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Equivalencies between beta-shifts and S-gap shifts

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Abstract: Let X_β be a β-shift for $\beta \in (1, 2]$ and $X(S)$ a S-gap shift for $S \subseteq \mathbb{N} \cup \{0\}$. We show that if X_β is SFT (resp. sofic), then there is a unique S-gap shift conjugate (resp. right-resolving almost conjugate) to this X_β, and if X_β is not SFT, then no S-gap shift is conjugate to X_β. For any synchronized X_β, an $X(S)$ exists such that X_β and $X(S)$ have a common synchronized 1-1 a.e. extension. For a nonsynchronized X_β, this common extension is just an almost Markov synchronized system with entropy preserving maps. We then compute the zeta function of X_β from the zeta function of that $X(S)$.

Key words: Shift of finite type, sofic, right-resolving, synchronized, finite equivalence, almost conjugacy, zeta function

1. Introduction

Two important classes of symbolic dynamics are β-shifts and S-gap shifts. Both are coded systems with applications in coding theory and number theory and a source of examples for symbolic dynamics. There have been some independent studies of these classes. See [7, 15, 18, 20] for β-shifts and [3, 7, 12] for S-gap shifts. There are some common properties between these two classes. For instance, both of them are at least half-synchronized, and every subshift factor of them is intrinsically ergodic [7]. There are disparities as well: β-shifts are all mixings, though this is not true in general for a S-gap shift [12], and S-gap shifts are synchronized, which is not true for all β-shifts. Even among sofic β-shifts and S-gap shifts, which are our primary interest here, there are some major differences. An important class of sofic S-gap shifts are almost-finite-type (AFT) [3], but no β-shift is AFT [20].

We let $\beta \in (1, 2]$ and search for an S-gap shift $X(S)$ that has some sort of equivalencies, such as conjugacy or right-resolving almost conjugacy, with our β-shift denoted by X_β. A main tool used here is almost conjugacy, introduced by Adler and Marcus [1], which is virtually a conjugacy between transitive points. This concept was defined for sofics and is now very much classic [13, 14]. It was then extended to nonsofics by Fiebig [8], and so our paper first deals with sofics and then nonsofics. Another tool is right-resolving, called *deterministic* in computer science, which is an important closing property in coded systems and in applications.

Here we summarize our results. Let $\beta \in (1, 2]$ and let X_β be the associated β-shift. We will associate to X_β a unique S-gap shift denoted by CORR(X_β) and will show that when X_β is sofic, then X_β and $X(S) =$CORR(X_β) are right-resolving almost conjugate, and when X_β is SFT, they are conjugate as well (Theorem 4.7). On the other hand, for a given S-gap shift, there does not necessarily exist a β-shift holding

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the same equivalencies with X(S). However, we give a necessary and sufficient condition on S to have this be true (Theorem 4.10).

In Subsection 3.2, we extend the results obtained for sofics to nonsofics. For instance, when X_β is synchronized, X_β and CORR(X_β) have a common synchronized 1-1 a.e. extension (Theorem 4.17). Additionally, in general, X_β and CORR(X_β) have a common extension, which is an almost Markov synchronized system whose maps are entropy-preserving (Theorem 4.18).

Theorem 5.1 gives the zeta function of X_β in terms of the zeta function of CORR(X_β) and Theorem 4.19 states that $\{\text{CORR}(X_\beta) : \beta \in (1, 2]\}$ is a Cantor dust in the set of all S-gap shifts.

2. Background and notations

The notations have been taken from [14] and the proofs of the claims in this section can be found there. Let $\mathcal{A}$ be an alphabet that is a nonempty finite set. The full $\mathcal{A}$-shift denoted by $\mathcal{A}^{\mathbb{Z}}$ is the collection of all bi-infinite sequences of symbols from $\mathcal{A}$. A block (or word) over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. The shift function σ on the full shift $\mathcal{A}^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ whose ith coordinate is $y_i = x_{i+1}$.

Let $\mathcal{B}_n(X)$ denote the set of all admissible n-blocks. The *language* of X is the collection $\mathcal{B}(X) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(X)$. A word $v \in \mathcal{B}(X)$ is *synchronizing* if whenever uv and vw are in $\mathcal{B}(X)$, we have $uvw \in \mathcal{B}(X)$.

Let $\mathcal{A}$ and $\mathcal{D}$ be alphabets and X a shift space over $\mathcal{A}$. Fix integers m and n with $-m \leq n$. Define the $(m+n+1)$*-block map* $\Phi : \mathcal{B}_{m+n+1}(X) \to \mathcal{D}$ by

$$y_i = \Phi(x_{i-m}x_{i-m+1}...x_{i+n}) = \Phi(x_{[i-m,i+n]}), \tag{2.1}$$

where y_i is a symbol in $\mathcal{D}$. The map $\phi : X \to \mathcal{D}^{\mathbb{Z}}$ defined by $y = \phi(x)$ with y_i given by (2.1) is called the *sliding block code* with *memory* m and *anticipation* n induced by Φ. An onto sliding block code $\phi : X \to Y$ is called a factor code. In this case, we say that Y is a factor of X. The map ϕ is a *conjugacy* if it is invertible.

An *edge shift*, denoted by X_G, is a shift space that consists of all bi-infinite walks in a directed graph G. A *labeled graph* $\mathcal{G}$ is a pair $(G, \mathcal{L})$ where G is a graph with edge set $\mathcal{E}$, vertex set $\mathcal{V}$, and the labeling $\mathcal{L} : \mathcal{E} \to \mathcal{A}$. Each $e \in \mathcal{E}$ starts at a vertex denoted by $i(e) \in \mathcal{V}$ and terminates at a vertex $t(e) \in \mathcal{V}$.

When the set of forbidden words is finite, the space is called a *subshift of finite type* (SFT). A *sofic shift* $X_{\mathcal{G}}$ is the set of sequences obtained by reading the labels of walks on G,

$$X_{\mathcal{G}} = \{\mathcal{L}_\infty(\xi) : \xi \in X_G\} = \mathcal{L}_\infty(X_G).$$

We say that $\mathcal{G}$ is a *presentation* or a *cover* of $X_{\mathcal{G}}$.

A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is *right-resolving* if for each vertex I of G the edges starting at I carry different labels. A *minimal right-resolving presentation* of a sofic shift X is a right-resolving presentation of X having the fewest vertices among all right-resolving presentations of X. Any two minimal right-resolving presentations of an irreducible sofic shift must be isomorphic as labeled graphs [14, Theorem 3.3.18]. Thus, we can speak of "the" minimal right-resolving presentation of an irreducible sofic shift X; we call it the *Fischer cover* of X.

Let $w \in \mathcal{B}(X)$. The *follower set* $F(w)$ of w is defined by $F(w) = \{v \in \mathcal{B}(X) : wv \in \mathcal{B}(X)\}$. A shift space X is sofic if and only if it has a finite number of follower sets [14, Theorem 3.2.10]. In this case, we have a labeled graph $\mathcal{G} = (G, \mathcal{L})$ called the *follower set graph* of X. The vertices of G are the follower sets and if $wa \in \mathcal{B}(X)$, then we draw an edge labeled a from $F(w)$ to $F(wa)$. If $wa \notin \mathcal{B}(X)$ then we do nothing.

Now we review the concept of the Fischer cover for a not necessarily sofic system [9]. Let $x \in \mathcal{B}(X)$. Then $x_+ = (x_i)_{i\in\mathbb{Z}^+}$ (resp. $x_- = (x_i)_{i<0}$) is called *right (resp. left) infinite* X*-ray.* For a left infinite X-ray, say x_-, its follower set is $\omega_+(x_-) = \{x_+ \in X^+ : x_-x_+ \text{ is a point in } X\}$. Consider the collection of all follower sets $\omega_+(x_-)$ as the set of vertices of a graph X^+. There is an edge from I_1 to I_2 labeled a if and only if there is an X-ray x_- such that x_-a is an X-ray and $I_1 = \omega_+(x_-)$, $I_2 = \omega_+(x_-a)$. This labeled graph is called the *Krieger graph* for X. If X is a synchronized system with synchronizing word α, the irreducible component of the Krieger graph containing the vertex $\omega_+(\alpha)$ is called the *right Fischer cover* of X. We are working only with coded synchronized systems, which are irreducible. In this situation, like irreducible sofics, the right Fischer cover is just called the Fischer cover.

Let $\phi = \Phi_\infty : X \to Y$ be a 1-block code. Then ϕ is *right-resolving* whenever ab and ac are 2-blocks in X with $\Phi(b) = \Phi(c)$, then $b = c$.

Let G and H be graphs. A *graph homomorphism* from G to H consists of a pair of maps $\partial\Phi : \mathcal{V}(G) \to \mathcal{V}(H)$ and $\Phi : \mathcal{E}(G) \to \mathcal{E}(H)$ such that $\partial\Phi(i(e)) = i(\Phi(e))$ and $\partial\Phi(t(e)) = t(\Phi(e))$ for all $e \in \mathcal{E}(G)$. A graph homomorphism is a *graph isomorphism* if both $\partial\Phi$ and Φ are one-to-one and onto. Two graphs G and H are graph isomorphic (written $G \cong H$) if there is a graph isomorphism between them. Let $\mathcal{E}_I(G)$ be the set of all the edges in $\mathcal{E}(G)$ starting from $I \in \mathcal{V}(G)$. A graph homomorphism $\Phi : G \to H$ maps $\mathcal{E}_I(G)$ into $\mathcal{E}_{\partial\Phi(I)}(H)$ for each vertex I of G. Thus, $\phi = \Phi_\infty$ is right-resolving if and only if for every vertex I of G the restriction Φ_I of Φ to $\mathcal{E}_I(G)$ is one-to-one. If G and H are irreducible and ϕ is a right-resolving code from X_G onto X_H, then each Φ_I must be a bijection. Thus, for each vertex I of G and every edge $f \in \mathcal{E}_{\partial\Phi(I)}(H)$, there exists a unique "lifted" edge $e \in \mathcal{E}_I(G)$ such that $\Phi(e) = f$. This lifting property inductively extends to paths: for every vertex I of G and every path w in H starting at $\partial\Phi(I)$, there is a unique path π in G starting at I such that $\Phi(\pi) = w$.

Points x and x' in X are *left-asymptotic* if there is an integer N for which $x_{(-\infty,\, N]} = x'_{(-\infty,\, N]}$. A sliding block code $\phi : X \to Y$ is *right-closing* if whenever x, x' are left-asymptotic and $\phi(x) = \phi(x')$, then $x = x'$. Similarly, *left-closing* will be defined. A sliding block code is *bi-closing* if it is simultaneously right-closing and left-closing. An irreducible sofic shift is called AFT if it has a biclosing presentation. The entropy of a shift space X is defined by $h(X) = \lim_{n\to\infty}(1/n)\log|\mathcal{B}_n(X)|$.

3. General properties of S-gap shifts and β-shifts

3.1. S-gap shifts

To define a S-gap shift $X(S)$, fix $S = \{s_i \in \mathbb{N}\cup\{0\} : 0 \le s_i < s_{i+1},\, i \in \mathbb{N}\cup\{0\}\}$. Define $X(S)$ to be the set of all binary sequences for which 1s occur infinitely often in each direction and such that the number of 0s between successive occurrences of a 1 is in S. When S is infinite, we need to allow points that begin or end with an infinite string of 0s. Note that $X(S)$ and $X(S^{'})$ are conjugate if and only if one of the S and $S^{'}$ is $\{0,\, n\}$ and the other $\{n,\, n+1,\, n+2, \ldots\}$ for some $n \in \mathbb{N}$ [3, Theorem 4.1]. We consider $X(S)$ up to conjugacy and by convention $\{0, n\}$ is chosen. Now let $d_0 = s_0$ and $\Delta(S) = \{d_n\}_n$ where $d_n = s_n - s_{n-1}$. Then an S-gap shift is SFT if and only if S is finite or cofinite, is AFT if and only if $\Delta(S)$ is eventually constant, and is sofic if and only if $\Delta(S)$ is eventually periodic [3]. Therefore, for sofic S-gap shifts we set

$$\Delta(S) = \{d_0,\, d_1, \ldots,\, d_{k-1}, \overline{g_0, g_1, \ldots, g_{l-1}}\}, \qquad g = \sum_{i=0}^{l-1} g_i \tag{3.1}$$

where $g_j = s_{k+j} - s_{k+j-1}, 0 \leq j \leq l-1$. Furthermore, k and l are the least integers such that (3.1) holds.

The Fischer cover of any irreducible sofic shift as well as S-gap shifts is the labeled subgraph of the follower set graph, which consists of the finite set of follower sets of synchronizing words as its vertices. For an S-gap shift this set is

$$\{F(1), F(10), \ldots, F(10^{n(S)})\}, \tag{3.2}$$

where $n(S) = \max S$ for $|S| < \infty$. If $|S| = \infty$, then $n(S)$ will be defined as follows.

1. For $k = 1$ and $g_{l-1} > s_0$,
   (a) if $g_{l-1} = s_0 + 1$, then $F(10^{s_{l-1}+1}) = F(1)$ and $n(S) = s_{l-1}$.
   (b) if $g_{l-1} > s_0 + 1$, then $F(10^g) = F(1)$ and $n(S) = g - 1$.
2. For $k \neq 1$, if $g_{l-1} > d_{k-1}$, then $F(10^{g+s_{k-2}+1}) = F(10^{s_{k-2}+1})$ and $n(S) = g + s_{k-2}$.
3. For $k \in \mathbb{N}$, if $g_{l-1} \leq d_{k-1}$, then $F(10^{s_{k+l-2}+1}) = F(10^{s_{k-1}-g_{l-1}+1})$ and $n(S) = s_{k+l-2}$.

For a view of the Fischer cover of a S-gap shift, we line up vertices in (3.2) horizontally starting from $F(1)$ on the left followed by $F(10)$ and then by $F(10^2)$, at last ending at $F(10^{n(S)})$ as the far right vertex. In all cases, label 0 the edge starting from $F(10^i)$ and terminating at $F(10^{i+1})$, $0 \leq i \leq n(S) - 1$; also, label 1 all edges from $F(10^s)$ to $F(1)$ for $s \in S$ and $s < n(S)$.

The only remaining edges to be taken care of are those starting at $F(10^{n(S)})$. In (1a), there are two edges from $F(10^{n(S)})$ to $F(1)$; label one 0 and the other 1. In (1b), there is only one edge from $F(10^{n(S)})$ to $F(1)$, which is labeled 0. In case (2) (resp. (3)), label 0 the edge from $F(10^{n(S)})$ to $F(10^{s_{k-2}+1})$ (resp. $F(10^{s_{k-1}-g_{l-1}+1})$) and label 1 the edge from $F(10^{n(S)})$ to $F(1)$. For a more detailed treatment see [2].

3.2. β-shifts

Rényi [16] was the first who considered the β-shifts. These shifts are symbolic spaces with rich structures and applications in theory and practice. We present here a brief introduction to β-shifts from [20]. For a more detailed treatment, see [6].

When t is a real number we denote by $\lfloor t \rfloor$ the largest integer smaller than t. Let β be a real number greater than 1. Set

$$1_\beta = a_1 a_2 a_3 \cdots \in \{0, 1, \ldots, \lfloor \beta \rfloor\}^{\mathbb{N}},$$

where $a_1 = \lfloor \beta \rfloor$ and

$$a_i = \lfloor \beta^i (1 - a_1 \beta^{-1} - a_2 \beta^{-2} - \cdots - a_{i-1} \beta^{-i+1}) \rfloor$$

for $i \geq 2$. The sequence 1_β is the expansion of 1 in the base β; that is, $1 = \sum_{i=1}^{\infty} a_i \beta^{-i}$. Let $\leq$ be the lexicographic ordering of $(\mathbb{N} \cup \{0\})^{\mathbb{N}}$. The sequence 1_β has the property that

$$\sigma^k 1_\beta \leq 1_\beta, \qquad k \in \mathbb{N}, \tag{3.3}$$

where σ denotes the shift on $(\mathbb{N} \cup \{0\})^{\mathbb{N}}$. It is a result of Parry [15] that this property characterizes the elements of $(\mathbb{N} \cup \{0\})^{\mathbb{N}}$, which are the β-expansion of 1 for some $\beta > 1$. Furthermore, it follows from (3.3) that

$$X_\beta = \{x \in \{0, 1, \ldots, \lfloor \beta \rfloor\}^{\mathbb{Z}} : x_{[i,\infty)} \leq 1_\beta, i \in \mathbb{Z}\} \tag{3.4}$$

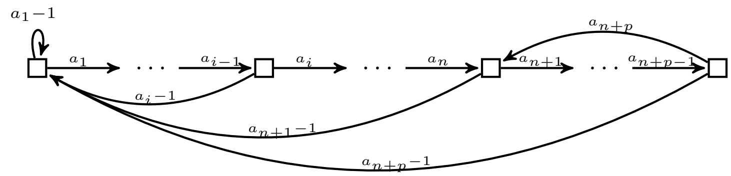

Figure 1. A typical Fischer cover of a strictly sofic β-shift for $1_\beta = a_1a_2\cdots a_n(a_{n+1}\cdots a_{n+p})^\infty$, $\beta \in (1, 2]$. The edges heading to α_1 exist if $a_i = 1$.

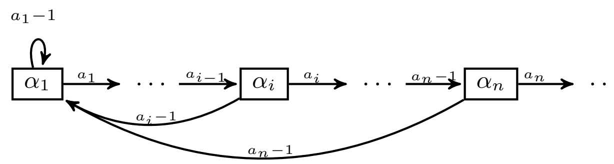

Figure 2. A typical Fischer cover of a nonsofic β-shift for $1_\beta = a_1a_2\cdots$, $\beta \in (1, 2]$. The edges ending at α_1 exist if $a_i = 1$.

is a shift space of $\{0, 1, \ldots, \lfloor\beta\rfloor\}^{\mathbb{Z}}$, called the β-shift. The β-shift is SFT if and only if the β-expansion of 1 is finite and it is sofic if and only if the β-expansion of 1 is eventually periodic [4]. Moreover, any β-shift is half-synchronized. See [9] for definition and properties of a half-synchronized system. Note that all half-synchronized system have a Fischer cover. We consider $\beta \in (1, 2]$, where in this case Fischer covers for a sofic and nonsofic β-shift are as in Figures 1 and 2, respectively.

4. Equivalencies between a beta-shift and an S-gap shift

We look for some sort of equivalencies for a given X_β and some S-gap shift. We use these equivalencies to do some computations for X_β. Sofics and nonsofics are treated differently.

4.1. Sofic case

A sliding block code $\phi : X \to Y$ is *finite-to-one* if there is $M \in \mathbb{N}$ such that for all $y \in Y$, $|\phi^{-1}(y)| \le M$. Shift spaces X and Y are *finitely equivalent* if there is an SFT, say W, together with finite-to-one factor codes $\phi_X : W \to X$ and $\phi_Y : W \to Y$. One calls W a common extension and ϕ_X, ϕ_Y the *legs*. The triple (W, ϕ_X, ϕ_Y) is a finite equivalence between X and Y. Call a finite equivalence between sofic shifts in which both legs are right-resolving (resp. right-closing) a *right-resolving finite equivalence* (resp. *right-closing finite equivalence*).

Let G and H be two irreducible graphs. Write that $H \preceq G$ if X_H is a right-resolving factor of X_G and let $\mathcal{R}_G$ be the collection of graph-isomorphism classes of graphs H for which $H \preceq G$. This ordering naturally determines an ordering that we still call $\preceq$ on $\mathcal{R}_G$. It turns out that there is a smallest element M_G in this partial ordering $(\mathcal{R}_G, \preceq)$.

Now we recall from [14] how M_G can be constructed. Let $\mathcal{V} = \mathcal{V}(G)$ be the set of vertices of G and let us define a nested sequence of equivalence relations $\sim_n$ on $\mathcal{V}$ for $n \ge 0$ and denote by $\mathcal{P}_n$ the partition of $\mathcal{V}$ into $\sim_n$ equivalence classes. To define $\sim_n$, first let $I \sim_0 J$ for all $I, J \in \mathcal{V}$. For $n \ge 1$, let $I \sim_n J$ if and only if for each class (or atom) $P \in \mathcal{P}_{n-1}$ the total number of edges from I to vertices in P equals the total number of edges from J to vertices in P. Note that the partitions $\mathcal{P}_n$ are nested: each atom in $\mathcal{P}_n$ is a union of atoms in $\mathcal{P}_{n+1}$.

We have $\mathcal{V}$ finite and $\mathcal{P}_n$ nested, so the $\mathcal{P}_n$s will be equal for all sufficiently large n, and we denote by $\mathcal{P}$ the limiting partition. Then $\mathcal{P}$ will be the set of states of M_G. To prevent confusion between M_G and G, we call a vertex in M_G "state" and of G just "vertex".

Since for all large enough n, $\mathcal{P} = \mathcal{P}_n = \mathcal{P}_{n+1}$, for each pair $P, Q \in \mathcal{P}$ there is k such that for each $I \in P$ there are exactly k edges in G from I to vertices in Q. We then assign k edges in M_G from P to Q.

Therefore, to have M_G, for each n, we refine the atoms of $\mathcal{P}_n$, and when $\mathcal{P}_n = \mathcal{P}$, then for each $P, Q \in \mathcal{P}$ and $I, J \in P$, the total number of paths from I and J to vertices in Q and also the length of these paths (with respect to G) for both I and J are equal.

Briefly we have $\mathcal{P}_0 = \mathcal{V}(G)$. Furthermore, $\sim_1$ partitions vertices by their out-degrees where for X_β and $X = X(S)$, $\sim_1$ partitions vertices into two atoms, one atom containing the vertices with out-degree one and the other with out-degree two. If $\mathcal{P} \neq \mathcal{P}_1$, for the next step, if $P \in \mathcal{P}_1$ is refined, then it is the turn for Q to be refined where $Q \in \mathcal{P}_1$ is any atom having edges terminating to vertices in P.

Theorem 4.1 *[14, Theorem 8.4.7] Suppose that X and Y are irreducible sofic shifts. Let G_X and G_Y denote the underlying graphs of their Fischer covers respectively. Then X and Y are right-resolving finitely equivalent if and only if $M_{G_X} \cong M_{G_Y}$. Moreover, the common extension can be chosen to be irreducible.*

A point in X is *doubly transitive* if every word in $\mathcal{B}(X)$ occurs infinitely often to the left and to the right of its representation. Shift spaces X and Y are *almost conjugate* if there is a shift of finite type W and 1-1 a.e. factor codes $\phi_X : W \to X$ and $\phi_Y : W \to Y$ (1-1 a.e. means that any doubly transitive point has exactly one pre-image). Call an almost conjugacy between sofic shifts in which both legs are right-resolving (resp. right-closing) a *right-resolving almost conjugacy* (resp. *right-closing almost conjugacy*).

Let r-r and r-c stand for right-resolving and right-closing, respectively. We summarize the relations among the mentioned properties in the following diagram.

$$\begin{array}{ccccc} & & & & \text{conjugacy} \\ & & & & \Downarrow \\ \text{r-r almost conjugacy} & \Rightarrow & \text{r-c almost conjugacy} & \Rightarrow & \text{almost conjugacy} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{r-r finite equivalence} & \Rightarrow & \text{r-c finite equivalence} & \Rightarrow & \text{finite equivalence} \end{array} \tag{4.1}$$

There are examples to show that in general the converse to the above implications is not necessarily true [14].

Definition 4.2 *Let $m \in \mathbb{N}$ and let $w = w_0w_1 \dots w_{p-1} = (w_0w_1 \dots w_{q-1})^m$ be a block of length p. The* least period *of w is the smallest integer q such that $m = \frac{p}{q}$. The block w is* primitive *if its least period equals its length p.*

Now we will picture the graph M_G of $X(S)$. First suppose $|S| < \infty$. Let $S = \{s_0, s_1, \dots, s_{k-1}\} \subseteq \mathbb{N}_0$, $k > 1$ and

$$\mathcal{D}(S) = d_1d_2 \cdots d_{k-2}(d_{k-1} + s_0 + 1) \tag{4.2}$$

where $d_i = s_i - s_{i-1}$, $1 \leq i \leq k-1$. Note that if $I, J \in \mathcal{V}(G)$ are in the same state of M_G, then both I and J have the same out-degree, which is one or two. The out-degree of any vertex $F(10^{s_i})$, $0 \leq i \leq k-1$ is two, except the last one. Hence, d_i, $1 \le i \le k-2$ measures the distance between any two vertices with out-degree two.

To pick the next vertex after $F(10^{s_{k-2}})$ with out-degree two, we continue to the right to $F(10^{s_{k-1}})$ and then along the graph to $F(1)$, and then again to the right to $F(10^{s_0})$, which is after $d_{k-1}+s_0+1$ steps.

Theorem 4.3 *Let $|S| < \infty$. Then $\mathcal{D}(S)$ is primitive if and only if $M_G \cong G$.*

Proof Suppose that $\mathcal{D}(S)$ is not primitive. Let $\mathcal{V} = \mathcal{V}(M_G)$ be the set of states of M_G. By the Fischer cover of $X(S)$, each state in M_G then consists of $m = \frac{|S|-1}{q}$ vertices of graph G where q is the least period $\mathcal{D}(S)$ and $|\mathcal{V}| = \sum_{i=1}^{q} d_i = s_q - s_0$. In fact, if $\mathcal{V} = \{P_i : 0 \le i \le s_q - s_0 - 1\}$, then

$$P_i = \{F(10^{s_0+i}),\, F(10^{s_0+i+|\mathcal{V}|}), \ldots, F(10^{s_0+i+(m-1)|\mathcal{V}| \mod u})\}$$

where $u = s_{k-1}+1$. Since $|\mathcal{V}| = s_q - s_0 < s_{k-1}+1 = |\mathcal{V}(G)|$, $M_G \ncong G$.

Now suppose that $M_G \ncong G$. There are thus at least two different vertices of G, say $I = F(10^p)$ and $J = F(10^q)$, such that I and J are in the same state of M_G. Assume $p < q$. There exists an edge from I (resp. J) to $F(10^{(p+1)})$ (resp. $F(10^{(q+1) \mod u})$). Therefore, by the fact that I and J are equivalent, we have that the vertices $F(10^{(p+1)})$ and $F(10^{(q+1) \mod u})$ are equivalent. By the same reasoning, for each $i \ge 2$, $F(10^{(p+i) \mod u})$ and $F(10^{(q+i) \mod u})$ are equivalent. Therefore, $\mathcal{D}(S)$ is not primitive. □

Theorem 4.4 *Let $X(S)$ be a sofic S-gap shift with $|S| = \infty$ and the Fischer cover $\mathcal{G} = (G, \mathcal{L})$. Then $M_G \cong G$.*

Proof We consider our three cases appearing for $|S| = \infty$ in Subsection 3.1. We claim that the last vertex $F(10^{n(S)})$ is not equivalent with any other vertex. Otherwise, we will show that at least one of k or l will not be the least integer in (3.1). Thus, the state of M_G containing this last vertex contains only this vertex, which in turn implies that other states of M_G also have one vertex. Therefore, $M_G \cong G$.

We prove our claim for the most involved case, i.e. case (3). First suppose there is a vertex

$$v_0 = F(10^{t_0}) \sim F(10^{n(S)}), \qquad s_{k-1} - g_{l-1} + 1 \le t_0 < n(S). \tag{4.3}$$

In fact, if $t_0 < s_{k-1} - g_{l-1} + 1$, then k is not the least integer in (3.1). Without loss of generality assume that this t_0 is the largest integer with this property. Recall that there is an edge from $F(10^{n(S)})$ to $F(10^{s_{k-1}-g_{l-1}+1})$; it is thus convenient to set $t_1 := n(S)$, $t_1 + 1 := s_{k-1} - g_{l-1} + 1$ and $v_1 := F(10^{n(S)})$. By (4.3), $v_2 := F(10^{t_1+1}) \sim F(10^{t_0+1})$, and moving horizontally to the right, $v_{i+1} := F(10^{t_1+i}) \sim F(10^{t_0+i})$, $i \ge 2$. Moreover, none of $F(10^{t_0+i})$ will be equivalent to v_0, for this would violate the way we have picked t_0. If $v_2 \sim v_0$ we are done, for then l will not be the least integer. Observe that there are only finitely many vertices; therefore, there must be $v_i \nsim v_0$, $2 \le i < p$, and $v_p \sim v_0$. Applying the same reasoning, we deduce that again l is not the least integer. □

Theorems 4.3 and 4.4 imply the following.

Corollary 4.5 *Let $X(S)$ be a sofic S-gap shift with the Fischer cover $\mathcal{G} = (G, \mathcal{L})$. Then any state of M_G has the same number of vertices of G.*

When $|S| < \infty$, there may be cases with $M_G \ncong G$. The difference with $|S| = \infty$ is that for $|S| < \infty$, the last vertex $F(10^{n(S)})$ always has out-degree one with label 1, while for $|S| = \infty$, the label of the edge starting from the vertex with out-degree one is 0.

Now let X be a sofic shift with the Fischer cover $\mathcal{G} = (G, \mathcal{L})$. By definition, $\mathcal{L}_\infty$ is then right-resolving, and it is also almost invertible [14, Proposition 9.1.6]. Thus:

Lemma 4.6 *Suppose X and Y are sofic with Fischer covers $\mathcal{G}_X = (G_X, \mathcal{L}_X)$ and $\mathcal{G}_Y = (G_Y, \mathcal{L}_Y)$ respectively and such that $G_X \cong G_Y$. Then X and Y will be right-resolving almost conjugate with legs $\mathcal{L}_{X\infty} : W \to X$ and $\mathcal{L}_{Y\infty} : W \to Y$ where W is SFT such that $W \cong G_X \cong G_Y$.*

Theorem 4.7 *Let X_β be a sofic β-shift for $\beta \in (1, 2]$. Then there is $S \subseteq \mathbb{N}_0$ such that X_β and $X(S)$ are right-resolving almost conjugate. The set S will be explicitly determined in terms of coefficients of 1_β. Moreover, if X_β is SFT, then $X(S)$ can be chosen to be conjugate to X_β.*

Proof For a given sofic β-shift, $\beta \in (1, 2]$, we claim that there is $S \subseteq \mathbb{N}_0$ such that the S-gap shift $X(S)$ and X_β have the same underlying graph for their Fischer covers. By Lemma 4.6, X_β and $X(S)$ will then be right-resolving almost conjugate.

Let $1_\beta = a_1 a_2 \cdots a_n (a_{n+1} \cdots a_{n+p})^\infty$ and $\{i_1, i_2, \ldots, i_t\} \subseteq \{1, 2, \ldots, n\}$ where $a_{i_v} = 1$ for $1 \leq v \leq t$. Note that i_1 is always 1. Similarly, let $\{j_1, j_2, \ldots, j_u\} \subseteq \{n+1, \ldots, n+p\}$ where $a_{j_w} = 1$ for $1 \leq w \leq u$. We consider two cases:

1. X_β is SFT. In this case, $(a_{n+1} \cdots a_{n+p})^\infty = 0^\infty$ and $a_n = 1$, so $i_t = n$ and $X(S)$ with

$$S = \{0, i_2 - 1, \ldots, i_{t-1} - 1, i_t - 1\} \tag{4.4}$$

is the required S-gap shift, as has been claimed. Since both X_β and $X(S)$ are SFT with the same underlying graph G for their Fischer covers, they are both conjugate to X_G [14, Theorem 3.4.17], and so conjugate to each other.

2. X_β is strictly sofic. Then $(a_{n+1} \cdots a_{n+p})^\infty \neq 0^\infty$. Relabel any edge on G_β ending at the first vertex for 1 and other edges for 0. The shift space corresponding to this labeling is an S-gap shift where

$$S = \{0, i_2 - 1, \ldots, i_t - 1, j_1 - 1, \ldots, j_u - 1, j_1 + p - 1, \ldots\}. \tag{4.5}$$

(observe that then

$$\Delta(S) = \{0, i_2 - 1, \ldots, i_t - i_{t-1}, j_1 - i_t, \overline{j_2 - j_1, \ldots, j_u - j_{u-1}, j_1 - j_u + p}\}, \tag{4.6}$$

which shows that $X(S)$ is sofic [3, Theorem 3.4]).

Rewrite $\Delta(S)$ in (4.6) as

$$\Delta(S) = \{0, d_1, \ldots, d_t, \overline{g_0, \ldots, g_{u-1}}\}.$$

We claim that $\mathcal{G}_S = (G_S, \mathcal{L}_S)$ is follower-separated. Otherwise, there are two cases.

(a) There is $1 \leq i \leq t$ such that $d_{t+1-j} = g_{u-j}$, $1 \leq j \leq i$. Then $\mathcal{G}_\beta = (G_\beta, \mathcal{L}_\beta)$ is not follower-separated and so it is not the Fischer cover of X_β, which is absurd.

(b) $g_0g_2\cdots g_{u-1}$ is not primitive. This implies that $a_{n+1}\cdots a_{n+p}$ is not primitive, which is again absurd.

This establishes the claim and S is completely determined.

□

Now the following is immediate.

Corollary 4.8 *Let X_β be a sofic β-shift whose underlying graph of its Fischer cover is G. Then $M_G \cong G$.*

Proof Suppose $M_G \ncong G$. For this X_β, find the S-gap shift satisfying the conclusion of Theorem 4.7. Then by Theorem 4.4, this $X(S)$ (as well as our X_β) must be SFT and $\mathcal{D}(S)$ is not primitive. However, this will not allow us to have (3.3), which is a necessary condition. □

Lemma 4.9 *Let $|S|=\infty$ and let $X(S)$ be a sofic shift satisfying (1a) in Subsection 3.1. Then there does not exist any β-shift being right-resolving finite equivalent with $X(S)$.*

Proof Suppose there is some $\beta\in(1,2]$ such that $X(S)$ and X_β are right-resolving finite equivalent and $\mathcal{G}_S=(G_S,\mathcal{L}_S)$ and $\mathcal{G}_\beta=(G_\beta,\mathcal{L}_\beta)$ are the Fischer covers of $X(S)$ and X_β, respectively. By Theorem 4.4 and Corollary 4.8, $G_S\cong G_\beta$. Then G_β is the underlying graph of $\mathcal{G}_S$ and $1_\beta=(a_1a_2\cdots a_n)^\infty$.

Now by hypothesis, $g_{l-1}=1$, so $1\notin S$ and this implies that $a_2=0$ while $a_1=a_n=1$. This means $(a_1a_2\cdots a_n)^\infty$ does not satisfy (3.3), and we are done. □

Let $X(S)$ be an S-gap shift where $s_0=0$ and $d_i=s_i-s_{i-1}$, $i\in\mathbb{N}$ and also $\mathcal{D}(S)$ as (4.2). Define

$$d_1d_2d_3\cdots=\begin{cases}(d_1d_2\cdots(d_{k-1}+1))^{\mathbb{N}}=(\mathcal{D}(S))^{\mathbb{N}}, & |S|=k;\\ d_1d_2\cdots, & |S|=\infty.\end{cases}\tag{4.7}$$

Theorem 4.10 *Suppose $X(S)$ is a sofic shift where $s_0=0$. Then $X(S)$ is right-resolving almost conjugate to a β-shift if and only if*

$$d_nd_{n+1}\cdots\geq d_1d_2\cdots\tag{4.8}$$

for all $n\geq 1$.

Proof Let $\beta\in(1,2]$ with $1_\beta=a_1a_2\cdots$ be so that $X(S)$ and X_β are right-resolving almost conjugate. This means they are right-resolving finite equivalent. First suppose $M_{G_S}\cong G_S$. By Corollary 4.8, $G_S\cong G_\beta$ and so (4.8) follows from the fact that $a_1a_2\cdots$ satisfies (3.3).

If $M_{G_S}\ncong G_S$, then by Theorems 4.3 and 4.4, $|S|<\infty$. Thus, X_β is right-resolving finite equivalent to $X(S')$ with $S'=\{0,s_1,\dots,(s_q-1)\}$ and $\mathcal{D}(S)=\mathcal{D}(S')^m$ where $m=\frac{|S|-1}{q}$ as in the proof of Theorem 4.3. Moreover, $M_{G_{S'}}\cong G_{S'}$, which gives again $d'_nd'_{n+1}\cdots\geq d'_1d'_2\cdots$ for all $n\geq1$. Now this fact reflects to $\mathcal{D}(S)$ and (4.8) holds.

To prove the sufficiency, suppose that $\mathcal{G}_S=(G_S,\mathcal{L}_S)$ is the Fischer cover of $X(S)$ and $\mathcal{V}=\mathcal{V}(G_S)$ the set of vertices of G_S. Relabel G_S by labeling 0 any edge terminating at vertex $F(1)$ and any edge whose initial vertex has out-degree 1, and assign 1 all other edges.

Recall that we have lined up the vertices horizontally from $F(1)$ on the left to $F(10^{n(S)})$ on the right. First let $|S|<\infty$ and $a_1a_2\cdots a_{n(S)}$ be the assigned label of the horizontal path from $F(1)$ to the last vertex with $a_i = 0$ or 1 as determined above. Then (4.8) implies that $a_1a_2\cdots a_{n(S)}1$ is the β-expansion of 1 for some $\beta \in (1, 2]$ and $\mathcal{G}_\beta$ is the Fischer cover of X_β.

When $|S| = \infty$, assign the label $a_1a_2\cdots a_{n(S)}$ to the horizontal path from $F(1)$ to the last vertex and label $a_{n(S)+1}$ to the edge starting from $F(10^{n(S)})$ and terminating at $F(10^{n(S)+1})$. Again, (4.8) implies that $a_1a_2\cdots a_n(a_{n+1}\cdots a_{n(S)+1})^\infty$ is the β-expansion of 1 for some $\beta \in (1, 2]$ where the index n depends on S. Then $\mathcal{G}_\beta$ is the Fischer cover of X_β (one needs similar arguments as in the proof of Theorem 4.7 to see this fact). Thus, Lemma 4.6 implies that $X(S)$ and X_β are right-resolving almost conjugate. □

Remark 4.11 *X_β can be explicitly determined in terms of S. If $S = \{0, s_1, \ldots, s_{k-1}\}$, then it is sufficient to set $1_\beta = a_1a_2\cdots a_{s_{k-1}+1}$ such that $a_1 = a_{s_i+1} = 1$, $1 \le i \le k-1$. When $|S| = \infty$, different cases of Subsection 3.1 must be considered. Case (1a) has been ruled out by Lemma 4.9, so other cases will be considered.*

(1b) If $k = 1$ and $g_{l-1} > 1$, then $F(10^g) = F(1)$, so $1_\beta = a_1a_2\cdots a_g$ such that $a_{s_i+1} = 1$, $0 \le i \le l-1$.

(2) If $k \ne 1$ and $g_{l-1} > d_{k-1}$, then $F(10^{g+s_{k-2}+1}) = F(10^{s_{k-2}+1})$, so $1_\beta = a_1a_2\cdots a_{s_{k-2}+1}(a_{s_{k-2}+2}\cdots a_{g+s_{k-2}+1})^\infty$ for which $a_{s_i+1} = 1$, $0 \le i \le k+l-2$.

(3) If $g_{l-1} \le d_{k-1}$, then $F(10^{s_{k+l-2}+1}) = F(10^{s_{k-1}-g_{l-1}+1})$, so

$$1_\beta = a_1a_2\cdots a_{s_{k-1}-g_{l-1}+1}(a_{s_{k-1}-g_{l-1}+2}\cdots a_{s_{k+l-2}+1})^\infty$$

for which $a_{s_i+1} = 1$, $0 \le i \le k+l-2$ and $a_{s_{k+l-2}+1} = 1$.

Now we show that the conclusion of Theorem 4.7 about conjugacy is not true in non-SFT cases. Recall that when X is a shift space with nonwandering part $R(X)$, we can consider the shift space

$$\partial X = \{x \in R(X) : x \text{ contains no words that are synchronizing for } R(X)\},$$

which is called the *derived shift space* of X. The derived shift space is a conjugacy invariant.

Theorem 4.12 *A non-SFT β-shift is not conjugate to a S-gap shift for any $S \subseteq \mathbb{N}_0$.*

Proof All the S-gap shifts are synchronized; therefore, a possible conjugacy happens between synchronized β and S-gap shifts and so we assume that our non-SFT β-shift is synchronized.

Suppose that there is $S \subseteq \mathbb{N}_0$ such that $\varphi : X(S) \to X_\beta$ is a conjugacy map. By [19, Proposition 4.5], we then must have $\varphi(\partial X(S)) = \partial X_\beta$. Since 1 is a synchronizing word for any S-gap shift, and $X(S)$ is not SFT, $\partial X(S) = \{0^\infty\}$ (for a SFT S-gap shift, $\partial X(S) = \emptyset$). To prove the theorem, we show that

$$\varphi(\{0^\infty\}) \ne \partial X_\beta. \tag{4.9}$$

Recall that the ω-limit set of the sequence 1_β under the shift map is the derived shift space ∂X_β of X_β [20, Theorem 2.8]. First assume that X_β has the specification property. There then exists some $n \ge 0$ such that 0^n is not a factor of 1_β [5], so 0^n is a synchronizing word for X_β [5, Proposition 2.5.2] and $0^\infty \notin \partial X_\beta$. Therefore, $\partial X_\beta \cap P_1(X_\beta) = \emptyset$ ($P_1(X_\beta)$ denotes the set of fixed points for X_β) while $\varphi(0^\infty) \in P_1(X_\beta)$ and $\varphi(0^\infty) \in \varphi(\partial X(S)) = \partial X_\beta$, and (4.9) holds.

If X_β does not have specification, then $\{0^\infty, 10^\infty\} \subseteq \omega(1_\beta) = \partial X_\beta$ and again (4.9) holds. □

Corollary 4.13 *Let X_β be SFT and $X(S_0)$ the unique S-gap shift conjugate to X_β (Theorem 4.7). Then X_β is:*

1. *right-resolving almost conjugate to $X(S_0)$,*
2. *right-resolving finite equivalent to infinitely many S-gap shifts $(X(S_n))_{n\in\mathbb{N}}$ with $\mathcal{D}(S_n) = (\mathcal{D}(S_0))^{n+1}$, $n \in \mathbb{N}$,*
3. *right-resolving almost conjugate to a unique strictly sofic S-gap shift.*

If X_β is strictly sofic, then it is right-resolving almost conjugate to a unique S-gap shift.

Proof Let X_β be SFT and let $1_\beta = a_1a_2\cdots a_{n-1}a_n$ and

$$\{i_1, i_2, \ldots, i_t\} \subseteq \{1, 2, \ldots, n\}$$

where $a_{i_j} = 1$, $1 \le j \le t$. We will relabel the Fischer cover of X_β for possible presentation of a S-gap shift.

One of such SFT S-gap shifts is $X(S_0)$, characterized in the proof of Theorem 4.7. By that theorem, X_β and $X(S_0)$ are right-resolving almost conjugate and conjugate, which gives (1). For (2), relabel $\Delta(S_0) = \{0, i_2 - 1, i_3 - i_2, \ldots, i_t - i_{t-1}\}$ as $\Delta(S_0) = \{0, d_1, \ldots, d_{t-1}\}$ and observe that $\mathcal{D}(S_0) = d_1 \cdots d_{t-2}(d_{t-1} + 1)$. Set

$$S_1 = (S_0 \setminus \{i_t - 1\}) \cup (i_t + S_0).$$

Then $\mathcal{D}(S_1) = (\mathcal{D}(S_0))^2$ is not primitive and we have $M_{G_{S_1}} \cong M_{G_{S_0}}$.

Now for $j \in \mathbb{N}$, let $s_{i_j} = \max\{s : s \in S_{j-1}\}$ and use an induction argument to see that for

$$S_j = (S_{j-1} \setminus \{s_{i_j}\}) \cup ((s_{i_j} + 1) + S_0), \tag{4.10}$$

$\mathcal{D}(S_j) = (\mathcal{D}(S_0))^{j+1}$ and $M_{G_{S_j}} \cong M_{G_{S_0}}$.

To prove (3), note that there is a strictly sofic S-gap shift with $k = 1$ and $g_{l-1} > 1$ as in Subsection 3.1 where $S = \{0, i_2 - 1, \ldots, i_{t-1} - 1, i_t, i_t + i_2 - 1, \ldots\}$. The element i_t appears in S because the edge starting from the last vertex and terminating at the first vertex is labeled 0. In fact,

$$\Delta(S) = \{0, \overline{i_2 - 1, i_3 - i_2, \ldots, i_{t-1} - i_{t-2}, i_t - i_{t-1} + 1}\}.$$

Hence, X_β and $X(S)$ have the same underlying graph for their Fischer covers and, by Lemma 4.6, they are right-resolving almost conjugate.

If there is another S-gap shift such that X_β and $X(S)$ are right-resolving finite equivalent, then $M_{G_\beta} \cong M_{G_S}$ and so $M_{G_{S_0}} \cong M_{G_S}$. Now Theorems 4.3 and 4.4 imply that $|S| < \infty$ and $\mathcal{D}(S)$ is not primitive, which in turn implies that $\mathcal{D}(S) = (\mathcal{D}(S_0))^m$ for some $m \in \mathbb{N}$. Therefore, $S = S_{m-1}$ as defined in (4.10).

Now suppose X_β is strictly sofic. A typical Fischer cover of X_β is shown in Figure 1. The existence of a loop in the first vertex from the left implies that it is the vertex $F(1)$ in the Fischer cover of the S-gap shift. By Fischer cover of S-gap shifts [2], there is only one $X(S)$ with Fischer cover as appears in Figure 1. □

4.2. Nonsofic case

Thus far for sofics, we have used Diagram (4.1) to get some equivalencies between a sofic X_β and some S-gap shifts. However, the most considered equivalencies between two nonsofic subshifts are when they have a common extension with some nice properties and, in particular, when the legs are 1-1 a.e. This sort of equivalencies was considered by Fiebig in [8]. For instance, for two synchronized systems X and Y, she proves that they have a common synchronized 1-1 a.e. extension if and only if $D(X)$ and $D(Y)$ are hyperbolic conjugate if and only if $D(X_{G_X})$ and $D(Y_{G_Y})$ are hyperbolic conjugate where $D(X)$ denotes the set of doubly transitive points in X. This hyperbolic conjugacy is automatically at hand when G_X and G_Y are isomorphic. This assertion motivates the following construction and definition.

Let G_β be the underlying graph of the Fischer cover of X_β, $\beta \in (1, 2]$ and α_1 the starting vertex of G_β (see Figure 2). Relabel G_β by labeling 1 any edge terminating at vertex α_1 and 0 all other edges to get an S-gap shift with the same underlying graph as X_β. Note that this relabeled graph is follower-separated for our $X(S)$ and is in fact the Fischer cover for $X(S)$.

Definition 4.14 *We say that $X(S)$ is the corresponding S-gap shift to a β-shift and is denoted by $CORR(X_\beta)$, $\beta \in (1, 2]$ if $X(S)$ has the same underlying graph for its Fischer cover as X_β.*

Similarly, for $X(S)$ satisfying (4.8), a unique X_β exists such that X_β has the same underlying graph for its Fischer cover as $X(S)$ and is denoted by $X_\beta = CORR\ (X(S))$. This X_β is called the corresponding β-shift to $X(S)$.

Remark 4.15 *X_β and $CORR(X_\beta)$ have all equivalencies given in Diagram (4.1) when they are both SFT and all except conjugacy when they are strictly sofic.*

Theorem 4.16 $h(X_\beta) = h(CORR(X_\beta)), \beta \in (1, 2]$.

Proof Entropy is an invariant for all the properties given in Diagram (4.1), so when X_β is sofic, the proof is obvious (Theorem 4.7).

Now let X_β be a nonsofic shift and let $1_\beta = a_1 a_2 \cdots$. We have $a_i = 1$ if and only if $i - 1 \in S$, but for $1_\beta = \sum_{i=1}^{\infty} a_i \beta^{-i}$, $h(X_\beta) = \log \beta$ and $h(X(S)) = \log \lambda$ where λ is a nonnegative solution of $\sum_{n \in S} x^{-(n+1)} = 1$ [17], so $h(X_\beta) = h(CORR(X_\beta))$. □

Since G_β (resp. G_S) and $G_{CORR(X_\beta)}$ (resp. $G_{CORR(X(S))}$) are isomorphic, the presaid result in [8] implies that:

Theorem 4.17 *1. A synchronized X_β and $CORR(X_\beta)$ have a common synchronized 1-1 a.e. extension.*

2. Suppose $CORR(X(S))$ is synchronized. Then $X(S)$ and $CORR(X(S))$ have a common synchronized 1-1 a.e. extension if and only if (4.8) holds.

Now we look for some equivalencies for the nonsynchronized case. Let X and Y be two coded systems. Then there is a coded system Z factoring onto X and Y with entropy-preserving maps if and only if $h(X) = h(Y)$. In particular, Z can be chosen to be an almost Markov synchronized system [8, Theorem 2.1], so by Theorem 4.16 this is true for any $X = X_\beta$ and $Y = CORR(X_\beta), \beta \in (1, 2]$. We thus have:

Theorem 4.18 *1. A X_β and $CORR(X_\beta)$ have a common almost Markov synchronized extension with entropy-preserving legs.*

2. A $X(S)$ and $CORR(X(S))$ have a common almost Markov synchronized extension with entropy-preserving legs if and only if (4.8) holds.

Now we investigate the frequency of corresponding S-gap shifts in the space of all S-gap shifts by using the topology of S-gap shifts given in [3]. This topology is obtained by assigning a real number $x_S = [d_0; d_1, d_2, ...]$, where $[d_0; d_1, d_2, ...]$ is the continued fraction expansion of x_S, to any $X(S)$ with $d_0 = s_0$ and $d_n = s_n - s_{n-1}$. By that, a one-to-one correspondence between the S-gap shifts up to conjugacy and $\mathcal{R} = \mathbb{R}^{\geq 0} \setminus \{\frac{1}{n} : n \in \mathbb{N}\}$, up to homeomorphism, will be established and the subspace topology of $\mathcal{R}$ together with its measure structure will be induced on the space of all S-gap shifts.

Theorem 4.19 *Let $\mathcal{S}$ be the set of all S-gap shifts corresponding to some X_β. Then $\mathcal{S}$ is a Cantor dust (a nowhere dense perfect set) on the space of all S-gap shifts. Entropy is a complete invariant for the conjugacy classes of $\mathcal{S}$.*

Proof First suppose $X(S)$ does not satisfy (4.8) and $x_S = [d_0; d_1, \ldots]$ corresponds to $X(S)$ [3]. Let N be the least integer such that $d_N d_{N+1} \cdots < d_1 d_2 \cdots$ and set $\gamma_i := [d_0; d_1, \ldots, d_i]$, $i \in \mathbb{N}_0$. If N is even, set $U = (\gamma_N, \gamma_{N+1})$, and otherwise, $U = (\gamma_{N+1}, \gamma_N)$. Then no points of U satisfy (4.8) and so none are corresponding S-gap shifts. This shows that $\mathcal{S}$ is closed.

Now let $X(S) \in \mathcal{S}$ and V be a neighborhood of x_S. Note that two real numbers are close if sufficiently large numbers of their first partial quotients in their continued fraction expansion are equal. We can select two points $x_{S'}, x_{S''} \in V$ such that $X(S')$ satisfies (4.8) and $X(S'')$ does not satisfy (4.8). This implies that all points of $\mathcal{S}$ are limit points of themselves and $\mathcal{S}$ is nowhere dense.

The second part follows from the fact that the entropy is a complete invariant for the conjugacy classes of β-shifts. □

5. Applications

By [11, Theorem 4.22], for every $\beta > 1$ there exists $1 < \beta' < 2$ such that X_β and $X_{\beta'}$ are flow equivalent. However, any two flow equivalent shift spaces have the same Bowen–Franks groups. Therefore, by Theorem 4.7 and [2, Theorems 4.1 and 4.2], which gives a complete account of the Bowen–Franks groups of sofic S-gap shifts, we also have a complete characterization of such groups for sofic β-gap shifts for $\beta > 1$.

An adjacency matrix for a sofic shift can be read from the underlying graph of its Fischer cover. Thus, the adjacency matrix of a X_β and its right-resolving almost-conjugate $X(S)$ are the same by Theorem 4.7. We already have a formula for the characteristic polynomial of a sofic S-gap shift from [2, Theorem 2.2] and hence for a sofic X_β as well.

For a dynamical system (X, T), let p_n be the number of periodic points in X having period n. When $p_n < \infty$, the zeta function $\zeta_T(t)$ is defined as

$$\zeta_T(t) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n}{n} t^n\right).$$

The zeta functions of β-shifts have been determined in [10]. Here we will give the zeta function of ζ_{σ_β} in terms of ζ_{σ_S} where $X(S) = CORR(X_\beta)$.

Let X_β be sofic and $1_\beta = a_1 a_2 \cdots a_n (a_{n+1} \cdots a_{n+p})^\infty$ such that

$$\{i_1 = 1, i_2, \dots, i_t\} \subseteq \{1, 2, \dots, n\}, \quad \text{and } \{j_1, j_2, \dots, j_u\} \subseteq \{n+1, \dots, n+p\}$$

where $a_{i_v} = a_{j_w} = 1$ for $1 \leq v \leq t$, $1 \leq w \leq u$. Now we have the following.

Theorem 5.1 *Let $X(S) = CORR(X_\beta)$ for some $\beta \in (1, 2]$. If X_β is SFT, then*

$$\zeta_{\sigma_\beta}(r) = \zeta_{\sigma_S}(r). \tag{5.1}$$

If X_β is not SFT, then

$$\zeta_{\sigma_\beta}(r) = (1-r)\zeta_{\sigma_S}(r). \tag{5.2}$$

Furthermore, in the case of SFT,

$$\zeta_{\sigma_\beta}(r) = \frac{1}{1 - r^{i_1} - r^{i_2} - \cdots - r^{i_t}}, \tag{5.3}$$

and for strictly sofic cases,

$$\zeta_{\sigma_\beta}(r) = \frac{1}{(1 - r^{i_1} - r^{i_2} - \cdots - r^{i_t})(1-r^p) - (r^{j_1} + \cdots + r^{j_u})}. \tag{5.4}$$

Proof First let X_β be an SFT shift. Also let $S = \{0, i_2 - 1, \dots, i_{t-1} - 1, i_t - 1\}$; then by Theorem (4.7), $X(S) = CORR(X_\beta)$. Since X_β and $X(S)$ are conjugate, they have the same zeta function, that is:

$$\zeta_{\sigma_\beta}(r) = \frac{1}{f_S(r^{-1})} = \frac{1}{1 - r^{i_1} - r^{i_2} - \cdots - r^{i_t}}$$

where $f_S(x) = 1 - \sum_{s_n \in S} \frac{1}{x^{s_n+1}}$ [2, Theorem 2.3].

Now suppose X_β is a strictly sofic shift and let $1_\beta = a_1 a_2 \cdots a_n (a_{n+1} \cdots a_{n+p})^\infty$. An arbitrary periodic point $x \in X_\beta$ has one presentation in $\mathcal{G}_\beta$ unless

$$x = (a_{n+1} \cdots a_{n+p})^\infty,$$

where then it has exactly two presentations. This fact can be deduced from the proof of [11, Proposition 4.7]. Thus, if $m = pk$ $(k \in \mathbb{N})$, then every point in X_β of period m is the image of exactly one point in X_{G_β} of the same period, except p points in the cycle of $(a_{n+1} \cdots a_{n+p})^\infty$, which are the image of two points of period m. As a result, $p_m(\sigma_{\mathcal{G}_\beta}) = p_m(\sigma_{G_\beta}) - p$ where $p_m = |P_m|$ and P_m is the set of periodic points of period m. When p does not divide m, $p_m(\sigma_{\mathcal{G}_\beta}) = p_m(\sigma_{G_\beta})$. Therefore,

$$\begin{aligned}
\zeta_{\sigma_\beta}(r) &= \exp\left(\sum_{\substack{m=1 \\ p \nmid m}}^{\infty} \frac{p_m(\sigma_{G_\beta})}{m} r^m + \sum_{\substack{m=1 \\ p \mid m}}^{\infty} \frac{p_m(\sigma_{G_\beta}) - p}{m} r^m \right) \\
&= \exp\left(\sum_{m=1}^{\infty} \frac{p_m(\sigma_{G_\beta})}{m} r^m - p \sum_{\substack{m=1 \\ p \mid m}}^{\infty} \frac{r^m}{m} \right) \\
&= \zeta_{\sigma_{G_\beta}}(r) \times (1 - r^p).
\end{aligned}$$

However, $G_\beta \cong G_S$ for S as in (4.5). Therefore, by [2, Theorem 2.3],

$$\begin{aligned}\zeta_{\sigma_{G_\beta}}(r) &= \frac{1}{(1-r^p)f_S(r^{-1})}\\ &= \frac{1}{(1-r^{i_1}-r^{i_2}-\cdots-r^{i_t})(1-r^p)-(r^{j_1}+\cdots+r^{j_u})}.\end{aligned}$$

It remains to consider the case when X_β is not sofic. We claim that $P_n(X(S)) = P_n(X_\beta)+1$ for all $n \in \mathbb{N}$.

Observe that one may assume that the initial vertex of π, a cycle in the graph of G_β, is α_1 as in Figure 2. Now let $x = v^\infty \in P_n(X_\beta)$ with $v = v_1 \cdots v_n \in \mathcal{B}_n(X_\beta)$. Pick π_β a cycle in G_β such that $v = \mathcal{L}_\beta(\pi_\beta)$ and set π_S to be the associated cycle to π_β in G_S, and let $w = \mathcal{L}_S(\pi_S)$. Then $w^\infty \in P_n(X(S))$. Now define $\varphi_n : P_n(X_\beta) \setminus P_1(X_\beta) \to P_n(X(S)) \setminus P_1(X(S))$ for all $n \geq 2$ such that $\varphi_n(v^\infty) = w^\infty$. Clearly, φ_n is well-defined. It is also one-to-one; otherwise, for $w^\infty \in P_n(X(S))$, there are two different cycles π_S and γ_S such that $w = \mathcal{L}_S(\pi_S) = \mathcal{L}_S(\gamma_S)$.

However, any 1 in w is characterized by one passing of π_S and γ_S through α_1, so $\pi_S = \gamma_S$ and φ_n is one-to-one. Now the claim follows by the fact that $P_1(X_\beta) = \{0^\infty\}$ and $P_1(X(S)) = \{0^\infty, 1^\infty\}$. Hence:

$$\begin{aligned}\zeta_{\sigma_S}(r) &= \exp\left(\sum_{m=1}^{\infty} \frac{p_m(\sigma_{G_S})}{m} r^m\right)\\ &= \exp\left(\sum_{m=1}^{\infty} \frac{p_m(\sigma_{G_\beta})+1}{m} r^m\right) = \zeta_{\sigma_{G_\beta}}(r) \times \frac{1}{1-r}.\end{aligned}$$

□

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