

## Companion inequalities to Ostrowski–Grüss type inequality and applications

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**Abstract:** The aim of this paper is to give some companion inequalities to the Ostrowski-Grüss type inequality for  $n$ -time differentiable absolutely continuous functions by using recently obtained bounds for the Chebyshev functional.

**Key words:** Chebyshev functional, Ostrowski–Grüss type inequality

### 1. Introduction

For two functions  $f, g : [a, b] \rightarrow \mathbb{R}$  such that  $f, g, f \cdot g \in L^1[a, b]$ , the Chebyshev functional [8] is defined by

$$T(f, g) = \frac{1}{b-a} \int_a^b f(s)g(s) ds - \frac{1}{b-a} \int_a^b f(s) ds \cdot \frac{1}{b-a} \int_a^b g(s) ds.$$

The following integral inequality is well known as the Grüss inequality [8]:

$$\left| \frac{1}{b-a} \int_a^b f(s)g(s) ds - \frac{1}{b-a} \int_a^b f(s) ds \cdot \frac{1}{b-a} \int_a^b g(s) ds \right| \leq \frac{1}{4} (\Gamma - \gamma) (\Phi - \phi),$$

where  $f, g : [a, b] \rightarrow \mathbb{R}$  are two integrable functions such that  $\gamma \leq f(s) \leq \Gamma$ , and  $\phi \leq g(s) \leq \Phi$ , for all  $s \in [a, b]$ , and  $\gamma, \Gamma, \phi, \Phi$  are real constants. The result related to the Chebyshev functional is the Ostrowski inequality [9]:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(s) ds \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) M,$$

where  $f : [a, b] \rightarrow \mathbf{R}$  is a differentiable function such that  $|f'(x)| \leq M$  for all  $x \in [a, b]$ . More about Ostrowski type inequalities and companion inequalities to the Ostrowski type inequality can be found in papers [4, 5] and in monographs [1, 6].

If  $f$  is a differentiable function,  $f'$  is integrable, and  $\gamma \leq f'(s) \leq \Gamma$ , for all  $s \in [a, b]$ , then the following Ostrowski–Grüss inequality can be stated:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(s) ds - \left( x - \frac{a+b}{2} \right) \frac{f(b) - f(a)}{b-a} \right| \leq \frac{1}{4\sqrt{3}} (b-a) (\Gamma - \gamma), \quad (1.1)$$

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for all  $x \in [a, b]$ . Inequality (1.1) was proved in [7].

Cerone and Dragomir [3] proved the following result:

**Lemma 1** *If  $\phi : [a, b] \rightarrow \mathbf{R}$  is an absolutely continuous function with*

$$(\cdot - a)(b - \cdot)(\phi')^2 \in L^1[a, b],$$

*then the following inequality holds:*

$$T(\phi, \phi) \leq \frac{1}{2(b-a)} \int_a^b (s-a)(b-s)[\phi'(s)]^2 ds. \tag{1.2}$$

*The constant 1/2 is best possible.*

Let us recall that the nonnegative normalized weighted function  $w : [a, b] \rightarrow [0, \infty)$  is an integrable function satisfying  $\int_a^b w(s) ds = 1$  and  $W(s) = \int_a^s w(u) du$  for  $s \in [a, b]$ ,  $W(s) = 0$  for  $s < a$ , and  $W(s) = 1$  for  $s > b$ .

We shall use the usual convention  $f^{(0)} = f$ ,  $0! = 1$  and  $\sum_{i=0}^{-1} \cdot = 0$ .

In [2] Aljinović et al. proved the following identity using some generalizations of the Montgomery identity:

**Theorem 1** *Let  $I$  be an open interval in  $\mathbf{R}$ ,  $[a, b] \subset I$ , and let  $f : I \rightarrow \mathbf{R}$  be such that  $f^{(n-1)}$  is absolutely continuous for some  $n \geq 1$ . Let  $w : [a, b] \rightarrow [0, \infty)$  be some nonnegative normalized weighted function. For each  $x \in [a, \frac{a+b}{2}]$ , the following identity then holds:*

$$\begin{aligned} \int_a^b w(s)f(s)ds &= \frac{1}{2} \left[ \sum_{i=0}^{n-1} \frac{f^{(i)}(x)}{i!} \int_a^b w(s)(s-x)^i ds \right. \\ &\quad \left. + \sum_{i=0}^{n-1} \frac{f^{(i)}(a+b-x)}{i!} \int_a^b w(s)(s-a-b+x)^i ds \right] \\ &\quad + \frac{1}{(n-1)!} \int_a^b S_{w,n}(x,s)f^{(n)}(s)ds, \end{aligned} \tag{1.3}$$

where

$$S_{w,n}(x,s) = \begin{cases} -\int_a^s w(u)(u-s)^{n-1} du, & a \leq s \leq x, \\ -\frac{1}{2} \left[ \int_a^s w(u)(u-s)^{n-1} du - \int_s^b w(u)(u-s)^{n-1} du \right], & x < s \leq a+b-x, \\ \int_s^b w(u)(u-s)^{n-1} du, & a+b-x < s \leq b. \end{cases} \tag{1.4}$$

**Remark 1** *In the nonweighted case ( $w(s) = \frac{1}{b-a}$ ,  $s \in [a, b]$ ) equality (1.3) reduces to*

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(s)ds &= \frac{1}{2} \sum_{i=0}^{n-1} \left[ f^{(i)}(x) + (-1)^i f^{(i)}(a+b-x) \right] \frac{(b-x)^{i+1} - (a-x)^{i+1}}{(i+1)!(b-a)} \\ &\quad + \frac{1}{n!} \int_a^b S_n(x,s)f^{(n)}(s)ds, \end{aligned} \tag{1.5}$$

where

$$S_n(x, s) = \begin{cases} \frac{(a-s)^n}{b-a}, & a \leq s \leq x, \\ \frac{(a-s)^n + (b-s)^n}{2(b-a)}, & x < s \leq a + b - x, \\ \frac{(b-s)^n}{b-a}, & a + b - x < s \leq b. \end{cases}$$

In this paper we will obtain some companion inequalities to the Ostrowski–Grüss type inequality by using Lemma 1 and identities (1.3) and (1.5).

## 2. Main results

Here and hereafter the symbol  $[f^{(n)}; a, b]$  denotes the divided difference of function  $f^{(n)}$ :

$$[f^{(n)}; a, b] = \frac{f^{(n)}(b) - f^{(n)}(a)}{b - a}.$$

**Theorem 2** Let  $I$  be an open interval in  $\mathbf{R}$ ,  $[a, b] \subset I$ , and let  $w : [a, b] \rightarrow [0, \infty)$  be some nonnegative normalized weighted function. Let  $f : I \rightarrow \mathbf{R}$  be such that  $f^{(n)}$  is absolutely continuous. For each  $x \in [a, \frac{a+b}{2}]$ , the following representation then holds:

$$\begin{aligned} \int_a^b w(s)f(s)ds &= \frac{1}{2} \left[ \sum_{i=0}^{n-1} \frac{f^{(i)}(x)}{i!} \int_a^b w(s)(s-x)^i ds + \sum_{i=0}^{n-1} \frac{f^{(i)}(a+b-x)}{i!} \int_a^b w(s)(s-a-b+x)^i ds \right] \\ &+ \frac{1}{(n-1)!} \int_a^b S_{w,n}(x, s) ds [f^{(n-1)}; a, b] + R_{w,n}(f, x). \end{aligned} \tag{2.1}$$

The remainder  $R_{w,n}(f, x)$  satisfies the estimation

$$\begin{aligned} |R_{w,n}(f, x)| &\leq \frac{\sqrt{2(b-a)}}{2(n-1)!} [T(S_{w,n}(x, \cdot), S_{w,n}(x, \cdot))]^{1/2} \\ &\cdot \left[ \int_a^b (s-a)(b-s) (f^{(n+1)}(s))^2 ds \right]^{1/2}, \end{aligned} \tag{2.2}$$

where  $S_{w,n}(x, \cdot)$  is defined by (1.4).

**Proof** Identity (1.3) can be rewritten as

$$\begin{aligned} \int_a^b w(s)f(s)ds &= \frac{1}{2} \left[ \sum_{i=0}^{n-1} \frac{f^{(i)}(x)}{i!} \int_a^b w(s)(s-x)^i ds \right. \\ &+ \left. \sum_{i=0}^{n-1} \frac{f^{(i)}(a+b-x)}{i!} \int_a^b w(s)(s-a-b+x)^i ds \right] \\ &+ \frac{1}{(n-1)!(b-a)} \int_a^b S_{w,n}(x, s) ds \int_a^b f^{(n)}(s)ds + R_{w,n}(f, x), \end{aligned}$$

where

$$R_{w,n}(f, x) = \frac{1}{(n-1)!} \int_a^b S_{w,n}(x, s) f^{(n)}(s) ds - \frac{1}{(n-1)!} \cdot \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \int_a^b S_{w,n}(x, s) ds$$

because

$$\int_a^b f^{(n)}(s) ds = f^{(n-1)}(b) - f^{(n-1)}(a).$$

Using Cauchy–Schwartz inequality for double integrals and the inequality between arithmetic and quadratic integral means [8], we obtain:

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b S_{w,n}(x, s) f^{(n)}(s) ds - \frac{1}{b-a} \int_a^b S_{w,n}(x, s) ds \cdot \frac{1}{b-a} \int_a^b f^{(n)}(s) ds \right| \\ & \leq [T(S_{w,n}(x, \cdot), S_{w,n}(x, \cdot))]^{1/2} \cdot [T(f^{(n)}, f^{(n)})]^{1/2}. \end{aligned} \tag{2.3}$$

If we apply Lemma 1 with  $f^{(n)}$  in place of  $\phi$ , we get

$$\begin{aligned} & [T(S_{w,n}(x, \cdot), S_{w,n}(x, \cdot))]^{1/2} \cdot [T(f^{(n)}, f^{(n)})]^{1/2} \\ & \leq \frac{1}{\sqrt{2(b-a)}} [T(S_{w,n}(x, \cdot), S_{w,n}(x, \cdot))]^{1/2} \cdot \left[ \int_a^b (s-a)(b-s) (f^{(n+1)}(s))^2 ds \right]^{1/2}, \end{aligned}$$

which completes the proof. □

In the next theorem we will use the beta function and the incomplete beta function of Euler type defined by

$$B(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} dt, \quad B_r(u, v) = \int_0^r t^{u-1} (1-t)^{v-1} dt, \quad u, v > 0.$$

**Theorem 3** Let  $I$  be an open interval in  $\mathbf{R}$ ,  $[a, b] \subset I$ , and let  $f : I \rightarrow \mathbf{R}$  be such that  $f^{(n)}$  is absolutely continuous. For each  $x \in [a, \frac{a+b}{2}]$ , the following identity then holds:

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(s) ds &= \frac{1}{2} \sum_{i=0}^{n-1} \left[ f^{(i)}(x) + (-1)^i f^{(i)}(a+b-x) \right] \frac{(b-x)^{i+1} - (a-x)^{i+1}}{(i+1)!(b-a)} \\ &+ \frac{(-1)^n + 1}{2} \cdot \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)!(b-a)} [f^{(n-1)}; a, b] + R_n(f, x). \end{aligned} \tag{2.4}$$

The remainder  $R_n(f, x)$  satisfies the estimation

$$|R_n(f, x)| \leq \frac{\sqrt{2(b-a)}}{2n!} [T(S_n(x, \cdot), S_n(x, \cdot))]^{1/2} \cdot \left[ \int_a^b (s-a)(b-s) (f^{(n+1)}(s))^2 ds \right]^{1/2}, \tag{2.5}$$

where

$$\begin{aligned}
 T(S_n(x, \cdot), S_n(x, \cdot)) &= \frac{3(x-a)^{2n+1} + (b-x)^{2n+1}}{2(2n+1)(b-a)^3} \\
 &+ \frac{(-1)^n(b-a)^{2n-2}}{2} \left[ B_{\frac{b-x}{b-a}}(n+1, n+1) - B_{\frac{x-a}{b-a}}(n+1, n+1) \right] \\
 &- \left[ \frac{(-1)^n + 1}{2} \cdot \frac{(x-a)^{n+1} + (-1)^n(b-x)^{n+1}}{(n+1)(b-a)^2} \right]^2, \tag{2.6}
 \end{aligned}$$

**Proof** Identity (1.5) can be written as

$$\frac{1}{n!} \int_a^b S_n(x, s) f^{(n)}(s) ds = \frac{1}{b-a} \int_a^b f(s) ds - \frac{1}{2} \sum_{i=0}^{n-1} \left[ f^{(i)}(x) + (-1)^i f^{(i)}(a+b-x) \right] \frac{(b-x)^{i+1} - (a-x)^{i+1}}{(i+1)!(b-a)}.$$

By elementary calculations we get

$$\begin{aligned}
 \int_a^b S_n(x, s) ds &= \int_a^x \frac{(a-s)^n}{b-a} ds + \int_x^{a+b-x} \frac{(a-s)^n + (b-s)^n}{2(b-a)} ds + \int_{a+b-x}^b \frac{(b-s)^n}{b-a} ds \\
 &= \frac{(-1)^n + 1}{2} \cdot \frac{(x-a)^{n+1} + (-1)^n(b-x)^{n+1}}{(n+1)(b-a)}.
 \end{aligned}$$

Then

$$R_n(f, x) = \frac{1}{n!} \int_a^b S_n(x, s) f^{(n)}(s) ds - \frac{1}{n!} \cdot \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \int_a^b S_n(x, s) ds. \tag{2.7}$$

From Cauchy–Schwartz inequality for double integrals and the inequality between arithmetic and quadratic integral means [8], we obtain

$$\begin{aligned}
 &\left| \frac{1}{b-a} \int_a^b S_n(x, s) f^{(n)}(s) ds - \frac{1}{b-a} \int_a^b S_n(x, s) ds \cdot \frac{1}{b-a} \int_a^b f^{(n)}(s) ds \right| \\
 &\leq [T(S_n(x, \cdot), S_n(x, \cdot))]^{1/2} \cdot [T(f^{(n)}, f^{(n)})]^{1/2}. \tag{2.8}
 \end{aligned}$$

If we apply Lemma 1 with  $f^{(n)}$  in place of  $\phi$ , we obtain

$$\begin{aligned}
 &[T(S_n(x, \cdot), S_n(x, \cdot))]^{1/2} \cdot [T(f^{(n)}, f^{(n)})]^{1/2} \\
 &\leq \frac{1}{\sqrt{2(b-a)}} [T(S_n(x, \cdot), S_n(x, \cdot))]^{1/2} \cdot \left[ \int_a^b (s-a)(b-s) (f^{(n+1)}(s))^2 ds \right]^{1/2}.
 \end{aligned}$$

Using the incomplete beta function of Euler type we get the following identity:

$$\begin{aligned}
 \int_a^b S_n^2(x, s) ds &= \frac{3(x-a)^{2n+1} + (b-x)^{2n+1}}{2(2n+1)(b-a)^2} \\
 &+ \frac{(-1)^n(b-a)^{2n-1}}{2} \left[ B_{\frac{b-x}{b-a}}(n+1, n+1) - B_{\frac{x-a}{b-a}}(n+1, n+1) \right].
 \end{aligned}$$

Finally, we have

$$\begin{aligned}
 T(S_n(x, \cdot), S_n(x, \cdot)) &= \frac{1}{b-a} \int_a^b S_n^2(x, s) ds - \frac{1}{(b-a)^2} \left( \int_a^b S_n(x, s) ds \right)^2 = \frac{3(x-a)^{2n+1} + (b-x)^{2n+1}}{2(2n+1)(b-a)^3} \\
 &+ \frac{(-1)^n (b-a)^{2n-2}}{2} \left[ B_{\frac{b-x}{b-a}}(n+1, n+1) - B_{\frac{x-a}{b-a}}(n+1, n+1) \right] \\
 &- \left[ \frac{(-1)^n + 1}{2} \cdot \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)(b-a)^2} \right]^2.
 \end{aligned}$$

□

**Corollary 1** Let  $I$  be an open interval in  $\mathbf{R}$ ,  $[a, b] \subset I$ , and let  $f : I \rightarrow \mathbf{R}$  be such that  $f'$  is absolutely continuous. For each  $x \in [a, \frac{a+b}{2}]$ , the following inequality then holds:

$$\left| \frac{1}{b-a} \int_a^b f(s) ds - \frac{f(x) + f(a+b-x)}{2} \right| \leq K(x) \cdot \left[ \int_a^b (s-a)(b-s)(f''(s))^2 ds \right]^{1/2}, \tag{2.9}$$

where

$$K(x) = \frac{1}{b-a} \left[ \frac{8(x-a)^3 + (a+b-2x)^3}{24} \right]^{1/2}.$$

**Proof** Applying Theorem 3 with  $n = 1$ , we get above inequality. □

**Remark 2** For  $x = a$  inequality (2.9) reduces to the following inequality:

$$\left| \frac{1}{b-a} \int_a^b f(s) ds - \frac{f(a) + f(b)}{2} \right| \leq \frac{\sqrt{b-a}}{2\sqrt{6}} \cdot \left[ \int_a^b (s-a)(b-s)(f''(s))^2 ds \right]^{1/2}.$$

For  $x = \frac{a+b}{2}$  inequality (2.9) reduces to the following midpoint inequality:

$$\left| \frac{1}{b-a} \int_a^b f(s) ds - f\left(\frac{a+b}{2}\right) \right| \leq \frac{\sqrt{b-a}}{2\sqrt{6}} \cdot \left[ \int_a^b (s-a)(b-s)(f''(s))^2 ds \right]^{1/2}$$

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