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A decomposition of transferable utility games: structure of transferable utility games

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Abstract: We define a decomposition of transferable utility games based on shifting the worth of the grand coalition so that the associated game has a nonempty core. We classify the set of all transferable utility games based on that decomposition and analyze their structure. Using the decomposition and the notion of minimal balanced collections, we give a set of necessary and sufficient conditions for a transferable utility game to have a singleton core.

Key words: Transferable utility games, cooperative games, core, single-valued core, balanced collections

1. Introduction

In cooperative game theory, given a set of players, players cooperate in order to optimize their payoffs. A transferable utility game (also called a cooperative game in characteristic function form with side payments) with player set N is a function that assigns a real number to each coalition.

Many different solution concepts have been established to determine how the payoffs should be distributed between the players in the case of cooperation. [3, 7, 8, 9, 11] are only a few examples of remarkable works in the literature. One solution concept that has received a great deal of attention in the literature and is widely accepted as the major stability notion in cooperative game theory is the core, which is defined by Gillies [5]. The core of a transferable utility game is the set of all feasible outcomes upon which no coalition can improve. Yet, the core of a game can be empty. Shapley and Bondareva [2, 10] give a set of necessary and sufficient conditions for a transferable utility game to have a nonempty core via balanced collections, which is known as the Shapley–Bondareva theorem.

In this paper, we define a decomposition of transferable utility games based on shifting the worth of the grand coalition so that the associated game has a nonempty core. We classify the set of all transferable utility games according to this decomposition and analyze the structure of them. We give a set of necessary and sufficient conditions for a transferable utility game to have a singleton core via balanced collections. We show that most of the transferable utility games have some specific structure, namely, most of them can be written as a sum of 2 games, one being the associated game that has a singleton core and the other being the remaining, trivial game.

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2. Preliminaries

Given $n \in \mathbb{N}$, let $N := \{1, \ldots, n\}$ denote the set of finite players. *A transferable utility game* (or simply a *game*), with player set N is a function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) := 0$. For each $T \subseteq N$, we refer to $v(T)$ as the *worth of coalition* T. Let G denote the set of all games (with player set N).

Nonempty subsets of the player set are called *coalitions.*

A vector $x \in \mathbb{R}^N$ assigning payoff $x_i \in \mathbb{R}$ to player $i \in N$ is called a *payoff vector.* For a payoff vector $x \in \mathbb{R}^N$ and $\emptyset \neq S \subseteq N$, the total payoff of the players in coalition S is $x(S) := \sum_{i \in S} x_i$.

A payoff vector x is *feasible* if $x(N) \leq v(N)$, and *stable* if for each $\emptyset \neq S \subseteq N$, $x(S) \geq v(S)$. The set of all feasible and stable payoff vectors is called the *core of the game* v, denoted by $C(v)$; i.e.

$$C(v) := \{x \in \mathbb{R}^N : x(N) \leq v(N) \text{ and for each } \emptyset \neq S \subseteq N,\, x(S) \geq v(S)\}.$$

The set of all games with nonempty cores is denoted by G_c, and the set of all games with empty cores is denoted by $\overline{G}_c$, i.e. $\overline{G}_c \equiv G \setminus G_c$.

It is well known that the core satisfies the following property, which is known as *covariant under strategic equivalence*: If $v, w \in G$, $\alpha > 0$, $\beta \in \mathbb{R}^N$, and $w = \alpha v + \beta$, then $C(w) = \alpha C(v) + \beta$. For the property, see for example [9].

A collection $\{S_1, \ldots, S_k\}$ of coalitions of N is *balanced* if there exists a collection of real numbers $\lambda_1, \ldots, \lambda_k \in [0, 1]$ such that for each $i \in N$, $\sum_{j \in \{1,\ldots,k\} : i \in S_j} \lambda_j = 1$. The numbers $\lambda_1, \ldots, \lambda_k$ are called *balancing coefficients.* A balanced collection $\{S_1, \ldots, S_k\}$ is a *minimal balanced collection* if no proper subcollection is balanced.

3. A decomposition of games

In this section, we give the definition of the decomposition of games, and classify the set of all games based on this decomposition.

Given a pair of games $v, \tilde{v} \in G$, for each $S \subseteq N$, $(v + \tilde{v})(S) := v(S) + \tilde{v}(S)$ and $(v - \tilde{v})(S) := v(S) - \tilde{v}(S)$.

Given $v \in G$, for each $r \in \mathbb{R}$, v_r is defined as follows:

$$v_r(S) := \begin{cases} v(S) & \text{if } S \subset N, \\ r & \text{if } S = N. \end{cases} \tag{1}$$

Let $M_v := \{r \in \mathbb{R} : C(v_r) \neq \emptyset\}$, and $r^* := \min_{r \in M_v} r$.

We briefly discuss the existence of r^*. It is well known that games with nonempty cores, that is G_c, is characterized by the Shapley–Bondareva theorem:
"For each $v \in G$, $C(v) \neq \emptyset$ if and only if for each minimal balanced collection $\{S_1, \ldots, S_k\}$ with balancing coefficients $\lambda_1, \ldots, \lambda_k$, inequality $\sum_{i=1}^k \lambda_i v(S_i) \leq v(N)$ holds."

Let $\mathbb{B}$ be the set of all minimal balanced collections of the player set N, except the minimal balanced collection $\{N\}$. For each $\mathcal{B} \in \mathbb{B}$, say $\mathcal{B} = \{S_1, \ldots, S_k\}$ with balancing coefficients $\lambda_1, \ldots, \lambda_k$, $\sum_{j=1}^k \lambda_j v(S_j) \leq \max_{\mathcal{B} \in \mathbb{B}} \sum_{S_j \in \mathcal{B}} \lambda_j v(S_j)$. By the Shapley–Bondareva theorem, one can easily check $r^* = \min_{r \in M_v}$

$r = \max_{\mathcal{B}\in\mathbb{B}} \sum_{S_j\in\mathcal{B}} \lambda_j v(S_j)$. In other words, the value r^* is a 'boundary value' with the property that for each $r \geq r^*$, the game v_r has a nonempty core and for each $r < r^*$, the game v_r has an empty core.

We call the game v_{r^*} the *minimal game associated with the game* v. Note that for each game there is a unique minimal game associated with that game, but not vice versa.

Given $v \in G$, we define a new game w as follows:

$$w(S) := \begin{cases} 0 & \text{if } S \subset N, \\ |v(N) - v_{r^*}(N)| & \text{if } S = N. \end{cases}$$

Given $v \in G$, $v := v_{r^*} \oplus |w|$ is called *the decomposition associated with the game* v where

$$v_{r^*} \oplus |w| = \begin{cases} v_{r^*} + w & \text{if } v(N) \geq v_{r^*}(N), \\ v_{r^*} - w & \text{if } v(N) < v_{r^*}(N). \end{cases}$$

If $v \in G_c$, then the decomposition associated with the game v is called *the decomposition of the game* v. If $v \in G_c$, then the minimal game associated with v is called the *root game associated with the game* v (*root game of* v). Note that the idea of root game of a game is also used by Calleja et al. [4], where they introduce and characterize the aggregate monotonic core. Our definition of 'root game of v' is the same as their definition of 'root game associated to the game v'.

If the root game of v is itself, then it is called a *root game*, that is, if v is a root game, then $v = v_{r^*} + w$ is the decomposition of v with $v_{r^*} \equiv v$ and for each $S \subseteq N$, $w(S) = 0$. The set of all root games is denoted by G_r.

Remark 1 *Given $v \in G$, via Eq. (1), one observes that G_r is a small subset of G. For comparing their sizes, we can formalize the class of all games and the class of all root games as follows: Consider any labeling $S_1, \ldots, S_{2^N-2}, S_{2^N-1}$ of the nonempty subsets of the player set N such that S_{2^N-1} corresponds to N, i.e. $S_{2^N-1} = N$. Let f be a function that assigns the (2^N-1)-tuple $(v(S_1), \ldots, v(S_{2^N-2}), v(S_{2^N-1})) \in \mathbb{R}^{2^N-1}$ to each $v \in G$. The function f shows that there is a one-to-one correspondence between the games in G and the elements in $\mathbb{R}^{2^N-1}$. Similarly, let f_r be a function that assigns the (2^N-2)-tuple $(v(S_1), \ldots, v(S_{2^N-2})) \in \mathbb{R}^{2^N-2}$ to each $v_{r^*} \in G_r$. The function f_r shows that there is a one-to-one correspondence between the games in G_r and the elements in $\mathbb{R}^{2^N-2}$. Thus, the dimension of G_r is one less than the dimension of G.*

While it is a small class, G_r allows us to understand the structure of G. For that, we classify G into groups with the help of the following classification of G_r.

G_r is divided into 2 disjoint groups depending on the size of their cores.

(i) G_{sin} : The set of all root games each of which has a single vector in its core is denoted by G_{sin}, that is, $G_{sin} := \{v \in G_r : |C(v)| = 1\}$.

(ii) G_{mul} : The set of all root games each of which has more than one vector in its core is denoted by G_{mul}, that is, $G_{mul} := \{v \in G_r^N : |C(v)| > 1\}$.

Note that $G_r = G_{sin} \cup G_{mul}$ where $G_{sin} \cap G_{mul} = \emptyset$. Figure 1 shows the classification of G_r.

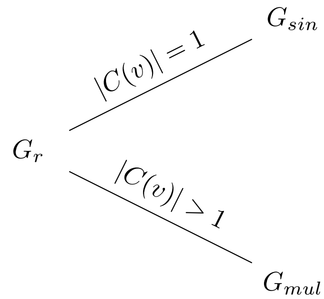

Figure 1. Summary of classification of G_r.

Now we classify the set of all games depending on their associated minimal games. First, G is divided into the two disjoint groups, G_c and $\overline{G}_c$. Next, G_c is divided into 2 disjoint groups depending on the size of cores of root games.

(i) G_s : The set of all games with nonempty cores each of which has a singleton in the core of its root game is denoted by G_s, that is, $G_s := \{v \in G_c : v = v_{r^*} \oplus w \Rightarrow v_{r^*} \in G_{sin}\}$.

Note that $G_{sin} \subset G_s$.

(ii) G_m : The set of all games with nonempty cores each of which has more than one element in the core of its root game is denoted by G_m, that is, $G_m := \{v \in G_c : v = v_{r^*} \oplus w \Rightarrow v_{r^*} \in G_{mul}\}$.

Note that $G_{mul} \subset G_m$.

Note also that if $v \in G_s$, then there is a unique element, say x, such that $C(v_{r^*}) = \{x\}$, and $C(v) = \{x + (a_1, \dots, a_n) : \forall i \in \{1, \dots, n\}, a_i \geq 0 \text{ and } \sum_{i+1}^{n} a_i = v(N) - v_{r^*}(N)\}$.

Lastly, the set of all games with empty cores, that is $\overline{G}_c$, is divided into 2 disjoint groups depending on the size of cores of their associated minimal games.

(i) $\overline{G}_s$: The set of all games with empty cores each of which has a singleton in the core of its associated minimal game is denoted by $\overline{G}_s$, that is, $\overline{G}_s := \{v \in \overline{G}_c : v = v_{r^*} \oplus w \Rightarrow v_{r^*} \in G_{sin}\}$.

(ii) $\overline{G}_m$: The set of all games with empty cores each of which has more than one element in the core of its associated minimal game is denoted by $\overline{G}_m$, that is, $\overline{G}_m := \{v \in \overline{G}_c : v = v_{r^*} \oplus w \Rightarrow v_{r^*} \in G_{mul}\}$.

Figure 2 shows the classification of all games.

Since each $v \in G$ has a unique minimal game associated with v, the set of all games (for a fixed N or as N varies), can be partitioned into equivalence classes according to this. For each pair of games $v, \hat{v}$, let the decompositions of v and $\hat{v}$ be $v = v_{r^*} \oplus |w|$ and $\hat{v} = \hat{v}_{r^*} \oplus |\hat{w}|$, respectively. For example, one can define an *equivalence relation between 2 games v and $\hat{v}$*, denoted by $v\mathcal{R}\hat{v}$, if $C(v_{r^*}) = C(\hat{v}_{r^*})$. *Two games v and $\hat{v}$ belong to the same equivalence class*, if $v\mathcal{R}\hat{v}$. A similar argument of partitioning the set of all games into equivalence is also used in [1], where they use their partitioning to study the core of combined games.

Remark 2 *Our classification of the set of all games is based on the core and the root game of a game. A similar argument can be used for other classifications of the set of all games by changing the worth of some*

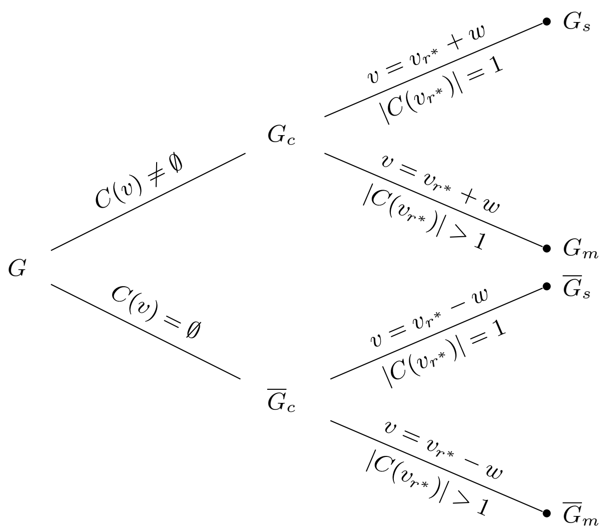

Figure 2. Summary of classification of G.

other coalition instead of the grand coalition. In general, similar to Eq. (1), given $v \in G$ and $\emptyset \neq T \subseteq N$, for each $r \in \mathbb{R}$, let $v_{(r,T)}$ be defined as follows:

$$v_{(r,T)}(S) := \begin{cases} v(S) & \text{if } T \neq S \subseteq N, \\ r & \text{if } S = T. \end{cases}$$

Let $M_{(v,T)} := \{r \in \mathbb{R} : C(v_{(r,T)}) \neq \emptyset\}$, and $r^*_T := \min_{r \in M_{(v,T)}} r$. Note that $v_{r^*_N} = v_{r^*}$. Now, using $v_{r^*_T}$ instead of v_{r^*} for any $\emptyset \neq T \subset N$, other classifications of the set of all games can done similar to our classification in Figure 2. Here, we are working with v_{r^*}, because geometrically, the change in the core is given by a hyperplane, while it will be given by a region bounded by a hyperplane otherwise.

4. Structure of games

In this section, we examine the class of all games defined in Section 3 in terms of minimal balanced collections. First, we give the geometric intuition behind our theorems.

Geometrically, it is not hard to see that nearly all the games in G_c are in G_s. Let a game that has a nonempty core be given. Roughly, if one shifts/changes the worth of the grand coalition as much as possible to obtain the root game of the given game, then the probability of ending up with a single point is higher than that of ending up with a line segment (or a hyperplane segment). In other words, the probability of getting a root game in G_{sin} is higher than that of getting a root game in G_{mul}.

As an example, consider $N = \{1,2,3\}$ and the game $v(12) = 1$, $v(123) = 5$, and $v(S) = 0$ otherwise. Note that $C(v) = \{(a+b, (1-a)+c, d) : 0 \leq a \leq 1, 0 \leq b, c, d \text{ and } b+c+d = 4\}$. Figure 3 below shows the cores of v_r for $r = \{1,2,3,4,5\}$. Note that v_1 corresponds to v_{r^*}, and $C(v_{r^*}) = \{(a, 1-a, 0 : 0 \leq a \leq 1)\}$ is a line segment; thus $v \in G_m$.

As another example, consider $N = \{1,2,3\}$ and the game $v(1) = v(2) = 0.5$, $v(123) = 5$, and $v(S) = 0$ otherwise. Note that $C(v) = \{(0.5+b, 0.5+c, d) : 0 \leq b, c, d \text{ and } b+c+d = 4\}$. Figure 4 shows the cores of v_r for $r = \{1,2,3,4,5\}$. Note that v_1 corresponds to v_{r^*}, and $C(v_{r^*}) = \{(0.5, 0.5, 0)\}$ is a singleton; thus $v \in G_s$.

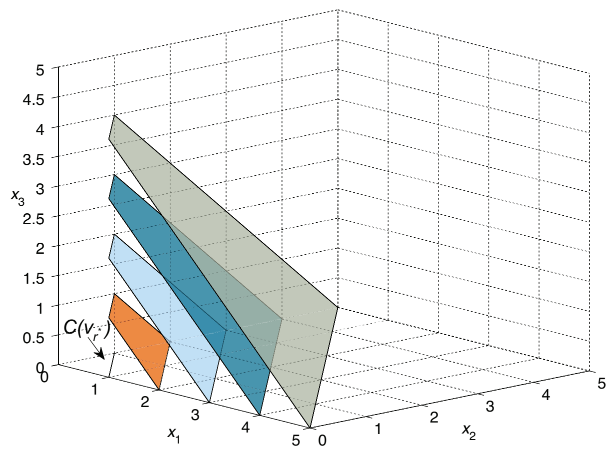

Figure 3. Left to right: $C(v_r)$ for $r = \{1, 2, 3, 4, 5\}$.

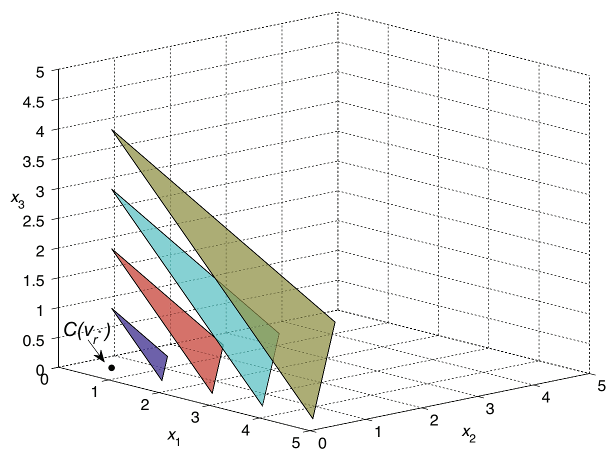

Figure 4. Left to right: $C(v_r)$ for $r = \{1, 2, 3, 4, 5\}$.

In general, given $v \in G_c$, let R be the region defined by the collection of the inequalities $(x(S) \geq v(S))_{\emptyset \neq S \subset N}$. For each $r \in \mathbb{R}$, let P_r denote the hyperplane $x(N) = r$. Note that for each $r \in \mathbb{R}$, the normal vector of P_r is $(1, \ldots, 1)$. Remember that $C(v) = R \cap P_{v(N)}$. Moreover, in order to find the root game of v, one looks for the minimum value of $r \in \mathbb{R}$ such that $R \cap P_r \neq \emptyset$, which in fact is denoted by r^*. Note that $v_{r^*} \in G_{mul}$ if there is a line segment (or a hyperplane segment) on the boundary of R with the normal vector $(1, \ldots, 1)$, and $v_{r^*} \in G_{sin}$ otherwise. Geometrically, given $v \in G_c$, the probability of having a line segment (or a hyperplane segment) on the boundary of R with the normal vector $(1, \ldots, 1)$ is nearly zero. Thus, probabilistic measure of the set G_{mul} is zero. Therefore, nearly all of the games in G_c are in G_s. Similar geometric results hold for the set $\overline{G}_c$.

Our next results in this paper explain the above geometric reasonings more precisely by minimal balanced collections. They allow us to compare the cardinalities of the sets and understand the structure of games more precisely.

The first theorem concerns the games in G_r. It is pretty straightforward to drive the next theorem by the (strong version of) Shapley–Bondarev theorem. Yet, it gives an obvious characterization of the games in G_r, and helps us to consider the latter theorems given in this section.

Theorem 1 *$v \in G_r$ if and only if the following conditions hold:*

(i) for each minimal balanced collection $\{S_1, \dots, S_k\}$ with balancing coefficients $\lambda_1, \dots, \lambda_k$, inequality

$$\sum_{j=1}^{k} \lambda_j v(S_j) \leq v(N) \tag{2}$$

holds,

(ii) there is at least one minimal balanced collection different than $\{N\}$, say $\{S_1, \dots, S_k\}$ with balancing coefficients $\lambda_1, \dots, \lambda_k$, inequality (2) is an equality; that is $\sum_{j=1}^{k} \lambda_j v(S_j) = v(N)$.

The proof of the theorem is easy and thus omitted.

Next, we give necessary and sufficient conditions for the set of games each of which has a single element in it is core, i.e. for G_{sin}. The result leads also to sufficient conditions for G_{mul}. Using these results, we compare the cardinalities of the set of games given via decomposition.

We first analyze the special case $|N| = 3$.

For $N = \{1, 2, 3\}$, the minimal balanced collections that are different than $\{N\}$ are $\{\{1\}, \{2, 3\}\}$, $\{\{2\}, \{1, 3\}\}$, $\{\{3\}, \{1, 2\}\}$, $\{\{1\}, \{2\}, \{3\}\}$ and $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$.

Theorem 2 *Let the player set be $N = \{1, 2, 3\}$. $v \in G_{mul}$ if and only if the inequality (2) of Theorem 1 is a strict inequality at the minimal balanced collections $\{\{1\}, \{2\}, \{3\}\}$ and $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, and it is an equality at only one of the minimal balanced collections below:*
(i) $\{\{1\}, \{2, 3\}\}$, (ii) $\{\{2\}, \{1, 3\}\}$, (iii) $\{\{3\}, \{1, 2\}\}$.

Proof We know $G_{mul} \subset G_r$; thus by Theorem 1, to have $v \in G_{mul}$, we only need to show that the inequality (2) of Theorem 1 is an equality at only one of the minimal balanced collections given in the theorem. For $N = \{1, 2, 3\}$, the minimal balanced collections that are different than $\{N\}$ are $\{\{1\}, \{2, 3\}\}$, $\{\{2\}, \{1, 3\}\}$, $\{\{3\}, \{1, 2\}\}$, $\{\{1\}, \{2\}, \{3\}\}$ and $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$.

We consider the cases one by one.

Assume that the inequality (2) of Theorem 1 is an equality at $\{\{1\}, \{2\}, \{3\}\}$; then $v(1) + v(2) + v(3) = v(123)$. Then, for $x \in C(v)$, obviously, for each $i \in N$, $x_i = v(i)$. Thus, $v \in G_{sin}$. Therefore, to have $v \in G_{mul}$, the inequality (2) of Theorem 1 cannot be an equality at $\{\{1\}, \{2\}, \{3\}\}$.

Now, assume that the inequality (2) of Theorem 1 is an equality at $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Then $v(12) + v(13) + v(23) = 2v(123)$. For $x \in C(v)$, we know

$$x_1 + x_2 \geq v(12),$$

$$x_1 + x_3 \geq v(13),$$

$$x_2 + x_3 \geq v(23).$$

By adding them, we get $2v(1,2,3) \geq 2[x_1 + x_2 + x_3] \geq v(12) + v(13) + v(23) = 2v(123)$. However, for each $i, j, k \in N$, we have $x_i + x_j = v(ij)$. Therefore,

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} v(12) \\ v(13) \\ v(23) \end{bmatrix}$$

which has a unique solution, because the determinant of the 3×3 matrix on the left-hand side is nonzero. Therefore, to have $v \in G_{mul}$, the inequality (2) of Theorem 1 cannot be an equality at $\{\{1,2\},\{1,3\},\{2,3\}\}$.

Now, assume that the inequality (2) of Theorem 1 is an equality at any 2 of the balanced collections, $\{\{1\},\{2,3\}\}$, $\{\{2\},\{1,3\}\}$ and $\{\{3\},\{1,2\}\}$. Without loss in generality, let the inequality (2) of Theorem 1 be an equality at $\{\{1\},\{2,3\}\}$ and $\{\{2\},\{1,3\}\}$. Then $v(1) + v(23) = v(123) = v(2) + v(13)$. Now, for $x \in C(v)$, we have $x_1 \geq v(1)$ and $x_2 + x_3 \geq v(23)$, but by adding them up and using the previous equality, we get $x_1 = v(1)$. Similarly, $x_2 = v(2)$. Then, since $x_1 + x_2 + x_3 = v(123)$, we have x_3 uniquely determined as well. Thus, $v \in G_{sin}$. Hence, the inequality (2) of Theorem 1 cannot be an equality at any 2 (or 3) of the balanced collections, $\{\{1\},\{2,3\}\}$, $\{\{2\},\{1,3\}\}$ and $\{\{3\},\{1,2\}\}$. Thus, given $v \in G_r$, to have a game $v \in G_{mul}$, due to symmetry, the only possibilities are to have an equality at only one of the minimal balanced collections: (i) $\{\{1\},\{2,3\}\}$, (ii) $\{\{2\},\{1,3\}\}$, (iii) $\{\{3\},\{1,2\}\}$. $\square$

By the negation of this theorem combined with Theorem 1, we also get a characterization of G_{sin} for $N = \{1,2,3\}$.

For the case $|N| = 3$: For $N = \{1,2,3\}$, there are 5 different balanced collections different than $\{N\}$. Thus, there are $2^5 - 1 = 31$ possible cases that the inequality (2) of Theorem 1 is an equality (since equality can hold at a unique minimal balanced collection or at multiple minimal balanced collections). Thus, by Theorem 2, given $v \in G_r$, the probability of v being in G_{mul} is $3/31 \approx 0.1$ and the probability of v being in G_{sin} is $28/31 \approx 0.9$. Thus in fact, given $v \in G_c$, the probability of v being in G_s is approximately 0.9.

Before giving our general result for any $|N| \geq 3$, we first study the special case $|N| = 4$, which provides insight into the general case.

For $N = \{1,2,3,4\}$, the minimal balanced collections different than $\{N\}$ up to symmetries are given by Table 1.

Table 1. Minimal balanced collections for $N = \{1,2,3,4\}$ (up to symmetries)

Type	Collection	Balancing coefficients	Number
1	$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}$	$1/3, 1/3, 1/3, 1/3$	1
2	$\{1,2,3\},\{1,4\},\{2,4\},\{3,4\}$	$2/3, 1/3, 1/3, 1/3$	4
3	$\{1,2,3\},\{1,4\},\{2,4\},\{3\}$	$1/2, 1/2, 1/2, 1/2$	12
4	$\{1,2\},\{1,3\},\{2,3\},\{4\}$	$1/2, 1/2, 1/2, 1$	4
5	$\{1\},\{2\},\{3\},\{4\}$	$1, 1, 1, 1$	1
6	$\{1,2,3\},\{1,2,4\},\{3,4\}$	$1/2, 1/2, 1/2$	6
7	$\{1,2\},\{3\},\{4\}$	$1, 1, 1$	6
8	$\{1,2,3\},\{4\}$	$1, 1$	4
9	$\{1,2\},\{3,4\}$	$1, 1$	3
	Total:		41

Table 1 shows all the minimal balanced collections different than $\{N\} = \{\{1,2,3,4\}\}$, their corresponding balancing coefficients, and the number of minimal balanced collections of that type considering its symmetries. Since there exist 41 different minimal balanced collections in total different than $\{N\}$, there exist $2^{41}-1$ possible cases that the inequality (2) of Theorem 1 can be an equality and thus can satisfy the conditions of Theorem 1.

For $|N|=4$, given a game $v \in G_r$, one can check that if the inequality (2) of Theorem 1 is an equality for *any* of the minimal balanced collections that is of type i, $i \in \{1,2,3,4,5\}$, then $|C(v)|=1$, and thus $v \in G_{sin}$. Similarly, if the inequality (2) of Theorem 1 is an equality for *only 1* minimal balanced collection that is either type 6 or 7 or 8 or 9, then $|C(v)|>1$; thus $v \in G^N_{mul}$. For example, if *the only* equality is at $\{1,2\},\{3\},\{4\}$, because of the dependence of the payoffs of player 1 and player 2, $|C(v)|>1$.

The above information in the last 2 paragraphs gives us the following. If $|N|=4$ and $v \in G_r$, then the probability of v being in G_{mul} is less than $(2^{19}-1)/(2^{41}-1) \approx 2.3 \times 10^{-7}$, and the probability of v being in G_{sin} is approximately $1-(2.3 \times 10^{-7}) \approx 1$. Thus in fact, when $|N|=4$, given $v \in G_c$, the probability of v being in G_s is approximately $1-(2.3 \times 10^{-7}) \approx 1$. In other words, for $|N|=4$, nearly all the games in G_c have the structure $v = v_{r^*} + w$, where $v_{r^*} \in G_{sin}$ and $w \in G_z$.

This gives us the information that given $v \in G_c$ it is more likely to have $v \in G_s$; thus, by definition of the core, there is a unique payoff vector, say $x_v \in \mathbb{R}^4$ such that $C(v_{r^*}) = \{x_v\}$, and $C(v) = \{x_v + (a_1,a_2,a_3,a_4) : 0 \le a_1,a_2,a_3,a_4,\ \sum_{i=1}^4 a_i = v(N) - v_{r^*}(N)\}$.

In light of the case $N=\{1,2,3,4\}$ discussed above, we have the following general result.

Theorem 3 *Let $v \in G_r$. If $v \in G_{sin}$, then for each pair $i,j \in N$, there is at least one minimal balanced collection different than $\{N\}$, say $\mathcal{P}_{ij} = \{S_1,\ldots,S_k\}$ with balancing coefficients $\lambda_1,\ldots,\lambda_k$ satisfying $\sum_{l=1}^k \lambda_l v(S_l) = v(N)$, at which there is at least one coalition $S \in \mathcal{P}_{ij}$ such that $i \in S$, but $j \notin S$.*

Proof Let $v \in G_r$. Without loss in generality, let $C(v) = \{x\} = \{(x_1,\ldots,x_n)\}$ such that for each pair $i,j \in N$, $x_i + x_j \neq 0$.* For each pair $i,j \in N$, let

$\mathcal{A}_{ij} := \{S \subset N : i \in S,\ j \notin S\}$,

$\mathcal{B}_{ij} := \{S \subset N : i \notin S,\ j \in S\}$,

$\mathcal{C}_{ij} := \{S \subset N : i,j \notin S\}$,

$\mathcal{D}_{ij} := \{S \subseteq N : i,j \in S\}$.

Note that any subset of N is in $A_{ij} \cup B_{ij} \cup C_{ij} \cup D_{ij}$, and $A_{ij}, B_{ij}, C_{ij}, D_{ij}$ are pairwise disjoint.

For each pair $i,j \in N$, define $w_{ij} \in G^N$ as follows:

$$w_{ij}(S) := \begin{cases} v(S) - x(S) + x_i & \text{if } S \in A_{ij}, \\ v(S) - x(S) + x_j & \text{if } S \in B_{ij}, \\ v(S) - x(S) & \text{if } S \in C_{ij}, \\ v(S) - x(S) + x_i + x_j & \text{if } S \in D_{ij}. \end{cases}$$

Note, since $x \in C(v)$, for each $S \in A_{ij}$, $w_{ij}(S) \le x_i$; for each $S \in B_{ij}$, $w_{ij}(S) \le x_j$; for each $S \in C_{ij}$, $w_{ij}(S) \le 0$; for each $S \in D_{ij}$, $w_{ij}(S) \le x_i + x_j$, and $w_{ij}(N) = x_i + x_j$.

For each pair $i,j \in N$, let $\tilde{x}_{ij} := (x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_{j-1},0,x_{j+1},\ldots x_n)$. For each $S \subseteq N$, $v(S) = w_{ij}(S) + \tilde{x}_{ij}(S)$. Since core is a solution concept satisfying covariant under strategic equivalence property,

*If $x_i + x_j = 0$, then nothing will change in the proof. In fact, we take $x_i + x_j \neq 0$ just for clarity of the proof.

$C(v) = C(w_{ij}) + \tilde{x}_{ij}$. Thus, $C(w_{ij}) = \{x - \tilde{x}_{ij}\}$. Now, one can easily see that there is at least one minimal balanced collection different than $\{N\}$, say $\tilde{\mathcal{P}}_{ij} = \{S_1, \ldots, S_k\}$ with balancing coefficients $\lambda_1, \ldots, \lambda_k$ satisfying $\sum_{l=1}^{k} \lambda_l w_{ij}(S_l) = w_{ij}(N)$, at which there is at least one coalition $S \in \tilde{\mathcal{P}}_{ij}$ such that $i \in S$, but $j \notin S$, because otherwise $|C(w_{ij})| \neq 1$. We claim that $\tilde{\mathcal{P}}_{ij}$ satisfies the condition given in the conclusion of the theorem. For that, we prove the following lemma.

Lemma 1 *Let $v \in G_{sin}$, $i, j \in N$, w_{ij} be the game defined as above and $\mathcal{P} = \{S_1, \ldots, S_k\}$ be a minimal balanced collection with balancing coefficients $\lambda_1, \ldots, \lambda_k$. Now, $\sum_{j=1}^{k} \lambda_l w_{ij}(S_l) = w(N)$ if and only if $\sum_{j=1}^{k} \lambda_l v(S_l) = v(N)$.*

Proof Let the hypothesis of the lemma hold. Note by definition of w_{ij} and definition of balancedness, we have

$$\begin{aligned}
\sum_{S_l \in \mathcal{P}_{ij}} \lambda_l w_{ij}(S_l) = & \sum_{S_l \in \mathcal{P}_{ij} \cap A_{ij}} \lambda_l w_{ij}(S_l) + \sum_{S_l \in \mathcal{P}_{ij} \cap B_{ij}} \lambda_l w_{ij}(S_l) + \sum_{S_l \in \mathcal{P}_{ij} \cap C_{ij}} \lambda_l w_{ij}(S_l) + \sum_{S_l \in \mathcal{P}_{ij} \cap D_{ij}} \lambda_l w_{ij}(S_l) \\
& = \sum_{S_l \in \mathcal{P}_{ij} \cap (A_{ij} \cup B_{ij} \cup C_{ij} \cup D_{ij})} \lambda_l (v(S_l) - x(S_l)) + x_i \sum_{S_l \in \mathcal{P}_{ij} \cap (A_{ij} \cup D_{ij})} \lambda_l + x_j \sum_{S_l \in \mathcal{P}_{ij} \cap (B_{ij} \cup D_{ij})} \lambda_l \\
& = \sum_{S_l \in \mathcal{P}_{ij}} \lambda_l v(S_l) - \sum_{S_l \in \mathcal{P}_{ij}} \lambda_l x(S_l) + x_i \sum_{S_l \in \mathcal{P}_{ij} : i \in S_l} \lambda_l + x_j \sum_{S_l \in \mathcal{P}_{ij} : j \in S_l} \lambda_l \\
& = \sum_{S_l \in \mathcal{P}_{ij}} \lambda_l v(S_l) - x(N) + x_i + x_j \\
& = \sum_{S_l \in \mathcal{P}_{ij}} \lambda_l v(S_l) - v(N) + w(N).
\end{aligned}$$

The above equality gives us the desired result of the lemma. $\square$

Finally, using the above lemma, for each pair $i, j \in N$, $\tilde{\mathcal{P}}_{ij}$ satisfies the necessary condition given in the theorem. $\square$

Theorem 3 provides a necessary condition for a singleton core in root games, but it is not a sufficient condition, as the next example shows.

First, we need some definitions. Let $v \in G_r$ be a game that satisfies the conclusion of Theorem 3. $\mathbb{BC}_v$ will denote the set of all minimal balanced collections that satisfy the condition given in the conclusion of Theorem 3. Formally, for each pair $i, j \in N$, $i \neq j$, define the set $\mathbb{B}_{ij}^{v} \subseteq \mathbb{B}$ as follows: $\mathcal{P} = \{S_1, \ldots, S_k\} \in \mathbb{B}_{ij}^{v}$ (with balancing coefficients $\lambda_1, \ldots, \lambda_k$) if $\sum_{l=1}^{k} \lambda_l v(S_l) = v(N)$, and if there is at least one coalition $S \in \mathcal{P}$ such that $i \in S$, but $j \notin S$. For each pair $i, j \in N, i \neq j$, we have $\mathbb{B}_{ij} \neq \emptyset$, because v satisfies the conclusion of Theorem 3. Now, let $\mathbb{BC}_v := \bigcup_{i,j \in N, i \neq j} \mathbb{B}_{ij}^{v}$. Note that $\mathbb{B}_{ij}^{v} = \mathbb{B}_{ji}^{v}$, and $\mathbb{BC}_v$ is well defined only for v that satisfies the conclusion of Theorem 3.

Example 1 Consider $N = \{1, 2, 3, 4\}$ and the game $v(12) = v(13) = v(24) = v(34) = 1$, $v(1234) = 2$, and $v(S) = 0$ otherwise.

Consider the minimal balanced collections $\mathcal{P}_1 = \{\{1, 2\}, \{3, 4\}\}$ and $\mathcal{P}_2 = \{\{1, 3\}, \{2, 4\}\}$ (both with balancing coefficients 1 and 1).

For $\mathcal{P}_1$, we have $v(12) + v(34) = v(1234)$, and for $\mathcal{P}_2$, we have $v(13) + v(24) = v(1234)$.

Note that $\mathbb{B}_{12} = \{\mathcal{P}_2\}$, $\mathbb{B}_{13} = \{\mathcal{P}_1\}$, $\mathbb{B}_{14} = \{\mathcal{P}_1, \mathcal{P}_2\}$, $\mathbb{B}_{23} = \{\mathcal{P}_1, \mathcal{P}_2\}$, $\mathbb{B}_{24} = \{\mathcal{P}_1\}$, $\mathbb{B}_{34} = \{\mathcal{P}_2\}$. Thus, $\mathbb{BC}_v = \{\mathcal{P}_1, \mathcal{P}_2\}$.

Yet, $C(v)=\{(a,1-a,1-a,a):0\leq a\leq 1\}$. Thus, $v\in G_{mul}\subset G_r$.

The example shows that the conclusion of Theorem 3 is not enough for a sufficient condition, yet it gives us the intuition for sufficiency. For a sufficient condition, let $v\in G_r$ satisfy the conclusion of Theorem 3. Let $E_v:=\{\emptyset\neq S\subset N: S\in\mathcal{P},\mathcal{P}\in\mathbb{BC}_v\}$. Now, for each $S\in E_v$, let $\delta_S=(\delta_1,\ldots,\delta_n)$ where $\delta_i=\begin{cases}1 & \text{if } i\in S\\ 0 & \text{if } i\notin S\end{cases}$.
Note that for each $\emptyset\neq T\subseteq N$, $x(S)=\delta_S\cdot x$. Without loss in generality, let $E_v=\{T_1,\ldots,T_m\}$.

Now, define

$$A_v=\begin{bmatrix}\delta_{T_1}\\ \vdots\\ \delta_{T_m}\end{bmatrix},\qquad x^t=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix},\qquad b_v=\begin{bmatrix}v(T_1)\\ \vdots\\ v(T_m)\end{bmatrix},$$

where A_v is a $m\times n$ matrix, x^t is the $n\times 1$ matrix formed by writing $x\in C(v)$ as a column matrix, and b_v is a $m\times 1$ matrix. Note that E_v; thus, A_v and b_v are well defined, because v satisfies the conclusion of Theorem 3. If $x\in C(v)$ and $\mathcal{P}=\{S_1,\ldots,S_k\}\in\mathbb{BC}_v$ with balancing coefficients $\lambda_1,\ldots,\lambda_k$, then we have

$$\begin{array}{rcccccl}
 & & \lambda_1 x(S_1) & \geq & \lambda_1 v(S_1) & & \\
 & & & \vdots & & & \\
 & & \lambda_1 x(S_l) & \geq & \lambda_1 v(S_l) & & \\
 & & & \vdots & & & \\
 & + & \lambda_1 x(S_k) & \geq & \lambda_1 v(S_k) & & \\
\hline
x(N) & = & \sum_{l=1}^k \lambda_l x(S_l) & \geq & \sum_{l=1}^k \lambda_l v(S_l) & = v(N)=x(N). &
\end{array}$$

Thus, for each $T\in E_v$, we have $x(T)=v(T)$. Hence, $A_v x^t=b_v$. Thus, if the solution of the system of equations given by $A_v x^t=b_v$ is unique, then x is the only element in $C(v)$, i.e. $v\in G_{sin}$. Hence, we have shown the following.

Theorem 4 *Let $v\in G_r$. If for each pair $i,j\in N$, there is at least one minimal balanced collection different than $\{N\}$, say $\mathcal{P}_{ij}=\{S_1,\ldots,S_k\}$ with balancing coefficients $\lambda_1,\ldots,\lambda_k$ satisfying $\sum_{l=1}^k\lambda_l v(S_l)=v(N)$, at which there is at least one coalition $S\in\mathcal{P}_{ij}$ such that $i\in S$, but $j\notin S$ and if $A_v x^t=b_v$ has a unique solution, then $v\in G_{sin}$.*

In light of Theorem 3, for G_{mul}, we also have the following result.

Theorem 5 *Let $v\in G_r$. If the condition*

- *there is at least one pair $i,j\in N$, for each minimal balanced collection $\mathcal{P}=\{S_1,\ldots,S_k\}$ with balancing coefficients $\lambda_1,\ldots,\lambda_k$ satisfying $\sum_{l=1}^k\lambda_l v(S_l)=v(N)$, if $i\in S\in\mathcal{P}$, then $j\in S$,*

holds, then $v\in G_{mul}$.

The proof of the theorem is omitted, because the theorem is simply the contrapositive of Theorem 3 combined with the fact that $G_r=G_{sin}\cup G_{mul}$ where $G_{sin}\cap G_{mul}\neq\emptyset$.

In light of Theorems 3 and 5, similar to the case in $|N|=4$, given $v\in G_c$, the probability of v being in G_s is approximately 1. Also note that, as $|N|$ increases, this probability tends to 1 more rapidly. Thus, nearly all games that are in G_c are in G_s, and thus they have the structure $v = v_{r^*} + w$, where $v_{r^*} \in G_{sin}$ and $w \in G_z$. Similar results hold for the set of games in $\overline{G}_c$, i.e. nearly all games that are in $\overline{G}_c$ are in $\overline{G}_s$, and thus they have the structure $v = v_{r^*} - w$, where $v_{r^*} \in G_{sin}$ and $w \in G_z$.

Lastly, we discuss the importance of our results for core selective allocation rules. An allocation rule for transferable utility games is a function that assigns a payoff vector to each game in G. An allocation rule is core selective if it assigns an element from $C(v)$ when $v \in G_c$. Many core selective allocation rules have been constructed to determine which core element should be chosen whenever the core is nonempty. Given $v = v_{r^*} + w \in G_c$, by our findings, we see that with a high probability v is in G_s. In that case, there exists a unique payoff vector, $\{x\} = C(v_{r^*})$, which is a common element for any core selective allocation rule whenever the allocation rule is additive[†]. Thus, when we restrict ourselves to the games in G_s, additive core selective allocation rules will differ only at the selection of a payoff vector from $C(w)$. Let Γ be an additive core selective allocation rule that is also egalitarian on w, i.e. $\Gamma(w) = \frac{w(N)}{|N|}$. Then Γ corresponds to the well-known allocation rule, the per-capita nucleolus defined by Grotte [6], when we restrict ourselves to G_s. In general, our results in this paper also give information about these allocation rules. A more extensive study about the relations between our results and allocation rules is part of an ongoing project.

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[†]An allocation rule Γ is additive if for each $v, w \in G$, $\Gamma(v) + \Gamma(w) = \Gamma(v + w)$.