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Some remarks on distributional chaos for bounded linear operators

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Abstract: The aim of this paper is to study distributional chaos for bounded linear operators. We show that distributional chaos of type $k \in \{1, 2\}$ is an invariant of topological conjugacy between two bounded linear operators. We give a necessary condition for distributional chaos of type 2 where it is possible to distinguish distributional chaos and Li–Yorke chaos. Following this condition, we compare distributional chaos with other well-studied notions of chaos for backward weighted shift operators and give an alternative proof to the one where strong mixing does not imply distributional chaos of type 2 (Martínez-Giménez F, Oprocha P, Peris A. Distributional chaos for operators with full scrambled sets. Math Z 2013; 274: 603–612.). Moreover, we also prove that there exists an invertible bilateral forward weighted shift operator such that it is DC1 but its inverse is not DC2.

Key words: Distributional chaos, operators, weighted shifts, topological conjugacy, strong mixing

1. Introduction and preliminaries

A discrete dynamical system is simply a continuous mapping $f : X \to X$ where X is a complete separable metric space. For $x \in X$, the orbit of x under f is $Orb(f, x) = \{x, f(x), f^2(x), \ldots\}$ where $f^n = f \circ f \circ \cdots \circ f$ is the nth iteration of f.

In 1975, Li and Yorke [13] observed complicated dynamical behavior for the class of interval maps with period 3. This phenomenon is currently known as Li–Yorke chaos. Thereafter, several kinds of chaos were well studied. In this paper we focus on distributional chaos.

Now there are three versions of distributional chaos denoted by DC1, DC2, and DC3 in brief. DC1 was originally introduced in [17], and the generalizations of DC2 and DC3 were introduced in [1, 18].

For any pair $\{x, y\} \subset X$ and any $n \in \mathbb{N}$, define the distributional function $F^n_{xy} : \mathbb{R} \to [0, 1]$,

$$F^n_{xy}(\tau) = \frac{1}{n} Card\{0 \le i \le n-1; d(f^i(x), f^i(y)) < \tau\}.$$

Furthermore, define

$$F_{xy}(\tau) = \liminf_{n\to\infty} F^n_{xy}(\tau),$$

$$F^*_{xy}(\tau) = \limsup_{n\to\infty} F^n_{xy}(\tau).$$

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Both F_{xy} and F_{xy}^* are nondecreasing functions and may be viewed as cumulative probability distributional functions satisfying $F_{xy}(\tau) = F_{xy}^*(\tau) = 0$ for $\tau < 0$. If $F_{xy}^*(t) > F_{xy}(t)$ for all t in an interval, we simply write $F_{xy}^* > F_{xy}$.

Definition 1.1 *A pair $\{x, y\} \subset X$ is called distributionally chaotic of type $k \in \{1, 2, 3\}$ (briefly, DC1, DC2, and DC3, respectively) if it satisfies condition (k) as follows:*

(1)$F_{xy}^* \equiv 1$ and $\exists\ \tau_0 > 0,\ F_{xy}(\tau_0) = 0$.

(2)$F_{xy}^* \equiv 1$ and $F_{xy}^* > F_{xy}$.

(3)$F_{xy}^* > F_{xy}$.

Furthermore, f is called distributionally chaotic of type $k \in \{1, 2, 3\}$ if there exists an uncountable subset $D \subseteq X$ such that each pair of two distinct points is a distributionally chaotic pair of type k. Moreover, D is called a distributionally scrambled set of type k.

Given $A \subseteq \mathbb{N}$, its upper and lower densities are defined by

$$\overline{dens}(A) = \limsup_{n\to\infty} \frac{Card\{A \cap [0, n-1]\}}{n}$$

and

$$\underline{dens}(A) = \liminf_{n\to\infty} \frac{Card\{A \cap [0, n-1]\}}{n},$$

respectively. With these concepts in mind, one can equivalently write

$$F_{xy}^*(\tau) = \overline{dens}\{n \in \mathbb{N}; d(f^n(x), f^n(y)) < \tau\}$$

and

$$F_{xy}(\tau) = \underline{dens}\{n \in \mathbb{N}; d(f^n(x), f^n(y)) < \tau\},$$

for any $x, y \in X$ and any $\tau > 0$.

In this paper, we are interested in the dynamical systems induced by bounded linear operators on Banach spaces. The dynamics of linear operators have been widely studied; one can see the survey [9] or the books [2, 10]. The aim of this paper is to study distributional chaos for bounded linear operators. We show that distributional chaos of type $k \in \{1, 2\}$ is an invariant of topological conjugacy between two bounded linear operators. We also give a necessary condition for DC2 where it is possible to distinguish distributional chaos and Li–Yorke chaos. Finally, we compare distributional chaos with other well-studied notions of chaos for backward weighted shift operators. In particular, we give an alternative proof to the one in [15] where strong mixing does not imply DC2.

Let (X, f) and (Y, g) be two dynamical systems. Recall that f is called topologically conjugate to g if there exists a homeomorphism $h : X \to Y$ such that $g = h \circ f \circ h^{-1}$. We also say that h is a topological conjugacy from f to g. The map f is transitive if for any two nonempty open sets U and V in X there exists an integer $n \geq 1$ such that $f^n(U) \cap V \neq \phi$. It is well known that, in a complete metric space without isolated points, transitivity is equivalent to the existence of dense orbit. The map f is weakly mixing if $(f \times f, X \times X)$ is transitive. The map f is strongly mixing if for any two nonempty open sets U, V in X there exists an integer $m \geq 1$ such that $f^n(U) \cap V \neq \phi$ for every $n \geq m$. f has sensitive dependence on initial conditions (or simply

f is sensitive) if there is a constant $\delta > 0$ such that for any $x \in X$ and any neighborhood U of x there exists a point $y \in U$ such that $d(f^n(x), f^n(y)) > \delta$, where d denotes the metric on X. f is called Devaney chaotic if it is transitive and sensitive, and its periodic points are dense in X. The map f is called Li–Yorke chaotic if there exists an uncountable subset $\Gamma \subseteq X$ such that each pair of two distinct points x and y in Γ is a Li–Yorke chaotic pair, which means

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0 \quad \textit{and} \quad \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0.$$

From now on, we always use $\mathcal{L}(X)$ to denote the collection of all bounded linear operators on the Banach space X. Without confusion, we also use 0 to denote the zero vector of the Banach space X.

2. Distributional chaos for linear operators

First of all, let us see some equivalent descriptions of distributional chaotic operators, in which the linearity of operators plays a major role.

Lemma 2.1 *Let X be a Banach space and $T \in \mathcal{L}(X)$. Let $k \in \{1,2,3\}$. The following conditions are equivalent:*

(I) T is distributionally chaotic of type k;

(II) There exists a vector x in X such that $\{x, 0\}$ is a distributionally chaotic pair of type k.

For continuous maps on compact metric spaces, DC1 and DC2 are invariants of topological conjugacy [18], but not DC3 [1]. Martínez-Giménez et al. showed that uniform DC1 is an invariant of topological uniform conjugacy for bounded linear operators on Banach spaces [14]. In fact, we can do without the additional assumption of uniform continuity for the topological conjugacy.

Theorem 2.2 *Let X and Y be two Banach spaces. Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$. Suppose that T and S are topologically conjugate. If T is distributionally chaotic of type $k \in \{1,2\}$, so is S.*

Proof This proof is similar to the proof of an analogous result for continuous maps on compact metric spaces in [18], which is easy to obtain by the uniform continuity of T and S and the commutative diagram induced by a topological conjugacy from T to S.

Let $f : X \to Y$ be a topological conjugacy from T to S. Without loss of generality we may assume $f(0) = 0$; otherwise, take $h : X \to Y$ defined by $h(x) = f(x) - f(0)$ [12]. Since f is continuous at 0, for any $\tau > 0$ there exists $\delta > 0$ such that for any $x \in X$, $\|x\| < \delta$ implies $\|f(x)\| < \tau$. Then $\|T^n x\| < \delta$ implies $\|f(T^n x)\| = \|S^n f(x)\| < \tau$. Consequently,

$$\overline{dens}\{n \in \mathbb{N}; \|T^n x\| < \delta\} \leq \overline{dens}\{n \in \mathbb{N}; \|S^n f(x)\| < \tau\}. \tag{2.1}$$

Similarly, by the continuity of f^{-1} at 0, for any $\epsilon > 0$ there exists $\mu > 0$ such that for any $x \in X$, $\|f(x)\| < \mu$ implies $\|x\| < \epsilon$ and hence

$$\underline{dens}\{n \in \mathbb{N}; \|S^n f(x)\| < \mu\} \leq \underline{dens}\{n \in \mathbb{N}; \|T^n x\| < \epsilon\}. \tag{2.2}$$

Now let $\{x, 0\}$ be a DC1 pair for T, i.e. $\overline{dens}\{n \in \mathbb{N}; \|T^n x\| < \delta\} = 1$ for any $\delta > 0$ and $\underline{dens}\{n \in \mathbb{N}; \|T^n x\| < \epsilon\} = 0$ for some $\epsilon > 0$. Then it follows from (2.1) and (2.2) that $\overline{dens}\{n \in \mathbb{N}; \|S^n f(x)\| < \tau\} = 1$ for any $\tau > 0$ and $\underline{dens}\{n \in \mathbb{N}; \|S^n f(x)\| < \mu\} = 0$ for some $\mu > 0$. Thus, $\{f(x), 0\}$ is a DC1 pair for S.

Notice that if $\underline{dens}\{n \in \mathbb{N}; \|T^n x\| < \epsilon\} < 1$ then, again by (2.2), $\underline{dens}\{n \in \mathbb{N}; \|S^n f(x)\| < \mu\} < 1$. Therefore, if T is DC2 then S is also DC2. □

In the research on distributionally chaotic operators, there have been several sufficient conditions for DC1 (see [3, 4, 11, 14]). In particular, it is worth noting a powerful sufficient condition for dense distributional chaos given by Bermúdez et al. in [3].

Definition 2.3 *Let X be a Banach space and let $T \in \mathcal{L}(X)$. A vector $x \in X$ is said to be distributionally irregular for T if there are increasing sequences of integers $A = \{n_k\}_k$ and $B = \{m_k\}_k$ such that $\overline{dens}(A) = \overline{dens}(B) = 1$, $\lim_{k\to\infty} T^{n_k}x = 0$, and $\lim_{k\to\infty} T^{m_k}x = \infty$. Moreover, a linear manifold $Y \subseteq X$ is called a distributionally irregular manifold for T if every nonzero vector $y \in Y \setminus \{0\}$ is a distributionally irregular vector for T.*

Lemma 2.4 ([3]) *Let $T : X \to X$ be an operator such that there exist a dense subset $X_0 \subseteq X$ with $\lim_{n\to\infty} T^n x = 0$, for each $x \in X_0$, and an increasing sequence of integers $B = \{m_k\}_k$ with $\overline{dens}(B) = 1$ satisfying*

(i) either $\sum_{k=1}^{\infty} \|T^{m_k}\|^{-1} < \infty$

(ii) or X is a complex Hilbert space and $\sum_{k=1}^{\infty} \|T^{m_k}\|^{-2} < \infty$.

Then T has a dense distributionally irregular manifold.

Wu et al. discussed an invariant DC2-scrambled linear manifold for backward shift on Köthe sequence space in [19]. To describe distributional chaos more precisely, we will give a necessary condition for DC2. Firstly, let us review a result of Downarowicz (see [7]). Let (X, f) be a dynamical system and $x, y \in X$. $\{x, y\}$ a distributionally chaotic pair of type 2 if and only if

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} d(f^i(x), f^i(y)) = 0 \text{ and } \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} d(f^i(x), f^i(y)) > 0.$$

Then we can write the following form for bounded linear operators.

Lemma 2.5 *Let X be a Banach space and let $T \in \mathcal{L}(X)$. T is DC2 if and only if there exists a vector x in X such that*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \|T^i(x)\| = 0 \text{ and } \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \|T^i(x)\| > 0.$$

Recall that a bounded linear operator T in a Banach space X is power bounded when $\sup_{n\geq 1} \|T^n\| < \infty$. It is said to be Cesàro bounded when

$$\sup_{n\geq 1} \frac{1}{n} \|\sum_{i=0}^{n-1} T^i\| < \infty.$$

Inspired by Lemma 2.5, we introduce a new definition of boundedness called absolutely Cesàro boundedness.

Definition 2.6 *Let X be a Banach space and let $T \in \mathcal{L}(X)$. T is said to be absolutely Cesàro bounded if there exists a constant C such that for every $x \in X$,*

$$\sup_{n\geq 1} \frac{1}{n}\sum_{i=1}^{n} \|T^i(x)\| \leq C\|x\|.$$

Otherwise, T is said to be absolutely Cesàro unbounded. Moreover, C is called an absolutely Cesàro upper bound of T.

Theorem 2.7 *Let X be a Banach space and let $T \in \mathcal{L}(X)$. Suppose that the set $\{x \in X : \lim_{n\to\infty} \|T^n x\| = 0\}$ is dense in X. If T is DC2, then T must be absolutely Cesàro unbounded.*

Proof Suppose that T is absolutely Cesàro bounded. According to Lemma 2.5, it suffices to prove

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \|T^i(x)\| = 0 \ for \ every \ x \in X,$$

which is in contradiction to T being DC2.

Let C be an absolutely Cesàro upper bound of T. Given any $x \in X$, for any $\epsilon > 0$, there exists $y \in X$ such that

$$\|x - y\| < \epsilon/C \quad and \quad \limsup_{n\to\infty} \|T^n y\| = 0.$$

Then

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \|T^i(x)\| &\leq \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \|T^i(x-y)\| + \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \|T^i(y)\| \\ &\leq C\|x-y\| + 0 \\ &< \epsilon.\end{aligned}$$

□

3. Remarks on distributional chaos for weighted shifts

In this section, we restrict our attention to weighted shift operators on l^p for $1 \leq p < \infty$. Here l^p is the classical Banach space of absolutely p^{th} power summable sequences $x = (x_1, x_2, \ldots)$ (or $x = (\ldots, x_{-1}, \underline{x_0}, x_1, \ldots)$). Let $\{\omega_n\}_{n=1}^{\infty}$ be a bounded sequence of nonzero complex numbers. A unilateral backward weighted shift operator T with weight sequence $\{\omega_n\}_{n=1}^{\infty}$ on l^p is defined by

$$T(x_1, x_2, \ldots) = (\omega_1 x_2, \omega_2 x_3, \ldots)$$

for any $x = (x_1, x_2, \ldots) \in l^p$.

Similarly, a bilateral backward (or forward) weighted shift operator T with weight sequence $\{\omega_n\}_{-\infty}^{\infty}$ on l^p is defined by

$$T(\ldots, x_{-1}, \underline{x_0}, x_1, \ldots) = (\ldots, \omega_{-1}x_0, \underline{\omega_0 x_1}, \omega_1 x_2, \ldots)$$

$$(or \quad T(\ldots, x_{-1}, \underline{x_0}, x_1, \ldots) = (\ldots, \omega_{-1}x_{-2}, \underline{\omega_0 x_{-1}}, \omega_1 x_0, \ldots))$$

for any $(\ldots, x_{-1}, \underline{x_0}, x_1, \ldots) \in l^p$.

Let us review some equivalent descriptions of some dynamical properties and their relations for weighted shift operators.

Proposition 3.1 *Suppose T is a backward weighted shift operator on l^p, $1 \le p < \infty$, with weight sequence $\{\omega_n\}_{n=1}^{\infty}$. Denote $\beta(n)$ by*

$$\beta(n) = \prod_{i=1}^{n} \omega(i), \quad for \quad n = 1, 2, \ldots \quad .$$

Then:

(I) ([8]) T is Devaney chaotic if and only if $\sum_{n=1}^{\infty} |\beta(n)|^{-p} < \infty$;

(II) ([5]) T is strongly mixing if and only if $\lim_{n\to\infty} |\beta(n)| = \infty$.

(III) ([16]) T is transitive if and only if T is weakly mixing, if and only if $\limsup_{n\to\infty} |\beta(n)| = \infty$.

(IV) ([12, 14]) T is sensitive if and only if T is Li–Yorke chaotic, if and only if $\sup_{n\ge 1} \|T^n\| = \infty$.

Proposition 3.2 ([14]) *Devaney chaos implies DC1 for backward weighted shift operators.*

Remark 3.3 *The original proof was given by Martínez-Giménez et al. in [14]. By Theorem 2.7 and Proposition 3.1, we also can obtain this conclusion.*

Proposition 3.4 ([14]) *DC1 does not imply transitivity for backward weighted shift operators.*

Distributional chaos of type $k \in \{1, 2\}$ obviously implies Li–Yorke chaos. In contrast, it is not easy to say an operator is distributionally chaotic of type $k \in \{1, 2\}$ or not when it is Li–Yorke chaotic (transitive or mixing). Martínez-Giménez et al. constructed a sequence of weights such that the backward weighted shift operator with these weights on l^p for any $1 \le p < \infty$ is strongly mixing but not DC3 (see [15]). By Theorem 2.7, we can also prove that strong mixing does not imply DC2.

Theorem 3.5 *Strong mixing does not imply DC2 for backward weighted shift operators.*

Proof Let T be a unilateral backward weighted shift operator on l^1 with weight sequence

$$\omega_n = \frac{2n}{2n-1}, \quad for \quad n = 1, 2, \ldots.$$

For convenience, let $\omega_n = 0$ for $n \le 0$. We will show that T is strongly mixing but not DC2.

Since

$$\lim_{n\to\infty} |\beta(n)| = \lim_{n\to\infty} \frac{(2n)!!}{(2n-1)!!} = \infty,$$

T is strongly mixing by (II) in Proposition 3.1.

Notice that the set $\{x : \lim_{n\to\infty} \|T^n x\| = 0\}$ is dense in l^1. To complete this proof, by Theorem 2.7, it suffices to prove the absolutely Cesàro boundedness of T. For any $x = (x_1, x_2, \ldots) \in l^1$. Since the sequence

$\{\omega_n\}_{n=1}^{\infty}$ is decreasing and greater than 1,

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\|T^i(x)\| \\
=&\frac{1}{n}\sum_{i=1}^{n}\sum_{k=1}^{\infty}|\omega_{k-1}\ldots\omega_{k-i}||x_k| \\
=&\frac{1}{n}\sum_{k=1}^{\infty}(\sum_{i=1}^{n}|\omega_{k-1}\ldots\omega_{k-i}|)|x_k| \\
\leq&\frac{1}{n}\sum_{k=1}^{\infty}(\sum_{i=1}^{n}|\omega_{n}\ldots\omega_{n+1-i}|)|x_k| \\
=&\frac{1}{n}[\frac{2n}{2n-1}+\frac{(2n)(2n-2)}{(2n-1)(2n-3)}+\ldots+\frac{(2n)(2n-2)\ldots 2}{(2n-1)(2n-3)\ldots 1}]\sum_{k=1}^{\infty}|x_k| \\
=&2\|x\|.
\end{aligned}$$

□

Remark 3.6 *In fact, for each $1 \leq p < \infty$ we can get a backward weighted shift operator on l^p such that it is topologically conjugate to the above operator T. For any $1 \leq p, q < \infty$, define $g_{p,q} : \mathbb{C} \to \mathbb{C}$ by*

$$g_{p,q}(0) = 0 \quad \text{and} \quad g_{p,q}(z) = \frac{z}{|z|} \cdot |z|^{\frac{p}{q}} \ \textit{for}\ z \neq 0.$$

Furthermore, define $G_{p,q} : l^p \to l^q$ by

$$G_{p,q}(x_1, x_2, \ldots) = (g_{p,q}(x_1), g_{p,q}(x_2), \ldots).$$

As is well known, $G_{p,q}$ is a natural homeomorphism from l^p onto l^q (see [6]).

Moreover, if A is a backward weighted shift operator on l^p with weight sequence $\{\omega_n\}$ and B is a backward weighted shift operator on l^q with weight sequence $\{g_{p,q}(\omega_n)\}$, then $G_{p,q}$ is a topological conjugacy from A onto B. Since DC2 and strong mixing are topologically conjugate invariants, it is not difficult to obtain a backward weighted shift operator on l^p that is strongly mixing but not DC2. For instance, denote S by a backward weighted shift operator on l^2 with weight sequence $\{\omega_n = \sqrt{2n/(2n-1)}\}$. Then $G_{1,2}$ is a topological conjugacy from above T to S and consequently S is strongly mixing but not DC2.

Similar to the construction in the proof of Theorem 3.5, we may obtain an invertible bilateral forward weighted shift operator such that it is DC1 but its inverse is not DC2.

Theorem 3.7 *There exists an invertible bilateral forward weighted shift operator T such that T is DC1 but T^{-1} is not DC2.*

Proof Let T be a bilateral forward weighted shift operator on l^1 with weight sequence

$$\omega_n == \begin{cases} \frac{2n-1}{2n}, & \text{for} \quad n \geqslant 1. \\ 2, & \text{for} \quad n < 1. \end{cases}$$

By Lemma 2.4, we obtain that T is DC1. Notice that T^{-1} is a bilateral backward weighted shift operator on l^1 with weight sequence

$$\lambda_n = \frac{1}{\omega_{n-1}}, \quad for \quad n \in \mathbb{Z}.$$

Following from an estimation similar to what we have done in the proof of Theorem 3.5, one can see that T^{-1} is absolutely Cesàro bounded and hence T^{-1} is not DC2. □

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