

## Spreading speeds in a lattice differential equation with distributed delay

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**Abstract:** This paper studies the spreading speed for a lattice differential equation with infinite distributed delay and we find that the spreading speed coincides with the minimal wave speed of traveling waves. Here the model has been proposed to describe a single species living in a 1D patch environment with infinite number of patches connected locally by diffusion.

**Key words:** Lattice differential equation, infinite distributed delay, spreading speeds

### 1. Introduction

In biological invasions, the spreading speed (short for the asymptotic speed of spread/propagation) is a very important notion, since it is used to describe the speed at which the geographic range of the species population expands [14, 20, 22, 29, 36, 38]. The concept of the spreading speed was first introduced by Aronson and Weinberger [2, 3] for reaction-diffusion equations and applied by Aronson [1] to an integrodifferential equation. A general theory of spreading speeds has been developed for monotone semiflows [19, 20, 21, 33], for integral and integrodifferential population models [8, 9, 29, 30, 31], for time-delayed reaction-diffusion equations [31, 37], and for lattice differential equations [4, 34]. Recently, Hsu and Zhao [14] and Li et al. [18] extended the theory of spreading speeds in nonmonotone integrodifference equations and Fang et al. [10] in nonmonotone discrete-delayed lattice equations.

Lattice differential equations arise in many applied subjects, such as chemical reaction, image processing, material science, and biology [7, 15, 17, 28]. In the models of lattice differential equations, the spatial structure has a discrete character, and lattice dynamics have recently been extensively used to model biological problems [4, 5, 6, 10, 12, 16, 22, 23, 25, 36] since the environment in which the species population lives may be discrete but not continuous. In 2003, Weng et al. [34] considered a single-species population with two age classes distributed over a patchy environment consisting of all integer nodes of a one-dimensional lattice, and they derived a time-delayed lattice differential equation. For a fixed delay, Weng et al. [34] proved that the minimal wave speed of traveling waves is also the spreading speed when the birth function satisfies the monostable assumption. In 2006, Kyrychko et al. [16] derived a stage-structured model for a single species on a finite one-dimensional lattice, provided that there was no migration into or from the lattice. Furthermore, they proved that the model has a positivity-preserving property and that the positive equilibrium is globally stable by establishing comparison principles for the cases where the birth function is increasing and where the birth function is nonmonotone. In

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2008, Cheng et al. [4] extended the model of Weng et al. [34] to a 2D lattice. They derived a lattice model for a single species in a 2D patchy environment with infinite number of patches connected locally by diffusion and global interaction by delay. In particular, they also showed that the minimal wave speed  $c_*(\theta)$  coincides with the asymptotic speed of spread for any fixed direction  $\theta$ . Then, in 2010, Cheng et al. [5] established the asymptotic stability of traveling wave fronts when the immature population is not mobile. Also in 2010, Fang et al. [10] considered a nonmonotone time-delayed lattice system with global interaction, using Schauder’s fixed-point theorem and limiting process, and they obtained the existence of traveling waves. Moreover, they also found that the minimal wave speed of traveling waves coincided with the spreading speed. Other related results on lattice models can be found in the works of Cheng et al. [6], Gourley and Wu [12], Ma et al. [22], Ma and Zou [23], Weng et al. [36], and the references therein.

Note that not all individuals in a population necessarily always mature at the same age, and the time from birth to maturity may be rather imperfectly known or it might vary from individual to individual, as described by Gourley and So [11]. Therefore, it was observed that distributed delays are more reasonable than discrete delays in modeling maturation periods. The works of Weng and Wu [35] also studied a nonlocal reaction-diffusion population model with general distributive maturity. In our previous paper [25], we used arguments similar to those of So et al. [27] and Weng et al. [34], and we derived a lattice model with infinite distributed delay in a 2D patchy environment to describe the growth of a single-species population. We proved the existence of traveling wave solutions when the birth rate is large enough that each can sustain a positive equilibrium.

In our previous paper [25], using discrete Fourier transformation and inverse discrete Fourier transformation, we derived a lattice model with infinite distributed delay to describe the growth of a single-species population in a 2D patchy environment. The corresponding lattice model in 1D form can be written as

$$\begin{aligned} \frac{dw_k(t)}{dt} = & D[w_{k+1}(t) + w_{k-1}(t) - 2w_k(t)] - dw_k(t) \\ & + \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \left[ \int_0^{\infty} \beta_{\alpha}(l)b(w_{k+l}(t-a))e^{-da}f(a)da \right], \end{aligned} \tag{1.1}$$

where  $\alpha = Da$  and

$$\beta_{\alpha}(l) = \operatorname{Re} \int_{-\pi}^{\pi} e^{il\omega - 4\alpha \sin^2 \frac{\omega}{2}} d\omega = 2e^{-\nu} \int_0^{\pi} \cos(l\omega)e^{\nu \cos \omega} d\omega, \quad (\nu := 2\alpha) \tag{1.2}$$

for any  $l \in \mathbb{Z}$ . In (1.1),  $w_k(t)$  is the total mature population at the  $k$ th patch at time  $t$ , and  $D$  and  $d$  are the diffusion coefficient and the death rate, respectively.  $b(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is the birth function, and the probability density function  $f(a) \in L^1([0, +\infty), \mathbb{R}^+)$  describes the probability of maturing at age  $a$  and satisfies  $\int_0^{\infty} f(a)da = 1$ ,  $f(a)da$  being the probability, given that an individual matures, of its maturation time being between  $a$  and  $a + da$ , with  $da$  infinitesimal. The purpose of this paper is to establish the theory of spreading speeds for system (1.1).

The following lemma gives the properties of  $\beta_{\alpha}(l)$ .

**Lemma 1.1** *Let  $\beta_{\alpha}(l)$  be given in (1.2). Then:*

- (i)  $\beta_{\alpha}(l) = \beta_{\alpha}(|l|), \forall l \in \mathbb{Z}$ ;

- (ii)  $\frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \beta_{\alpha}(l) = 1$ ;
- (iii)  $\beta_{\alpha}(l) \geq 0$  if  $\alpha = 0$  and  $l \in \mathbb{Z}$ ;  $\beta_{\alpha}(l) > 0$  if  $\alpha > 0$  and  $l \in \mathbb{Z}$ ;
- (iv)  $\frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \beta_{\alpha}(l)e^{-\lambda l} < +\infty$  for any  $\lambda > 0$ .

For the proof of Lemma 1.1, we refer to Lemma 2.1 and (3.6) in the work of Weng et al. [34].

**Remark 1.2** Here we note that the kernel function  $\frac{1}{2\pi}\beta_{\alpha}(l)$  in (1.2) was exactly derived in [25] by using the discrete Fourier transformation and inverse discrete Fourier transformation, which is similar to those in [4, 34, 36]. It should be pointed out that the properties listed by Lemma 1.1 are sufficient for this paper and the main results of this paper remain valid for general kernel functions under similar assumptions. Ma et al. [22] considered the following lattice differential system:

$$u'_n(t) = D \sum_{i \in \mathbb{Z} \setminus \{0\}} J(i)[u_{n-i}(t) - u_n(t)] - du_n(t) + \sum_{i \in \mathbb{Z}} K(i)b(u_{n-i}(t-r),)$$

and they established the asymptotic speed of propagation and traveling wave fronts, where  $K$  is a general kernel function and is just requested to satisfy the following assumptions: (i)  $K(i) = K(-i) \geq 0$  for all  $i \in \mathbb{Z} \setminus \{0\}$ ; (ii)  $\sum_{i \in \mathbb{Z}} K(i) = 1$ ; (iii)  $\sum_{i \in \mathbb{Z}} K(i)e^{-\lambda i} < +\infty$  for any  $\lambda \in \mathbb{R}$ . Moreover, the kernel function  $J$  was also requested to satisfy the above three assumptions. Using the same argument in this paper, we can prove that Theorems 3.3, 3.8, and 3.9 still hold even if we replace the kernel function  $\frac{1}{2\pi}\beta_{\alpha}(l)$  in (1.2) with the above kernel function  $K(i)$ .

This paper is organized as follows: in Section 2, we present some preliminaries. In Section 3, we establish the existence of the spreading speed for initial-value problem (1.1) and (2.1) for both monotone and nonmonotone birth functions, and we prove that the spreading speed coincides with the minimal wave speed of the nontrivial traveling waves. When the birth function  $b(w)$  is monotone, we prove that the positive equilibrium is stable if it exists. Finally, we apply our results to a Nicholson blowfly model.

## 2. Preliminaries

In this paper, we always denote  $C_K^+(-\infty, T] = C((-\infty, T], [0, K])$ ,  $C_K^+(-\infty, \infty) = C((-\infty, \infty), [0, K])$ , where  $K$  and  $T$  are positive constants. Assume that the birth function  $b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfies the following conditions:

(H<sub>b</sub><sup>1</sup>)  $b$  is local Lipschitz continuous and

- (i)  $b(0) = 0$ ;
- (ii) there exists sufficiently large  $K_0 > 0$  such that  $dw > \bar{f}(d)b(w)$  for  $w > K_0$ , where  $\bar{f}(d) = \int_0^{\infty} e^{-da} f(a)da$ .

**Definition 2.1** ([10, Definition 2.1]) For any function  $u_k : \mathbb{R}_+ \times \mathbb{Z} \rightarrow \mathbb{R}_+$ , a number  $c^* > 0$  is called the spreading speed if  $\limsup_{t \rightarrow \infty, |k| \geq ct} u_k(t) = 0$  for every  $c > c^*$ , and there exists some  $\varepsilon > 0$  such that  $\liminf_{t \rightarrow \infty, |k| \leq ct} u_k(t) \geq \varepsilon$  for every  $c \in (0, c^*)$ .

For an initial condition

$$W^0(t) = \{w_k^0(t)\}_{k \in \mathbb{Z}}, w_k^0(t) \in C_K^+(-\infty, 0], k \in \mathbb{Z}, \tag{2.1}$$

we consider the following initial-value problem:

$$\begin{cases} \frac{dw_k(t)}{dt} = D[w_{k+1}(t) + w_{k-1}(t) - 2w_k(t)] - dw_k(t) \\ \quad + \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} [\int_0^{\infty} \beta_{\alpha}(l)b(w_{k+l}(t-a))e^{-da}f(a)da], \quad t > 0, \\ w_k(t) = w_k^0(t), \quad t \leq 0. \end{cases} \quad (2.2)$$

For convenience, let

$$\begin{aligned} \mathcal{A}_k(w)(s) &= D[w_{k+1}(s) + w_{k-1}(s)], \\ \mathcal{B}_k(w)(s) &= \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \left[ \int_0^{\infty} \beta_{\alpha}(l)b(w_{k+l}(s-a))e^{-da}f(a)da \right]. \end{aligned}$$

The initial-value problem (2.2) can then be rewritten as:

$$\begin{cases} w_k(t) = e^{-\delta t}w_k(0) + \int_0^t e^{-\delta(t-s)}(\mathcal{A}_k(w)(s) + \mathcal{B}_k(w)(s))ds, \quad t > 0, \\ w_k(t) = w_k^0(t), \quad t \leq 0, \end{cases} \quad (2.3)$$

where  $\delta = 2D + d$ .

Now we establish the existence and uniqueness of solutions to the initial-value problem (2.3) and the comparison principle.

**Theorem 2.2** *Assume that  $(H_b^1)$  holds, let  $K > K_0$ , and  $b(\cdot)$  is nondecreasing on  $[0, K]$ . Then for any given function*

$$W^0(t) = \{w_k^0(t)\}_{k \in \mathbb{Z}}, \quad w_k^0(t) \in C_K^+(-\infty, 0], \quad k \in \mathbb{Z},$$

the initial-value problem (2.3) has a unique solution  $W(t) = \{w_k(t)\}_{k \in \mathbb{Z}}$  with  $w_k(t) \in C_K^+(-\infty, +\infty)$ .

**Proof** For each fixed  $T \in (0, \infty)$  and  $W^0 = \{w_k^0\}_{k \in \mathbb{Z}}$  with  $w_k^0 \in C_K^+(-\infty, 0]$ , we define

$$S_T = \{W = \{w_k\}_{k \in \mathbb{Z}} \mid w_k \in C_K^+(-\infty, T], \quad w_k(t) = w_k^0(t), \quad t \in (-\infty, 0]\}.$$

Define an operator  $F^T = \{F_k^T\}_{k \in \mathbb{Z}}$  by

$$F_k^T[W](t) = \begin{cases} e^{-\delta t}w_k(0) + \int_0^t e^{-\delta(t-s)}(\mathcal{A}_k(w)(s) + \mathcal{B}_k(w)(s))ds, \quad t \geq 0, \\ w_k^0(t), \quad t \leq 0, \end{cases}$$

where  $W \in S_T$ . Obviously,  $F^T[W](t)$  is continuous for  $t \in (-\infty, T]$ . Then for any  $t \in [0, T]$  and  $k \in \mathbb{Z}$ , we have

$$0 \leq F_k^T[W](t) \leq e^{-\delta t}K + [2DK + \bar{f}(d)b(K)] \int_0^t e^{-\delta(t-s)}ds \leq K.$$

Therefore,  $F^T(S_T) \subseteq S_T$ .

For any  $W \in S_T$  and  $\lambda > 0$ , we define a norm  $\|\cdot\|_{\lambda}$  by  $\|W\|_{\lambda} := \sup_{t \in [0, T], k \in \mathbb{Z}} |w_k(t)|e^{-\lambda t}$ , and for any

$W, \bar{W} \in S_T$ , let  $g_k(t) = w_k(t) - \bar{w}_k(t)$  and  $G(t) = \{g_k(t)\}_{k \in \mathbb{Z}}$ . For  $t \in (0, T]$ , we then have

$$F_k^T[W](t) - F_k^T[\bar{W}](t) = \int_0^t e^{-\delta(t-s)}(\mathcal{A}_k(g)(s) + \mathcal{B}_k(w)(s) - \mathcal{B}_k(\bar{w})(s))ds.$$

Therefore,

$$\begin{aligned} & |F_k^T [W] (t) - F_k^T [\bar{W}] (t)| e^{-\lambda t} \\ & \leq D \int_0^t e^{-\lambda s} e^{-\lambda(t-s)} [|g_{k+1}(s)| + |g_{k-1}(s)|] ds + \\ & \quad \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \beta_\alpha(l) \int_0^t e^{-\lambda s} e^{-\lambda(t-s)} \int_0^s L_K |g_{k+l}(s-a)| f(a) e^{-da} da ds \\ & \leq \frac{2D}{\lambda} \|G\|_\lambda (1 - e^{-\lambda t}) + \frac{\bar{f}(d)L_K}{\lambda} \|G\|_\lambda (1 - e^{-\lambda t}), \end{aligned}$$

where  $L_K$  is a Lipschitz constant. Since

$$\lim_{\lambda \rightarrow \infty} \left[ \frac{2D}{\lambda} \|G\|_\lambda (1 - e^{-\lambda t}) + \frac{\bar{f}(d)L_K}{\lambda} \|G\|_\lambda (1 - e^{-\lambda t}) \right] = 0,$$

we have that  $F^T$  is a contracting map in  $(S_T, \|\cdot\|_\lambda)$  if  $\lambda > 0$  is sufficiently large. Since  $(S_T, \|\cdot\|_\lambda)$  is a Banach space, by the Banach fixed point theorem,  $F_T$  has a unique fixed point  $W$  in  $S_T$ . This shows that initial-value problem (2.3) has a unique solution on  $[0, T]$  for any  $T > 0$ .

By the arbitrariness of  $T$ , we can conclude that the initial-value problem (2.3) has a unique solution on  $[0, \infty)$ . This completes the proof of Theorem 2.2.  $\square$

Next we establish the comparison theorem for initial-value problem (2.3). Assume that

$$X = \left\{ u = \{u_i\}_{i \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}} \mid \sup_{i \in \mathbb{Z}} |u_i| < +\infty \right\}$$

with the norm  $|u|_X = \sup_{i \in \mathbb{Z}} |u_i|$ . Let  $X^+ = \mathbb{R}_+^{\mathbb{Z}}$ , and then  $X^+$  is a positive closed cone in  $X$ . The partial order relation  $\leq_X$  is generated by  $X^+$  in a Banach lattice  $X$ . If we set  $T(t) = e^{-\delta t}$ , then  $T(t)$  is a  $C^0$  linear semigroup in  $X$ , and  $T(t)$  is strongly positive.

Let  $h : (-\infty, 0] \rightarrow [1, +\infty)$  satisfy the following conditions:

- (i)  $h$  is a continuous, nonincreasing function and  $h(0) = 1$ ;
- (ii)  $\frac{h(s+\theta)}{h(s)} \rightarrow 1$  uniformly for  $s \in (-\infty, 0)$  as  $\theta \rightarrow 0^+$ ;
- (iii)  $h(s) \rightarrow +\infty$  as  $s \rightarrow -\infty$ .

A typical function satisfying the above conditions is  $e^{-\lambda s}$ , where  $\lambda$  is a positive constant. Define

$$\mathcal{C}_h = \left\{ \phi \in C((-\infty, 0], X) \mid \frac{\phi}{h} \in UC((-\infty, 0]), \sup_{s \leq 0} \frac{|\phi(s)|_X}{h(s)} < \infty \right\}$$

with the norm  $|\phi|_{\mathcal{C}} = \sup_{s \leq 0} \frac{|\phi(s)|_X}{h(s)}$ ,  $\phi \in \mathcal{C}_h$ , where  $UC((-\infty, 0])$  is the space of uniformly continuous functions on  $(-\infty, 0]$ . Let

$$\mathcal{C}_h^+ = \{ \phi \in \mathcal{C}_h \mid \phi(s) \in X^+ \},$$

$\mathcal{C}_h^+$  generate a partial order relation  $\leq_{\mathcal{C}}$  in  $\mathcal{C}_h$ .

For any continuous function  $W : (-\infty, b) \rightarrow X$ , we define  $W_t(s) = W(t + s)$ , where  $b > 0$  and  $s \in (-\infty, 0]$ . Notice that  $W_t \in C((-\infty, 0], X)$ .

For any  $K > 0$ , define

$$\begin{aligned} X_K &= \{u = \{u_i\}_{i \in \mathbb{Z}} \in X \mid 0 \leq u_i \leq K, i \in \mathbb{Z}\}, \\ \mathcal{C}_{h,K} &= \{\phi \in \mathcal{C}_h \mid 0 \leq \phi_i(s) \leq K, i \in \mathbb{Z}, s \leq 0\}, \\ D &= [0, \infty) \times X_K, D(t) = X_K, \forall t \in [0, \infty), \\ \mathcal{D} &= [0, \infty) \times \mathcal{C}_{h,K}, \mathcal{D}(t) = \mathcal{C}_{h,K}, \forall t \in [0, \infty), \end{aligned}$$

and then conditions (D1)–(D3) [26, Section 4] are satisfied. It is easy to verify  $\mathcal{C}_{h,K} = C_K^+(-\infty, 0]$ .

Define  $F : \mathcal{C}_h^+ \rightarrow X^+$  by  $F(\Phi) = \{F_k(\Phi)\}_{k \in \mathbb{Z}}$ , where  $\Phi = \{\phi_k\}_{k \in \mathbb{Z}} \in \mathcal{C}_h^+$  and

$$F_k(\Phi) = D[\phi_{k+1}(0) + \phi_{k-1}(0)] + \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \left[ \int_0^{\infty} \beta_{\alpha}(l)b(\phi_{k+l}(-a))e^{-da} f(a)da \right].$$

For  $W^0 \in \mathcal{D}(0)$ , that is  $W^0 \in \mathcal{C}_{h,K}$ , the initial-value problem (2.3) has an equivalent form as follows:

$$\begin{cases} W(t) = T(t)W(0) + \int_0^t T(t-s)F(W_s)ds, & t > 0, \\ W(t) = W^0(t), & t \leq 0. \end{cases} \tag{2.4}$$

**Definition 2.3** A continuous function  $V : (-\infty, b) \rightarrow X$  is called an upper (a lower) solution of initial-value problem (2.4) if

$$V(t) \geq (\leq) T(t)V(0) + \int_0^t T(t-s)F(V_s)ds, \quad 0 < t < b.$$

If  $V$  is an upper and a lower solution, then  $V$  is a solution of initial-value problem (2.4) on  $[0, b)$ .

**Theorem 2.4 (Comparison principle)** Assume that  $(H_b^1)$  holds, and the birth function  $b(\cdot)$  is nondecreasing on  $[0, K]$ , where  $K > K_0$ . For a pair of upper and lower solutions  $W^+(t) = \{w_i^+(t)\}_{i \in \mathbb{Z}} \in C(\mathbb{R}, X)$  and  $W^-(t) = \{w_i^-(t)\}_{i \in \mathbb{Z}} \in C(\mathbb{R}, X)$  of initial-value problem (2.4), if  $0 \leq w_i^-(t) \leq w_i^+(t) \leq K$  for  $i \in \mathbb{Z}$  and  $t \leq 0$ , then we have  $0 \leq w_i^-(t) \leq w_i^+(t) \leq K$  for  $i \in \mathbb{Z}$  and  $t > 0$ .

**Proof** We can easily verify that  $F$  is Lipschitz continuous in  $\mathcal{C}_{h,K}$ . Since  $b(\cdot)$  is nondecreasing on  $[0, K]$ , then for any  $\Psi, \Phi \in \mathcal{C}_{h,K}$  and  $\Psi \geq_C \Phi$ , we have  $F_k(\Psi) - F_k(\Phi) \geq 0$ . Therefore,  $F(\Psi) \geq_X F(\Phi)$ , which implies that  $F : \mathcal{C}_{h,K} \rightarrow X^+$  is quasimonotone, that is

$$\lim_{\rho \rightarrow 0} \frac{1}{\rho} \text{dist} \left( (\Psi(0) - \Phi(0)) + \rho[F(\Psi) - F(\Phi)], X^+ \right) = 0.$$

If we define  $S(t, s)$  and  $T(t, s)$  [26, Equation (4.1)] by  $S(t, s) = T(t, s) = T(t-s) = e^{-\delta(t-s)}$ , where  $t > s \geq 0$ , then using [26, Theorem 5.2], we have

$$W^+(t) \geq_X W(t; W^+) \geq_X W(t; W^-) \geq_X W^-(t), \quad \forall t > 0,$$

where  $W(t; W^+)$  and  $W(t; W^-)$  are solutions of initial-value problem (2.3) with  $W^+(t) = \{w_k^+(t)\}_{k \in \mathbb{Z}}$  and  $W^-(t) = \{w_k^-(t)\}_{k \in \mathbb{Z}}$ ,  $t \in (-\infty, 0]$ , respectively. Therefore, we obtain  $w_k^+(t) \geq w_k^-(t)$  for any  $t \geq 0$  and  $k \in \mathbb{Z}$ . This completes the proof.  $\square$

### 3. Spreading speed

In this section, we study the spreading speed of solutions of initial-value problem (1.1) and (2.1). Here we consider two cases: the birth function  $b$  is monotone and nonmonotone. The main idea comes from Zhao and Xiao [38] and Fang et al. [10].

#### 3.1. Monotone birth function

In this subsection, we always assume that the birth function  $b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfies the following assumptions:

( $H_b^2$ )  $b$  is local Lipschitz continuous,  $b'(0)$  exists, and

- (i)  $b(0) = 0$ ,  $b'(0)\bar{f}(d) > d$ ,  $b(w) \leq b'(0)w$ ,  $\forall w \in \mathbb{R}_+$ ;
- (ii)  $\bar{f}(d)b(w) = dw$  admits a unique positive solution  $w^* > 0$  and  $b$  is nondecreasing on  $[0, w^*]$ ;
- (iii) there exist constants  $\rho \in (0, 1]$ ,  $M_0 > 0$ , and  $\eta \in (0, w^*)$  such that  $b'(0)w - b(w) < M_0w^{1+\rho}$  for  $w \in (0, \eta)$ ;
- (iv)  $b(w)\bar{f}(d) > dw$  for  $w \in (0, w^*)$ .

Before studying equation (1.1), we consider the following lattice differential equation with finite distributed delay:

$$\begin{aligned} \frac{dw_k(t)}{dt} = & D[w_{k+1}(t) + w_{k-1}(t) - 2w_k(t)] - dw_k(t) \\ & + \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \left[ \int_0^\tau \beta_\alpha(l)b(w_{k+l}(t-a))e^{-da}f(a)da \right], \end{aligned} \tag{3.1}$$

where  $\tau$  is a positive parameter. ( $H_b^2$ ) ensures that there exists  $\tau_0 > 0$  so large that for any  $\tau > \tau_0$ , equation  $dw - b(w) \int_0^\tau e^{-da}f(a)da = 0$  has a unique positive root  $w = w_\tau^* \in (0, w^*]$ , and furthermore  $\lim_{\tau \rightarrow \infty} w_\tau^* = w^*$ .

Denote  $\mathfrak{C} = C([-\tau, 0], X)$ ,  $\mathfrak{C}^+ = C([-\tau, 0], X^+)$ , and denote

$$\mathfrak{C}_{[0,K]} = \{ \phi \in \mathfrak{C} : 0 \leq \phi_i(s) \leq K, s \in [-\tau, 0], i \in \mathbb{Z} \}.$$

For any continuous function  $W(t) = \{w_k(t)\}_{k \in \mathbb{Z}} : [-\tau, \infty) \rightarrow X$ , we define  $W_t \in \mathfrak{C}$ ,  $t \in [0, \infty)$ , with  $W_t(s) = W(t+s)$ ,  $s \in [-\tau, 0]$ .

Defining  $F = \{F_k\}_{k \in \mathbb{Z}} : \mathfrak{C}_{[0,K]} \rightarrow X$  by

$$F_k(\varphi) = D[\varphi_{k+1}(0) + \varphi_{k-1}(0)] + \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \left[ \int_0^\tau \beta_\alpha(l)b(\varphi_{k+l}(-a))e^{-da}f(a)da \right],$$

equation (3.1) can be rewritten as:

$$\frac{dw_k(t)}{dt} = -\delta w_k(t) + F_k(W_t), \quad \forall t > 0, k \in \mathbb{Z},$$

where  $\delta = 2D + d$ . Since  $b(\cdot)$  is increasing on  $[0, w^*]$ , by Martin and Smith [24, Corollary 5] (the discussion is similar to Cheng et al. [4, Theorem 3.1]), equation (3.1) has a unique solution  $W(t; \varphi)$  for any initial data  $\varphi \in \mathfrak{C}_{[0,w^*]}$ . Furthermore, the solution semiflow of equation (3.1) is order-preserving on  $\mathfrak{C}_{[0,w^*]}$ .

Define

$$\begin{aligned} \Delta_\tau(\lambda, c) &:= D(e^\lambda + e^{-\lambda} - 2) - c\lambda - d + b'(0) \int_0^\tau \left[ \frac{1}{2\pi} \sum_{l=-\infty}^\infty \beta_\alpha(l) e^{\lambda l} \right] e^{-(c\lambda+d)a} f(a) da \\ &= D(e^\lambda + e^{-\lambda} - 2) - c\lambda - d + b'(0) \int_0^\tau e^{2\alpha(\cosh \lambda - 1) - (c\lambda+d)a} f(a) da. \end{aligned}$$

Similar to our previous paper [25, Lemma 3.2], we find that the system  $\Delta_\tau(\lambda, c) = 0, \frac{\partial}{\partial \lambda} \Delta_\tau(\lambda, c) = 0$  has a positive root  $(\lambda_\tau^*, c_\tau^*)$ .

For equation (3.1), the following results come from Fang et al. [10, Theorem 3.3], which imply that the minimal wave speed  $c_\tau^*$  is also the spreading speed.

**Lemma 3.1** *Let  $\tau \geq \tau_0$ , and then the following hold true:*

- (i) *assuming that  $\varphi = \{\varphi_k\}_{k \in \mathbb{Z}} \in \mathfrak{C}_{[0, w_\tau^*]}$ , if there exists  $N_0 \in \mathbb{N}$  such that  $\varphi_k(s) = 0$  for any  $|k| > N_0$  and  $s \in [-\tau, 0]$ , then  $\lim_{t \rightarrow \infty, |k| \geq ct} w_k(t) = 0$  for any  $c > c_\tau^*$ ;*
- (ii) *assuming that  $\varphi = \{\varphi_k\}_{k \in \mathbb{Z}} \in \mathfrak{C}_{[0, w_\tau^*]}$ , if  $\varphi(s) \not\equiv 0$  for any  $s \in [-\tau, 0]$ , then  $\lim_{t \rightarrow \infty, |k| \leq ct} w_k(t) = w_\tau^*$  for any  $0 < c < c_\tau^*$ .*

We now consider the spreading speed of initial-value problem (1.1) and (2.1). Assume that  $(\lambda^*, c^*)$  is a unique positive solution of the following system [25, Lemma 3.2]:

$$\Delta(\lambda, c) = 0, \frac{\partial}{\partial \lambda} \Delta(\lambda, c) = 0,$$

where

$$\begin{aligned} \Delta(\lambda, c) &= D(e^\lambda + e^{-\lambda} - 2) - c\lambda - d + b'(0) \int_0^\infty \left[ \frac{1}{2\pi} \sum_{l=-\infty}^\infty \beta_\alpha(l) e^{\lambda l} \right] e^{-(c\lambda+d)a} f(a) da \\ &= D(e^\lambda + e^{-\lambda} - 2) - c\lambda - d + b'(0) \int_0^\infty e^{2\alpha(\cosh \lambda - 1) - (c\lambda+d)a} f(a) da \end{aligned}$$

is the characteristic equation of equation (1.1).

**Proposition 3.2**  $\lim_{\tau \rightarrow \infty} (\lambda_\tau^*, c_\tau^*) = (\lambda^*, c^*)$ .

**Proof** For fixed  $\lambda \geq 0$  and  $c \geq 0$ ,  $\Delta_\tau(\lambda, c)$  and  $c_\tau^*$  are both strictly increasing for  $\tau > 0$ . It is easy to see that for any  $\lambda > 0$ ,

$$\Delta(\lambda, c_\tau^*) = \Delta_\tau(\lambda, c_\tau^*) + b'(0) \int_\tau^\infty \exp \{ 2\alpha(\cosh \lambda - 1) - (c_\tau^* \lambda + d)a \} f(a) da > 0.$$



Using [25, Lemma 3.2], we have  $c_\tau^* < c^*$  for  $\forall \tau > 0$ . Denote  $c_\infty^* := \lim_{\tau \rightarrow \infty} c_\tau^*$ , and then  $c_\infty^* \leq c^*$ . We now prove that  $\limsup_{\tau \rightarrow \infty} \lambda_\tau^* = \lambda_\infty^* < +\infty$ . Suppose that  $\limsup_{\tau \rightarrow \infty} \lambda_\tau^* = +\infty$ . Since

$$\begin{aligned} \Delta_\tau(\lambda_\tau^*, c_\tau^*) &= D(e^{\lambda_\tau^*} + e^{-\lambda_\tau^*} - 2) - c_\tau^* \lambda_\tau^* - d \\ &\quad + b'(0) \int_0^\tau \exp \{ 2\alpha(\cosh \lambda_\tau^* - 1) - (c_\tau^* \lambda_\tau^* + d)a \} f(a) da \\ &\geq D(e^{\lambda_\tau^*} + e^{-\lambda_\tau^*} - 2) - c_\infty^* \lambda_\tau^* - d \\ &\quad + b'(0) \int_0^\tau \exp \{ 2\alpha(\cosh \lambda_\tau^* - 1) - (c_\infty^* \lambda_\tau^* + d)a \} f(a) da, \end{aligned}$$

we obtain  $\limsup_{\tau \rightarrow \infty} \Delta_\tau(\lambda_\tau^*, c_\tau^*) = +\infty$ , a contradiction with  $\Delta_\tau(\lambda_\tau^*, c_\tau^*) \equiv 0$ . Therefore,  $\limsup_{\tau \rightarrow \infty} \lambda_\tau^* = \lambda_\infty^* < +\infty$ .

Assuming that  $\lim_{\tau \rightarrow \infty} \lambda_\tau^* = \lambda_\infty^*$  (taking a subsequence if necessary), then one has

$$\Delta(\lambda_\infty^*, c_\infty^*) = 0 \quad \text{and} \quad \frac{\partial}{\partial \lambda} \Delta(\lambda_\infty^*, c_\infty^*) = 0.$$

By the uniqueness of the solution of  $\Delta(\lambda, c) = 0$  and  $\frac{\partial}{\partial \lambda} \Delta(\lambda, c) = 0$ , we obtain  $(\lambda_\infty^*, c_\infty^*) = (\lambda^*, c^*)$ . This completes the proof.  $\square$

The following theorem shows that  $c^*$  is the spreading speed of equation (1.1) with initial data having compact support.

**Theorem 3.3** *Assume that  $(H_b^2)$  holds. Let  $W(t) = \{w_k(t)\}_{k \in \mathbb{Z}}$  be the solution of equation (1.1) with  $W^0(t) = \{w_k^0(t)\}_{k \in \mathbb{Z}}$ , where  $w_k^0(t) \in C((-\infty, 0], [0, w^*])$ . Then:*

(i) *if there exist  $M > 0$  and  $N \in \mathbb{N}$  such that  $w_k^0(t) = 0$  for any  $t \in (-\infty, -M]$  and  $|k| > N$ , then*

$$\lim_{t \rightarrow \infty, |k| \geq ct} w_k(t) = 0 \quad \text{for any } c > c^* ;$$

(ii) *if  $W^0(t) \not\equiv 0$  for any  $t \in (-\infty, 0]$ , then  $\lim_{t \rightarrow \infty, |k| \leq ct} w_k(t) = w^*$  for any  $0 < c < c^*$ .*

**Proof** (i) It is easy to verify that  $\Delta(\lambda, c) < 0$  if  $c > c^* > 0$  and  $\lambda > 0$ . For  $z = 1$  or  $-1$  and some  $\beta > 0$ , we define  $\bar{W}(t) = \{\bar{w}_k(t)\}_{k \in \mathbb{Z}}$ , where  $\bar{w}_k(t) := \min \{w^*, \beta e^{\lambda(ct-zk)}\}$ . Since  $b(w) \leq b'(0)w$  for any  $w > 0$ , we obtain

$$\begin{aligned} &\frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \left[ \int_0^\infty \beta_\alpha(l) b(\bar{w}_{k+l}(t-a)) e^{-da} f(a) da \right] \\ &\leq \frac{b'(0)}{2\pi} \sum_{l=-\infty}^{\infty} \left[ \int_0^\infty \beta_\alpha(l) \bar{w}_{k+l}(t-a) e^{-da} f(a) da \right]. \end{aligned}$$

Let  $u_k(t) := \beta e^{\lambda(ct-zk)}$ ; combining with  $\beta_\alpha(l) = \beta_\alpha(-l)$ , it is not difficult to prove that  $U(t) = \{u_k(t)\}_{k \in \mathbb{Z}}$  is an upper solution of the following linearized equation:

$$\begin{aligned} \frac{du_k(t)}{dt} = & D[u_{k+1}(t) + u_{k-1}(t) - 2u_k(t)] - du_k(t) + \\ & \frac{b'(0)}{2\pi} \sum_{l=-\infty}^{\infty} \left[ \int_0^\infty \beta_\alpha(l) u_{k+l}(t-a) e^{-da} f(a) da \right]. \end{aligned}$$

As a consequence, we have  $e^{-\delta t} w_k^0(0) + \int_0^t e^{-\delta(t-s)} (\mathcal{A}_k(\bar{w})(s) + \mathcal{B}_k(\bar{w})(s)) ds \leq \beta e^{\lambda(ct-zk)}$ . In addition, it is obvious that  $e^{-\delta t} w_k^0(0) + \int_0^t e^{-\delta(t-s)} (\mathcal{A}_k(\bar{w})(s) + \mathcal{B}_k(\bar{w})(s)) ds \leq w^*$ . Therefore,  $\bar{W}(t) = \{\bar{w}_k(t)\}_{k \in \mathbb{Z}}$  is an upper solution of equation (1.1).

For  $c > c^*$ , choose  $\bar{c} \in (c^*, c)$ . In our previous paper [25, Lemma 3.2], there existed  $\bar{\lambda} > 0$  such that  $\Delta(\bar{\lambda}, \bar{c}) < 0$ . Since  $w_k^0(t)$  has compact support, we choose  $\beta > 0$  large enough such that  $w_k^0(t) \leq \beta e^{\bar{\lambda}(\bar{c}t-zk)}$ ,  $\forall k \in \mathbb{Z}, t \in (-\infty, 0], z = 1$  or  $-1$ . The comparison principle (Theorem 2.4) implies that  $w_k(t) \leq \bar{w}_k(t) \leq \beta e^{\bar{\lambda}(\bar{c}t-zk)}$ ,  $\forall k \in \mathbb{Z}, t \in [0, \infty), z = 1$  or  $-1$ . Let  $z = \frac{|k|}{k}$  and  $k \neq 0$ , and then  $w_k(t) \leq \beta e^{\bar{\lambda}(\bar{c}t-|k|)}$ ,  $\forall k \in \mathbb{Z}, t \in [0, \infty)$ . As above, we obtain  $\lim_{t \rightarrow \infty, |k| \geq ct} w_k(t) = 0$  for  $c > c^*$ .

(ii) Let  $c \in (0, c^*)$ , choosing  $\hat{c} \in (c, c^*)$ . Since  $c_\tau^* \rightarrow c^*$  as  $\tau \rightarrow \infty$ , there exists  $\tau^* > 0$  such that for any  $\tau > \tau^*$  we have  $c_\tau^* \in (\hat{c}, c^*)$ . For any given  $\tau \geq \tau^*$ , let  $\tilde{W}(t; \tau) = \{\tilde{w}_k(t; \tau)\}_{k \in \mathbb{Z}}$  where  $\tilde{w}_k(t; \tau) := \min\{w_k^0(t), w_\tau^*\}, \forall k \in \mathbb{Z}, t \in [-\tau, 0]$ . Notice that  $0 \leq w_k(t) \leq w_\tau^*$  and  $W(t) = \{w_k(t)\}_{k \in \mathbb{Z}}$  is a solution of equation (1.1) with initial data  $W^0$ . Therefore,  $W(t)$  is an upper solution of equation (3.1). From the comparison principle (Theorem 2.4) we have that

$$w_k(t) \geq \tilde{w}_k(t; \tilde{W}^0), \forall k \in \mathbb{Z}, t \in [0, \infty),$$

where  $\tilde{W}(t; \tilde{W}^0) = \{\tilde{w}_k(t; \tilde{W}^0)\}_{k \in \mathbb{Z}}$  is a solution of equation (3.1) with initial data  $\tilde{W}^0(s; \tau)$ . Using Lemma 3.1

(ii), we have  $w^* \geq \lim_{t \rightarrow \infty, |k| \leq ct} w_k(t) \geq \lim_{t \rightarrow \infty, |k| \leq ct} \tilde{w}_k(t; \tilde{W}^0) = w_\tau^*$  for any  $\tau > \tau^*$ . By the arbitrariness of  $\tau$  and

$\lim_{\tau \rightarrow \infty} w_\tau^* = w^*$ , then  $\lim_{t \rightarrow \infty, |k| \leq ct} w_k(t) = w^*$ . This completes the proof. □

Define  $X_\lambda = \{u \in \mathbb{R}^{\mathbb{Z}} : \sup_{k \in \mathbb{Z}} |u_k| e^{-\lambda|k|} < +\infty\}$  with supremum norm  $\|u\|_{X_\lambda} = \sup_{k \in \mathbb{Z}} |u_k| e^{-\lambda|k|}$ , where

$\lambda > 0$  is a fixed constant.

**Theorem 3.4** Assume that  $(H_b^2)$  holds. Let  $W(t) = \{w_k(t)\}_{k \in \mathbb{Z}}$  be the solution of equation (1.1) with  $W^0(t) = \{w_k^0(t)\}_{k \in \mathbb{Z}}$ , where  $w_k^0(t) \in C((-\infty, 0], [0, w^*])$ ,  $W^0(t) \not\equiv 0$  for any  $t \in (-\infty, 0]$ . Then the positive equilibrium  $w^*$  of equation (1.1) is stable under the norm  $\|\cdot\|_{X_\lambda}$ .

**Proof** We only need to show that  $\lim_{t \rightarrow \infty} \|w_k(t) - w^*\|_{X_\lambda} = 0$ . Choosing  $c \in (0, c^*)$ , from Theorem 3.3 (ii) for any  $\varepsilon > 0$  there exists  $T' > 0$  such that

$$\sup_{|k| \leq ct} |w_k(t) - w^*| e^{-\lambda|k|} < \frac{\varepsilon}{2} \text{ for } t > T'.$$

Since  $|w_k(t) - w^*| \leq w^*$ , there exists  $N > 0$  such that  $|w_k(t) - w^*|e^{-\lambda|k|} \leq w^*e^{-\lambda|k|} < \frac{\varepsilon}{2}$  for all  $|k| > N$ , and since there exists  $T'' > 0$  such that  $ct > N$  as  $t > T''$ , we have

$$\sup_{|k| \geq ct} |w_k(t) - w^*|e^{-\lambda|k|} \leq \sup_{|k| > N} |w_k(t) - w^*|e^{-\lambda|k|} < \frac{\varepsilon}{2} \text{ for } t > T''.$$

For any  $\varepsilon > 0$ , let  $T = \max\{T', T''\}$ , and then we obtain

$$\begin{aligned} \|w_k(t) - w^*\|_{X_\lambda} &= \sup_{k \in \mathbb{Z}} |w_k(t) - w^*|e^{-\lambda|k|} \\ &\leq \sup_{|k| \leq ct} |w_k(t) - w^*|e^{-\lambda|k|} + \sup_{|k| > ct} |w_k(t) - w^*|e^{-\lambda|k|} < \varepsilon \end{aligned}$$

for  $t > T$ . This completes the proof. □

### 3.2. Nonmonotone birth function

In this subsection, we consider the spreading speed of solution of initial-value problem (1.1) and (2.1) when  $b$  is nonmonotone. For convenience, we define  $b_\pm(w)$  as follows:

$$b_+(w) := \max_{v \in [0, w]} b(v), \quad b_-(w) := \min_{v \in [w, w_+^*]} b(v).$$

Note that the constructions of  $b_+$  and  $b_-$  appeared in [29] for the first time. After that, similar techniques were also adopted by Fang et al. [10], Ma et al. [22], and Wang and Li [32]. Assume that the birth function  $b$  satisfies the following assumptions:

**(H<sub>b</sub><sup>3</sup>)**  $b : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is local Lipschitz continuous and

- (i)  $b(0) = 0$ ,  $b'(0)\bar{f}(d) > d$ ,  $b''(0)$  exists,  $b(w) \leq b'(0)w$  for  $\forall w > 0$ ;
- (ii)  $\bar{f}(d)b_+(w) = dw$  has a unique positive solution  $w_+^*$ ;
- (iii) there exist positive constants  $\rho \in (0, 1]$ ,  $M_0 > 0$  and  $\eta \in (0, w_+^*)$  such that  $b'(0)w - b(w) < M_0w^{1+\rho}$  and  $b_\pm(w) = b(w)$  for  $w \in (0, \eta)$ .

It is easy to see that  $b_-$  and  $b_+$  are nondecreasing and satisfy  $b_-(w) \leq b(w) \leq b_+(w)$  for  $w \in [0, w_+^*]$ . If  $b$  is nondecreasing, then  $b_\pm = b$  and  $(H_b^3)$  reduces to  $(H_b^2)$ . If  $\bar{f}(d)b(w) = dw$  has a unique positive solution  $w^*$ , then  $\bar{f}(d)b(w) > dw$  for  $0 < w < w^*$ ,  $\bar{f}(d)b(w) < dw$  for  $w > w^*$  and  $(H_b^3)$ (ii) holds.  $(H_b^3)$ (iii) implies that both  $\bar{f}(d)b_-(w) = dw$  and  $\bar{f}(d)b(w) = dw$  have minimum positive solutions in  $[0, w_+^*]$ , denoted by  $w_-^*$  and  $w^*$ , respectively. If  $b$  satisfies  $(H_b^3)$ , then  $b_\pm$  satisfy  $(H_b^2)$ . In the following, we always assume that  $w_-^*$  and  $w^*$  are the minimum positive solutions of  $\bar{f}(d)b_-(w) = dw$  and  $\bar{f}(d)b(w) = dw$  in  $[0, w_+^*]$ , respectively. In particular,  $b'_+(0) = b'_-(0) = b'(0)$ .

We consider the following two auxiliary equations:

$$\begin{aligned} \frac{d}{dt}w_k(t) &= D[w_{k+1}(t) + w_{k-1}(t) - 2w_k(t)] - dw_k(t) \\ &\quad + \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \int_0^{\infty} \beta_\alpha(l)b_+(w_{k+l}(t-a))e^{-da}f(a)da \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \frac{d}{dt}w_k(t) = & D[w_{k+1}(t) + w_{k-1}(t) - 2w_k(t)] - dw_k(t) \\ & + \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \int_0^{\infty} \beta_{\alpha}(l)b_{-}(w_{k+l}(t-a))e^{-da}f(a)da. \end{aligned} \tag{3.3}$$

Applying Theorem 2.2 to equations (3.2) and (3.3), respectively, we obtain that equation (3.2) has a global solution  $W^+(t; \phi) = \{w_{+,k}(t; \phi)\}_{k \in \mathbb{Z}}$  with initial data  $\phi \in C_{[0, w_+^*]}$ , equation (3.3) has a global solution  $W^-(t; \phi) = \{w_{-,k}(t; \phi)\}_{k \in \mathbb{Z}}$  with initial data  $\phi \in C_{[0, w_+^]}$ , and  $0 \leq w_{-,k}(t; \phi) \leq w_{+,k}(t; \phi) \leq w_+^*$  for  $t > 0$ .

Define  $F^{\pm} : \mathcal{C}_{h, w_+^*} \rightarrow X^+$  by  $F^{\pm}(\Phi) = \{F_k^{\pm}(\Phi)\}_{k \in \mathbb{Z}}$ , where

$$F_k^{\pm}(\Phi) = D[\phi_{k+1}(0) + \phi_{k-1}(0)] + \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \int_0^{\infty} \beta_{\alpha}(l)b_{\pm}(\phi_{k+l}(-a))e^{-da}f(a)da$$

and  $\Phi = \{\phi_k\}_{k \in \mathbb{Z}} \in [0, w_+^*]$ . Also define  $S^{\pm}(t, s) = T(t, s) = T(t - s) = e^{(2D+d)t}$ . Notice that  $b_+(w) \geq b(w) \geq b_-(w)$  for  $w \in [0, w_+^*]$  and  $b_{\pm}(\cdot)$  are nondecreasing, by Ruan and Wu [26, Theorem5.1]; we obtain the following theorem:

**Theorem 3.5** *Assume that  $(H_b^3)$  holds. We also assume  $w_{-,k}^0(t), w_k^0(t), w_{+,k}^0(t) \in C((-\infty, 0], [0, w_+^*])$ . If  $w_{-,k}^0(t) \leq w_k^0(t) \leq w_{+,k}^0(t)$  for any  $t \in (-\infty, 0]$ , then the initial-value problem (1.1) with (2.1) has a solution  $\{w_k(t, W^0)\}_{k \in \mathbb{Z}}$  on  $(0, +\infty)$  such that  $w_{-,k}(t, W_-^0) \leq w_k(t, W^0) \leq w_{+,k}(t, W_+^0)$  for  $\forall t > 0, k \in \mathbb{Z}$ , where  $\{w_{-,k}(t, W_-^0)\}_{k \in \mathbb{Z}}$  is a solution of equation (3.3) with initial data  $W_-^0(t) = \{w_{-,k}^0(t)\}$ , and  $\{w_{+,k}(t, W_+^0)\}_{k \in \mathbb{Z}}$  is a solution of equation (3.2) with initial data  $W_+^0(t) = \{w_{+,k}^0(t)\}_{k \in \mathbb{Z}}$ .*

Applying Theorem 3.3 to equations (3.2) and (3.3), respectively, we obtain the following two lemmas:

**Lemma 3.6** *Assume that  $(H_b^3)$  holds. Let  $W_+^0(t) = \{w_{+,k}^0(t)\}_{k \in \mathbb{Z}}$ , where  $w_{+,k}^0(t) \in C((-\infty, 0], [0, w_+^*])$ , and let  $W_+(t) = \{w_{+,k}(t)\}_{k \in \mathbb{Z}}$  be a solution of equation (3.2) with initial data  $W_+^0(t)$ . Then:*

(i) *if there exist  $M > 0$  and  $N \in \mathbb{N}$  such that  $w_{+,k}^0(t) = 0$  for any  $t \in (-\infty, -M]$  and  $|k| > N$ , then*

$$\lim_{t \rightarrow \infty, |k| \geq ct} w_{+,k}(t) = 0 \text{ for any } c > c^* ;$$

(ii) *if  $W_+^0(t) \not\equiv 0$  for any  $t \in (-\infty, 0]$ , then  $\lim_{t \rightarrow \infty, |k| \leq ct} w_{+,k}(t) = w_+^*$  for any  $0 < c < c^*$ .*

**Lemma 3.7** *Assume that  $(H_b^3)$  holds. Let  $W_-^0(t) = \{w_{-,k}^0(t)\}_{k \in \mathbb{Z}}$ , where  $w_{-,k}^0(t) \in C((-\infty, 0], [0, w_+^*])$ , and let  $W_-(t) = \{w_{-,k}(t)\}_{k \in \mathbb{Z}}$  be a solution of equation (3.3) with initial data  $W_-^0(t)$ . Then:*

(i) *if there exist  $M > 0$  and  $N \in \mathbb{N}$  such that  $w_{-,k}^0(t) = 0$  for any  $t \in (-\infty, -M]$  and  $|k| > N$ , then*

$$\lim_{t \rightarrow \infty, |k| \geq ct} w_{-,k}(t) = 0 \text{ for any } c > c^* ;$$

(ii) *if  $W_-^0(t) \not\equiv 0$  for any  $t \in (-\infty, 0]$ , then  $\lim_{t \rightarrow \infty, |k| \leq ct} w_{-,k}(t) = w_-^*$  for any  $0 < c < c^*$ .*

Now we establish the spreading speed of solutions of initial-value problem (1.1) and (2.1).

**Theorem 3.8** Assume that  $(H_b^3)$  holds. Let  $W^0(t) = \{w_k^0(t)\}_{k \in \mathbb{Z}}$ , where  $w_k^0(t) \in C((-\infty, 0], [0, w_+^*])$ , and let  $W(t) = \{w_k(t)\}_{k \in \mathbb{Z}}$  be a solution of equation (1.1) with initial data  $W^0(t)$ . Then:

(i) if there exist  $M > 0$  and  $N \in \mathbb{N}$  such that  $w_k^0(t) = 0$  for any  $t \in (-\infty, -M]$  and  $|k| > N$ , then

$$\lim_{t \rightarrow \infty, |k| \geq ct} w_k(t) = 0 \text{ for any } c > c^* ;$$

(ii) if  $W^0(t) \not\equiv 0$  for any  $t \in (-\infty, 0]$ , then  $w_-^* \leq \liminf_{t \rightarrow \infty, |k| \leq ct} w_k(t) \leq \limsup_{t \rightarrow \infty, |k| \leq ct} w_k(t) \leq w_+^*$  for any  $0 < c < c^*$ .

**Proof** Let  $\Phi(t) = \{\phi_k(t)\}_{k \in \mathbb{Z}}$ , where  $\phi_k(t) = \min \{w_k^0(t), w_-^*\}$ ,  $t \in (-\infty, 0]$ . Theorem 3.5 implies that

$$w_{-,k}(t; \Phi) \leq w_k(t; W^0) \leq w_{+,k}(t; W^0), \forall t > 0, k \in \mathbb{Z}.$$

Consequently, applying Lemma 3.6 and Lemma 3.7 yields the results. This completes the proof. □

**Theorem 3.9** For any  $0 < c < c^*$ , equation (1.1) has no traveling wave solution  $(\phi, c)$  that satisfies  $\phi \in C_{[0, w_+^*]}$  with  $\liminf_{s \rightarrow -\infty} \phi(s) < w_-^*$ .

**Proof** Suppose that for some  $c_0 \in (0, c^*)$ , equation (1.1) has a traveling wave solution  $(\phi, c_0)$  with  $\liminf_{s \rightarrow -\infty} \phi(s) < w_-^*$ . Let  $c_0 < c_1 < c_2 < c^*$ ,  $c_1 < 2c_0$  and  $[c_1 t]$  is the integer part of  $c_1 t$ . If we set  $m(t) := -[c_1 t] + c_0 t$  and  $I_k := [k/c_1, (k+1)/c_1)$  for  $k \geq 0$ , we obtain that  $m(t)$  is continuous and increasing on each  $I_k$ ,  $\sup_{t \in I_k} m(t) > \sup_{t \in I_{k+1}} m(t) > \inf_{t \in I_k} m(t)$ , and  $\liminf_{k \rightarrow \infty} \inf_{t \in I_k} m(t) = -\infty$ . Therefore,

$$\{m(t) : t \geq 0\} = \bigcup_{k=0}^{\infty} \{m(t) : t \in I_k\} = \bigcup_{k=0}^{\infty} [\inf_{t \in I_k} m(t), \sup_{t \in I_k} m(t)) = (-\infty, c_0/c_1].$$

Since  $\phi \not\equiv 0$ , we can find a constant  $\xi_0 \in \mathbb{R}$  such that  $\phi(\xi_0) > 0$ . Choosing  $j_0 \in \mathbb{Z}$  such that  $t_0 := (\xi_0 + j_0)/c_0 \geq 0$ , we define  $\psi := \{\psi_j\}_{j \in \mathbb{Z}}$ , where  $\psi_j(\theta) = \phi(j + c_0(t_0 + \theta))$  for  $\theta \in (-\infty, 0]$ . Then  $\psi_{-j_0}(0) = \phi(\xi_0) > 0$  and hence  $\psi \not\equiv 0$ . Let  $\{w_k(t; \psi)\}_{j \in \mathbb{Z}}$  be a solution of equation (1.1) with initial data  $\psi$ . The existence and uniqueness of the solutions imply that  $w_k(t; \psi) \equiv \phi(j + c_0(t + t_0))$  for  $t \geq 0$ . According to Theorem 3.8 (ii), we have

$$w_-^* \leq \liminf_{t \rightarrow \infty, |k| \leq c_2 t} w_k(t; \psi) = \liminf_{t \rightarrow \infty, |k| \leq c_2 t} \phi(j + c_0(t + t_0)) = \liminf_{t \rightarrow \infty, |k| \leq c_2 t} \phi(j + c_0 t).$$

Letting  $j = -[c_1 t]$ , from the above inequality, we obtain

$$w_-^* \leq \liminf_{t \rightarrow \infty} \phi(-[c_1 t] + c_0 t) = \liminf_{t \rightarrow \infty} \phi(m(t)) = \liminf_{s \rightarrow -\infty} \phi(s) < w_-^*.$$

This is a contradiction. The proof is complete. □

**Remark 3.10** In our previous paper [25], we proved that equation (1.1) has a traveling wave solution if  $c > c^*$ . Using a limiting argument, we get that equation (1.1) has a traveling wave solution if  $c = c^*$ . Combining with Theorem 3.9, we prove that  $c^*$  is also the minimal wave speed; that is, the spreading speed coincides with the minimal wave speed.

### 3.3. Application

In this subsection, we apply our results to a Nicholson blowfly model derived by Gurney et al. [13]. Taking  $b(w) = pwe^{-rw}$  and  $f(a) = \frac{2}{\sqrt{\pi}}e^{-a^2}$ , where  $p$  and  $r$  are positive constants, equation (1.1) becomes

$$\begin{aligned} \frac{dw_k(t)}{dt} = & D[w_{k+1}(t) + w_{k-1}(t) - 2w_k(t)] - dw_k(t) \\ & + \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \left[ \int_0^{\infty} \beta_{\alpha}(l) \left( pw_{k+l}(t-a)e^{-rw_{k+l}(t-a)} \right) e^{-da} \left( \frac{2}{\sqrt{\pi}} e^{-a^2} \right) da \right]. \end{aligned} \quad (3.4)$$

Let

$$\bar{f}(d) = \int_0^{\infty} e^{-da} f(a) da = e^{\frac{d^2}{4}} \left( 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{d}{2}} e^{-x^2} dx \right)$$

and

$$w^* = \frac{1}{r} \ln \left( \frac{p}{d} \bar{f}(d) \right).$$

Using the same arguments as in our previous paper [25, Section 3.3], it is easy to verify that the birth function  $b(w) = pwe^{-rw}$  in the Nicholson blowfly model satisfies assumptions  $(H_b^1)$ ,  $(H_b^2)$  and  $(H_b^3)$  when the parameters are in appropriate ranges. The proof is similar to our previous paper [25, section 3.3] and we omit it here, only listing the results. We consider two cases:

*Case (i):*  $1 < \frac{p}{d} \bar{f}(d) \leq e$ . In this case we can confirm that the assumption  $(H_b^2)$  holds. Consequently, applying Theorem 3.3 and Remark 3.10 yields that the spreading speed for initial-value problem (3.4) and (2.1) exists and it coincides with the minimal wave speed.

*Case (ii):*  $\frac{p}{d} \bar{f}(d) > e$ . Similar to Case (i), in this case we can easily show that the assumption  $(H_b^3)$  holds. Applying Theorem 3.8 and Remark 3.10, we have that the spreading speed for initial-value problem (3.4) and (2.1) exists and it coincides with the minimal wave speed.

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