended weak entwining structure in the conditions of Proposition 2.8. Let $(A, D_\alpha, \psi^\alpha)$ be the weak entwining structure where $\psi^\alpha$ is the morphism defined in (8). If $A_D \hookrightarrow A$ is a weak $(D, \alpha)$-Galois extension with normal basis, then $A_{D_\alpha} \hookrightarrow A$ is a weak $D_\alpha$-Galois extension with normal basis, where $\rho_A^\alpha = (A \otimes p_\alpha) \circ \rho_A$. Conversely, let $(A, D_\alpha, \Gamma)$ be a weak entwining structure, and let $(A, \varrho_A)$ be a right $D_\alpha$-comodule such that $(A, \mu_A, \varrho_A)$ is an object in $\mathcal{M}_A^{D_\alpha}(\Gamma)$. If $A_{D_\alpha} \hookrightarrow A$ is a weak $D_\alpha$-Galois extension with normal basis, we have that $A_D \hookrightarrow A$ is a weak $(D, \alpha)$-Galois extension with normal basis for $(A, D, {}^{\alpha}\Gamma, \alpha)$, where ${}^{\alpha}\Gamma$ is the morphism defined in (9).*

**Proof** Assume that $A_D \hookrightarrow A$ is a weak $(D, \alpha)$-Galois extension with normal basis. Then, by Proposition 4.2, $A_{D_\alpha} \hookrightarrow A$ is a weak $D_\alpha$-Galois extension for the weak entwining structure $(A, D_\alpha, \psi^\alpha)$, where $\psi^\alpha$ is the morphism defined in (8) and $\rho_A^\alpha = (A \otimes p_\alpha) \circ \rho_A$. Let $\beta_A : A_D \to A_{D_\alpha}$ be the isomorphism satisfying (34). By (44), the morphism

$$\Pi_A^\alpha = (\beta_A \otimes p_\alpha) \circ \Pi_A \circ (\beta_A^{-1} \otimes i_\alpha) : A_{D_\alpha} \otimes D_\alpha \to A_{D_\alpha} \otimes D_\alpha$$

is idempotent. It is also a morphism of left $A_{D_\alpha}$-modules, because $\beta_A$ is an algebra morphism and $\Pi_A$ is a morphism of left $A_D$-modules. Moreover, by (44) and the right $D$-comodule condition for $\Pi_A$, we obtain that $\Pi_A^\alpha$ is a morphism of right $D_\alpha$-comodules.

Let $A_\alpha \boxtimes D_\alpha$ be the image of $\Pi_A^\alpha$, and let $s_A^\alpha : A_{D_\alpha} \boxtimes D_\alpha \to A_{D_\alpha} \otimes D_\alpha$ and $r_A^\alpha : A_{D_\alpha} \otimes D \to A_{D_\alpha} \boxtimes D_\alpha$ be the morphisms such that $s_A^\alpha \circ r_A^\alpha = \Pi_A^\alpha$ and $r_A^\alpha \circ s_A^\alpha = id_{A_{D_\alpha} \boxtimes D_\alpha}$. The morphism $u_A^\alpha = r_A^\alpha \circ (\beta_A \otimes p_\alpha) \circ s_A :$ $A_D \boxtimes D \to A_{D_\alpha} \boxtimes D_\alpha$ is an isomorphism with inverse $(u_A^\alpha)^{-1} = r_A \circ (\beta_A^{-1} \otimes i_\alpha) \circ s_A^\alpha$ and, as a consequence, the morphism $g_A^\alpha = u_A^\alpha \circ g_A : A \to A_{D_\alpha} \boxtimes D_\alpha$, where $g_A$ is the isomorphism associated to $A_D \hookrightarrow A$, is an isomorphism. Moreover, it is a morphism of left $A_{D_\alpha}$-modules because:

$$\begin{aligned}
& g_A^\alpha \circ \varphi_A^\alpha \\
&= u_A^\alpha \circ g_A \circ \varphi_A \circ (\beta_A^{-1} \otimes A) \\
&= r_A^\alpha \circ (\beta_A \otimes p_\alpha) \circ \Pi_A \circ (\mu_{A_D} \otimes D) \circ (\beta_A^{-1} \otimes (s_A \circ g_A)) \\
&= r_A^\alpha \circ ((\beta_A \circ \mu_{A_D}) \otimes p_\alpha) \circ (\beta_A^{-1} \otimes (s_A \circ g_A)) \\
&= r_A^\alpha \circ (\mu_{A_{D_\alpha}} \otimes D_\alpha) \circ (A_{D_\alpha} \otimes ((\beta_A \otimes p_\alpha) \circ s_A \circ g_A)) \\
&= r_A^\alpha \circ (\mu_{A_{D_\alpha}} \otimes D_\alpha) \circ (A_{D_\alpha} \otimes (\Pi_A^\alpha \circ (\beta_A \otimes p_\alpha) \circ s_A \circ g_A)) \\
&= \varphi_{A_{D_\alpha} \boxtimes D_\alpha} \circ (A_{D_\alpha} \otimes g_A^\alpha).
\end{aligned}$$

In the previous calculus, the first equality follows by (34), the second one is a consequence of the left $A_D$-module condition for $g_A$, and the third one relies on the same condition for $\Pi_A$. In the fourth one we used that $\beta_A$ is a monoid morphism, and the fifth one follows because $\Pi_A^\alpha$ is a morphism of left $A_{D_\alpha}$-modules. Finally, the last equality follows by definition.

Moreover, $g_A^\alpha$ is a morphism of right $D_\alpha$-comodules. Indeed:

$$\begin{aligned}
& (g_A^\alpha \otimes D_\alpha) \circ \rho_A^\alpha \\
&= ((r_A^\alpha \circ (\beta_A \otimes p_\alpha) \circ s_A \circ g_A) \otimes p_\alpha) \circ \rho_A \\
&= ((r_A^\alpha \circ (\beta_A \otimes p_\alpha) \circ \Pi_A) \otimes p_\alpha) \circ (A_D \otimes \delta_D) \circ s_A \circ g_A \\
&= (r_A^\alpha \otimes D_\alpha) \circ (\beta \otimes (\delta_{D_\alpha} \circ p_\alpha)) \circ s_A \circ g_A \\
&= (r_A^\alpha \otimes D_\alpha) \circ (A_{D_\alpha} \otimes \delta_{D_\alpha}) \circ (\beta \otimes p_\alpha) \circ s_A \circ g_A \\
&= \rho_{A_{D_\alpha} \boxtimes D_\alpha} \circ g_A^\alpha.
\end{aligned}$$

The first and fifth equalities follow by the definitions, the second one follows because $g_A$ is a morphism of right $D$-comodules, in the third one we use that $\Pi_A$ is a morphism of right $D$-comodules, and the fourth one relies on the right $D_\alpha$-comodule condition for $\Pi_A^\alpha$.

Thus, $A_{D_\alpha} \hookrightarrow A$ is a weak $D_\alpha$-Galois extension with normal basis.

Conversely, let $(A, D_\alpha, \Gamma)$ be a weak entwining structure and let $(A, \varrho_A)$ be a right $D_\alpha$-comodule such that $(A, \mu_A, \varrho_A)$ is an object in $\mathcal{M}_A^{D_\alpha}(\Gamma)$. Let $A_{D_\alpha} \hookrightarrow A$ be a weak $D_\alpha$-Galois extension with normal basis with idempotent $\Pi_A : A_{D_\alpha} \otimes D_\alpha \to A_{D_\alpha} \otimes D_\alpha$ and associated isomorphism $g_A : A \to A_{D_\alpha} \boxtimes D_\alpha$. Let $\pi_A$ the unique monoid isomorphism satisfying (35). The morphism

$$^{\alpha}\Pi_A = (\pi_A \otimes i_\alpha) \circ \Pi_A \circ (\pi_A^{-1} \otimes p_\alpha) : A_D \otimes D \to A_D \otimes D$$

is idempotent, satisfies (44), and is a morphism of left $A_D$-modules and right $D$-comodules. Moreover, $^{\alpha}u_A = {}^{\alpha}r_A \circ (\pi_A^{-1} \otimes i_\alpha) \circ s_A$ is an isomorphism and $^{\alpha}g_A = {}^{\alpha}u_A \circ g_A : A \to A_D \boxtimes D$ is an isomorphism of left $A_D$-modules and right $D$-comodules. Therefore, $A_D \hookrightarrow A$ is a weak $(D, \alpha)$-Galois extension with normal basis for $(A, D, {}^{\alpha}\Gamma, \alpha)$. □

**Theorem 4.5** *Let $(A, D, \psi, \alpha)$ be a coextended weak entwining structure in the conditions of Proposition 2.8. The following are equivalent.*

*(i) $A_D \hookrightarrow A$ is a weak $(D, \alpha)$-cleft extension.*

*(ii) $A_D \hookrightarrow A$ is a weak $(D, \alpha)$-Galois extension and satisfies the normal basis condition.*

**Proof** (i) ⇒ (ii). If $A_D \hookrightarrow A$ is a weak $(D, \alpha)$-cleft extension, $A_{D_\alpha} \hookrightarrow A$ is a weak $D$-cleft extension for $(A, D_\alpha, \psi^\alpha)$. Then, by Theorem 2.11 of [2], $A_{D_\alpha} \hookrightarrow A$ is a weak $(D, \alpha)$-Galois extension for $(A, D_\alpha, \psi^\alpha)$ and satisfies the normal basis condition (note that in Theorem 2.11 of [2] the condition "$A \otimes -$ preserve coequalizers" can be dropped). Therefore, by Propositions 4.2 and 4.4, $A_D \hookrightarrow A$ is a weak $(D, \alpha)$-Galois extension for $(A, D, \psi, \alpha)$ and satisfies the normal basis condition.

The proof for (ii) ⇒ (i) is similar and the details are left to the reader. □

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