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The fundamental theorems of algebroid functions on annuli

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Abstract: An extension of Nevanlinna value distribution theory for algebroid functions on annuli is proposed. The main characteristics are one-parameter and possess the same properties as in the classical case. Analogs of the Cartan theorem, the first fundamental theorem, the second fundamental theorem, deficient values, and the uniqueness of algebroid functions on annuli are proved.

Key words: Nenanlinna theory, the first fundamental theorem, the second fundamental theorem, the Cartan theorem, deficient values and the uniqueness of algebroid functions on annuli

1. Introduction

Several problems lead us to study meromorphic functions in multiply connected domains. In particular, considering the composition $f \circ R$, f is transcendental meromorphic in $\mathbb{C}$ and a rational function R with $n-1$ distinct poles in $\mathbb{C}$. We obtain a meromorphic function in an n-connected domain. Many authors have studied meromorphic functions in multiply connected domains and generalized the Nevanlinna theory. In particular, they have generalized the first fundamental theorem, the second fundamental theorem, and other important theorems in doubly connected domains [2,3] similar to these theories on plane [4–6,8,9,12,14].

As the extension of meromorphic functions, we have many wonderful achievements on algebroid functions [7,10,11,13,15–18,20]. Naturally, we have this question: whether these wonderful achievements of algebroid functions on plane can be generalized to multiply connected domains. We want to answer this question. In this paper, we mainly study doubly connected domains. By the doubly connected mapping theorem [1] each doubly connected domain is conformally equivalent to the annulus $\mathbb{A}(R_1,R_2)=\{z : R_1<|z|<R_2\}, 0\le R_1<R_2\le +\infty$. We only consider two cases:

$$(1)\quad R_1=0, R_2=+\infty$$

$$(2)\quad 0<R_1<R_2<+\infty$$

In the latter case the homothety $z \longmapsto \frac{z}{\sqrt{R_1R_2}}$ reduces the given domain to the annulus $\{z : \frac{1}{R_0}<|z|<R_0\}$, where $R_0=\sqrt{\frac{R_2}{R_1}}$. Thus, in two cases every annulus is invariant with respect to the inversion $z \longmapsto \frac{1}{z}$.

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In this paper, we mainly introduce the Nevanlinna characteristics of algebroid functions, study their properties, and prove the first fundamental theorem, the second fundamental theorem, and the uniqueness theorem of value distribution theory on annuli.

Let $A_v(z), A_{v-1}(z), \cdots, A_0(z)$ be a group of holomorphic functions that have no common zeros and define on the annulus $\mathbb{A}(\frac{1}{R_0}, R_0)(1 < R_0 \leq +\infty)$,

$$\psi(z, W) = A_v(z)W^v + A_{v-1}(z)W^{v-1} + \cdots + A_1(z)W + A_0(z) = 0. \tag{1.1}$$

Then the irreducible equation (1.1) defines a v-valued algebroid function on the annulus $\mathbb{A}(\frac{1}{R_0}, R_0)(1 < R_0 \leq +\infty)$.

The following example illustrates the existence of an algebroid function on the annulus $\mathbb{A}(R_1, R_2) = \{z : R_1 < |z| < R_2\}, 0 \leq R_1 < R_2 \leq +\infty$.

Example 1 *Algebroid function defines on the annulus* $\mathbb{A}(R_1, R_2) = \{z : R_1 < |z| < R_2\}, 0 \leq R_1 < R_2 \leq +\infty$,

$$e^{\frac{1}{z-R_1} + \frac{1}{z-R_2}} W^2 + z = 0.$$

Let $W(z)$ *be a* v*-valued algebroid function on the annulus* $\mathbb{A}(\frac{1}{R_0}, R_0)(1 < R_0 \leq +\infty)$*; we use the notations:*

$$m(r, W) = \frac{1}{v}\sum_{j=1}^{v} m(r, w_j) = \frac{1}{v}\sum_{j=1}^{v} \frac{1}{2\pi}\int_0^{2\pi} \log^+ |w_j(re^{i\theta})| d\theta.$$

$$N_1(r, W) = \frac{1}{v}\int_{\frac{1}{r}}^{1} \frac{n_1(t, W)}{t} dt, \quad N_2(r, W) = \frac{1}{v}\int_1^r \frac{n_2(t, W)}{t} dt.$$

$$N_{x_1}(r, W) = \frac{1}{v}\int_{\frac{1}{r}}^{1} \frac{n_{x_1}(t, W)}{t} dt, \quad N_{x_2}(r, W) = \frac{1}{v}\int_1^r \frac{n_{x_2}(t, W)}{t} dt.$$

$$m_0(r, W) = m(r, W) + m(\frac{1}{r}, W) - 2m(1, W), \quad N_0(r, W) = N_1(r, W) + N_2(r, W),$$

$$N_x(r, W) = N_{x_1}(r, W) + N_{x_2}(r, W).$$

where $w_j(z)(j = 1, 2 \cdots v)$ *is a one-valued branch of* $W(z)$, $n_1(t, W)$ *is the counting function of poles of the function* $W(z)$ *in* $\{z : t < |z| \leq 1\}$ *and* $n_2(t, W)$ *is its counting function of poles in* $\{z : 1 < |z| \leq t\}$ *(both counting multiplicity), and* $n_{x_1}(t, W)$ *and* $n_{x_2}(t, W)$ *are the counting function of branch points of the function* $W(z)$ *in* $\{z : t < |z| \leq 1\}$ *and* $\{z : 1 < |z| \leq t\}$ *respectively.* $N_x(r, W)$ *is the density index of branch point of* $W(z)$ *on the annulus* $\mathbb{A}(\frac{1}{R_0}, R_0)(1 < R_0 \leq +\infty)$.

Let $W(z)$ be an algebroid function on the annulus $\mathbb{A}(\frac{1}{R_0}, R_0)(1 < R_0 \leq +\infty)$; if there are λ branches of $W(z)$ that take $a(\neq \infty)$ as the value in z_0 point, then we have the fractional power series,

$$W(z) = a + b_\tau(z - z_0)^{\frac{\tau}{\lambda}} + b_{\tau+1}(z - z_0)^{\frac{\tau+1}{\lambda}} + \cdots . \tag{1.2}$$

$n_0(r,a) = n_0(r,\frac{1}{W-a}) = \sum_{W=a} \tau$, where $n_0(r,a)$ is the counting function of zeros of $W(z)-a$ on the annulus $\mathbb{A}(\frac{1}{R_0},R_0)(1<R_0\le+\infty)$ (counting multiplicity). If there are λ branches of $W(z)$ that take ∞ as the value in z_0 point, then we have the fractional power series

$$W(z) = b_{-\tau}(z-z_0)^{-\frac{\tau}{\lambda}} + b_{-\tau+1}(z-z_0)^{\frac{-\tau+1}{\lambda}} + \cdots. \tag{1.3}$$

$n_0(r,\infty) = n_0(r,W) = \sum_{W=\infty} \tau$, where $n_0(r,\infty)$ is the counting function of poles of $W(z)$ on the annulus $\mathbb{A}(\frac{1}{R_0},R_0)(1<R_0\le+\infty)$ (counting multiplicity). $z=z_0$ is a branch point of $\lambda-1$ degree of $W(z)$ on its Riemann surface $\widetilde{\mathscr{M}}$. $n_x(r,W)=\Sigma(\lambda-1)$ denotes the branch points of $W(z)$ on its Riemann surface on the annulus $\mathbb{A}(\frac{1}{R_0},R_0)(1<R_0\le+\infty)$.

Let $W(z)$ be a v-valued algebroid function determined by (1.1) on the annulus $\mathbb{A}(\frac{1}{R_0},R_0)(1<R_0\le+\infty)$, when $a\in\mathbb{C}$, $n_0(r,\frac{1}{W-a}) = n_0(r,\frac{1}{\psi(z,a)})$, $N_0(r,\frac{1}{W-a}) = \frac{1}{v}N_0(r,\frac{1}{\psi(z,a)})$. In particular, when $a=0$, $N_0(r,\frac{1}{W}) = \frac{1}{v}N_0(r,\frac{1}{A_0})$. When $a=\infty$, $N_0(r,W)=\frac{1}{v}N_0(r,\frac{1}{A_v})$, where $n_0(r,\frac{1}{W-a})$ and $n_0(r,\frac{1}{\psi(z,a)})$ are the counting function of zeros of $W(z)-a$ and $\psi(z,a)$ on the annulus $\mathbb{A}(\frac{1}{R_0},R_0)(1<R_0\le+\infty)$.

Definition 1 *Let $W(z)$ be an algebroid function on the annulus $\mathbb{A}(\frac{1}{R_0},R_0)(1<R_0\le+\infty)$, the function*

$$T_0(r,W) = m_0(r,W) + N_0(r,W), \quad 1\le r<R_0$$

is called the Nevanlinna characteristic of $W(z)$.

2. Some lemmas

Lemma 1 (2) *Let f be a nonconstant meromorphic function on the annulus $\mathbb{A}(\frac{1}{R_0},R_0)(1<R_0\le+\infty)$, then*

$$T_0(r,f) = T_0(r,\frac{1}{f}), \quad T_0(r,f(z)) = T_0(r,f(\frac{1}{z})),$$

where $1\le r<R_0$.

Lemma 2 (3) *(Jensen theorem for meromorphic function on annuli) Let f be a nonconstant meromorphic function on the annulus $\mathbb{A}(\frac{1}{R_0},R_0)(1<R_0\le+\infty)$, then*

$$N_0(r,\frac{1}{f}) - N_0(r,f) = \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|d\theta + \frac{1}{2\pi}\int_0^{2\pi}\log|f(\frac{1}{r}e^{i\theta})|d\theta - \frac{1}{\pi}\int_0^{2\pi}\log|f(e^{i\theta})|d\theta,$$

where $1\le r<R_0$.

Lemma 3 *Let $W(z)$ be a v-valued algebroid function determined by (1.1) on the annulus $\mathbb{A}(\frac{1}{R_0},R_0)(1<R_0\le+\infty)$, then*

$$N_x(r,W) \le 2(v-1)T_0(r,W) + O(1).$$

Proof Firstly, we have

$$n_x(r,W) \le n_0(r,\frac{1}{J(z)}). \tag{2.1}$$

$J(z) = |A_v(z)|^{2(v-1)} \prod_{1\le j<k\le v} [w_j(z) - w_k(z)]^2$, $J(z)$ is the discriminant of $W(z)$. According to the higher algebra, we know it can be written in another form, that is $A_v(z)J(z) = (-1)^{\frac{v(v-1)}{2}} R(\psi, \psi_W)$, $R(\psi,\psi_W)$ is the resultant of $\psi(z,W)$ and $\psi_W(z,W)$. $n_0(r,\frac{1}{J(z)})$ denotes the counting function of zeros of $J(z)$ on the annulus, $J(z)$ can be expressed as the determinant of $A_v(z),\cdots A_0(z)$, that is

$$J(z) = (-1)^{\frac{v(v-1)}{2}} \begin{vmatrix} 1 & A_{v-1}(z) & A_{v-2}(z) & \cdots & A_0(z) & 0 & \cdots & 0 \\ 0 & A_v(z) & A_{v-1}(z) & \cdots & A_1(z) & A_0(z) & \cdots & 0 \\ \vdots & \vdots & & \vdots & & & & \\ 0 & 0 & 0 & \cdots & A_v(z) & A_{v-1}(z) & \cdots & A_0(z) \\ v & (v-1)A_{v-1}(z) & (v-2)A_{v-2}(z) & \cdots & 0 & 0 & \cdots & 0 \\ 0 & vA_v(z) & (v-1)A_{v-1}(z) & \cdots & A_1(z) & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & & & \\ 0 & 0 & 0 & \cdots & 0 & vA_v(z) & \cdots & A_1(z) \end{vmatrix}$$

Thus, from above determinant we know that $J(z)$ is holomorphic function on the annulus. In fact, by (1.2), if there are λ branches of $W(z)$ that take $a \in \mathbb{C}$ as the value in z_0 point, then there are $\frac{\lambda(\lambda-1)}{2}$ items including the factor $(z-z_0)^{\frac{2\tau}{\lambda}}$ in $J(z)$ (τ is the multiplicity of zero), that is: z_0 is a zero of $J(z)$, the multiplicity of z_0 is $\frac{\lambda(\lambda-1)}{2} \cdot \frac{2\tau}{\lambda} = \tau(\lambda-1) \ge (\lambda-1)$ at least. That is to say, the branch points of $\lambda-1$ degree of $W(z)$ are zeros of $\lambda-1$ degree of $J(z)$ at least. So (2.1) is true. By substituting $z = re^{i\theta}, z = \frac{1}{r}e^{i\theta}, z = e^{i\theta}$ into $J(z)$, using Jensen theorem for meromorphic function on annuli, we have

$$\begin{aligned} N_0(r,\frac{1}{J(z)}) =& \frac{1}{2\pi}\int_0^{2\pi} \log|J(re^{i\theta})|d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log|J(\frac{1}{r}e^{i\theta})|d\theta - 2\times\frac{1}{2\pi}\int_0^{2\pi}\log|J(e^{i\theta})|d\theta \\ =& \frac{2(v-1)}{2\pi}\int_0^{2\pi}\log|A_v(re^{i\theta})|d\theta + \frac{2(v-1)}{2\pi}\int_0^{2\pi}\log|A_v(\frac{1}{r}e^{i\theta})|d\theta \\ & - 2\times\frac{2(v-1)}{2\pi}\int_0^{2\pi}\log|A_v(e^{i\theta})|d\theta + \frac{1}{2\pi}\int_0^{2\pi}\log|\prod_{1\le j<k\le v}[w_j(re^{i\theta}) - w_k(re^{i\theta})]^2|d\theta \\ & + \frac{1}{2\pi}\int_0^{2\pi}\log|\prod_{1\le j<k\le v}[w_j(\frac{1}{r}e^{i\theta}) - w_k(\frac{1}{r}e^{i\theta})]^2|d\theta \\ & - 2\times\frac{1}{2\pi}\int_0^{2\pi}\log|\prod_{1\le j<k\le v}[w_j(e^{i\theta}) - w_k(e^{i\theta})]^2|d\theta \end{aligned}$$

$$
\begin{aligned}
\leq & 2(v-1)N_0(r,\frac{1}{A_v})+2(v-1)\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log^+|w_j(re^{i\theta})|d\theta \\
& +2(v-1)\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log^+|w_j(\frac{1}{r}e^{i\theta})|d\theta \\
& -2\times 2(v-1)\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log^+|w_j(e^{i\theta})|d\theta+O(1) \\
= & 2v(v-1)N_0(r,W)+2v(v-1)m_0(r,W)+O(1) \\
= & 2v(v-1)T_0(r,W)+O(1).
\end{aligned}
$$

So we have

$$N_x(r,W)\leq\frac{1}{v}N_0(r,\frac{1}{J(z)})\leq 2(v-1)T_0(r,W)+O(1).$$

□

Lemma 4 (19) *Let a be a finite complex number; then*

$$\log^+|a|=\frac{1}{2\pi}\int_0^{2\pi}\log|e^{i\varphi}-a|d\varphi.$$

3. Main results

Theorem 1 *(the first fundamental theorem on annuli) Let $W(z)$ be a v-valued algebroid function determined by (1.1) on the annulus $\mathbb{A}(\frac{1}{R_0},R_0)(1<R_0\leq+\infty)$, $a\in\mathbb{C}$*

$$m_0(r,a)+N_0(r,a)=T_0(r,W)+O(1).$$

Theorem 2 *(the second fundamental theorem on annuli) Let $W(z)$ be a v-valued algebroid function determined by (1.1) on the annulus $\mathbb{A}(\frac{1}{R_0},R_0)(1<R_0\leq+\infty)$, $a_k(k=1,2\cdots p)$ are p distinct complex numbers (finite or infinite), then we have*

$$(p-2v)T_0(r,W)\leq\sum_{k=1}^{p}N_0(r,\frac{1}{W-a_k})-N_1(r,W)+S_0(r,W). \tag{3.1}$$

$N_1(r,W)$ is the density index of all multiple values including finite or infinite, every τ multiple value counts $\tau-1$, and

$$S_0(r,W)=m_0(r,\frac{W'}{W})+\sum_{k=1}^{p}m_0(r,\frac{W'}{W-a_k})+O(1).$$

Theorem 3 *(Cartan theorem on annuli) Let $W(z)$ be a v-valued algebroid function determined by (1.1) on the annulus $\mathbb{A}(\frac{1}{R_0}, R_0)(1 < R_0 \leq +\infty)$; then we have*

$$T_0(r, W) = \frac{1}{2\pi v} \int_0^{2\pi} N_0(r, \frac{1}{\psi(z, e^{i\alpha})}) d\alpha. \tag{3.2}$$

Remark 1 *Let $A(z) = \max\limits_{0 \leq j \leq v} \{|A_j(z)|\}$,*

$$\mu(r, A) = \frac{1}{2\pi v} \int_0^{2\pi} \log A(re^{i\theta}) d\theta + \frac{1}{2\pi v} \int_0^{2\pi} \log A(\frac{1}{r} e^{i\theta}) d\theta - 2 \times \frac{1}{2\pi v} \int_0^{2\pi} \log A(e^{i\theta}) d\theta.$$

Theorem 4 *Let $W(z)$ be a v-valued algebroid function determined by (1.1) on the annulus $\mathbb{A}(\frac{1}{R_0}, R_0)(1 < R_0 \leq +\infty)$; then*

$$|T_0(r, W) - \mu(r, A)| \leq O(1).$$

Theorem 5 *Let $W(z)$ be a v-valued algebroid function determined by (1.1) on the annulus $\mathbb{A}(\frac{1}{R_0}, R_0)(1 < R_0 \leq +\infty)$. If the following conditions are satisfied*

$$\varliminf_{r \to \infty} \frac{T_0(r, W)}{\log r} < \infty, \qquad R_0 = +\infty,$$

$$\varliminf_{r \to R_0^-} \frac{T_0(r, W)}{\log \frac{1}{(R_0 - r)}} < \infty, \qquad R_0 < +\infty,$$

then $W(z)$ is an algebraic function.

Remark 2 *Now let $W(z)$ be a v-valued algebroid function determined by (1.1) on the annulus $\mathbb{A}(\frac{1}{R_0}, R_0)(1 < R_0 \leq +\infty)$ and $\widehat{W}(z)$ be a μ-valued algebroid function determined by the following equation on the annulus $\mathbb{A}(\frac{1}{R_0}, R_0)(1 < R_0 \leq +\infty)$:*

$$\varphi(z, \widehat{W}) = B_\mu(z) \widehat{W}^\mu + B_{\mu-1}(z) \widehat{W}^{\mu-1} + \cdots + B_1(z) \widehat{W} + B_0(z) = 0. \tag{3.3}$$

Without loss of generality, let $\mu \leq v$, $\overline{n}_\triangle(r, a)$ denotes the counting function of the common values of $W(z) = a$ and $\widehat{W}(z) = a$ on the annulus $\mathbb{A}(\frac{1}{R_0}, R_0)(1 < R_0 \leq +\infty)$, ignoring multiplicity. And let

$$\overline{N}_\triangle(r, a) = \frac{\mu + v}{2\mu v} \int_{\frac{1}{r}}^1 \frac{\overline{n}_{\triangle_1}(t, a)}{t} dt + \frac{\mu + v}{2\mu v} \int_1^r \frac{\overline{n}_{\triangle_2}(t, a)}{t} dt.$$

$$\overline{N}_{12}(r, a) = \overline{N}_0(r, \frac{1}{W - a}) + \overline{N}_0(r, \frac{1}{\widehat{W} - a}) - 2\overline{N}_\triangle(r, a).$$

Theorem 6 *Let $W(z)$ and $\widehat{W}(z)$ be v-valued and μ-valued algebroid functions on the annulus $\mathbb{A}(\frac{1}{R_0}, R_0)(1 < R_0 \leq +\infty)$, respectively, and $\mu \leq v$. There are $4v + 1$ distinct complex numbers $a_j (j = 1, 2, \cdots 4v + 1)$, if $W(z)$ and $\widehat{W}(z)$ take the same a_j value, ignoring multiplicity, then we must have $W(z) \equiv \widehat{W}(z)$.*

Theorem 7 *Let $W(z)$ be a v-valued algebroid function determined by (1.1) on the annulus $\mathbb{A}(\frac{1}{R_0}, R_0)(1 < R_0 \leq +\infty)$, and let*

$$\Theta(a) = 1 - \overline{\lim_{r\to R_0}} \frac{\overline{N}_0(r, \frac{1}{W-a})}{T_0(r, W)},$$

then (1) we have $0 \leq \Theta(a) \leq 1$ for all complex a,

(2) we have $\Theta(a) = 0$ except for countable a, and

$$\sum \Theta(a) \leq 2v,$$

in particular, the number of a that satisfies the equation $\Theta(a) = 1$ is at most $2v$.

4. Proofs of theorems

Proof of Theorem 1

By Viete theorem we have

$$W(z) - a = (w_1(z) - a)(w_2(z) - a)\cdots(w_v(z) - a) = (-1)^v \frac{\psi(z, a)}{A_v(z)}. \quad (4.1)$$

Using Jensen theorem for meromorphic function on annuli, we have

$$T_0(r, (-1)^v \frac{\psi(z, a)}{A_v(z)}) = T_0(r, (-1)^v \frac{A_v(z)}{\psi(z, a)}),$$

$$m_0(r, (-1)^v \frac{\psi(z, a)}{A_v(z)}) - m_0(r, (-1)^v \frac{A_v(z)}{\psi(z, a)}) = N_0(r, (-1)^v \frac{A_v(z)}{\psi(z, a)}) - N_0(r, (-1)^v \frac{\psi(z, a)}{A_v(z)}),$$

$$m_0(r, W - a) - m_0(r, \frac{1}{W - a}) = \frac{1}{v} N_0(r, (-1)^v \frac{A_v(z)}{\psi(z, a)}) - \frac{1}{v} N_0(r, (-1)^v \frac{\psi(z, a)}{A_v(z)}),$$

$$m_0(r, W - a) - m_0(r, \frac{1}{W - a}) = \frac{1}{v} N_0(r, \frac{1}{\psi(z, a)}) - \frac{1}{v} N_0(r, \frac{1}{A_v(z)}),$$

$$m_0(r, W - a) - m_0(r, \frac{1}{W - a}) = N_0(r, \frac{1}{W - a}) - N_0(r, W),$$

$$m_0(r, W - a) + N_0(r, W) = m_0(r, \frac{1}{W - a}) + N_0(r, \frac{1}{W - a}),$$

$$m_0(r, a) + N_0(r, a) = T_0(r, W) + \varepsilon(a, r).$$

Among them

$$\varepsilon(a, r) = m_0(r, W - a) - m_0(r, W).$$

because

$$\log^+ |w_j(z) - a| \leq \log^+ |w_j(z)| + \log^+ |a| + \log 2.$$

$$\log^+ |w_j(z)| \leq \log^+ |w_j(z) - a| + \log^+ |a| + \log 2.$$

So

$$|\varepsilon(a,r)| = O(1).$$

□

Proof of Theorem 2

Let $a_k \in \mathbb{C}(k=1,2\cdots p)$, $w_j = w_j(z)(j=1,2\cdots v)$ are v branches of $W(z)$, by the following identity:

$$\prod_{j=1}^{v}\prod_{k=1}^{p}\frac{1}{w_j-a_k} = \prod_{j=1}^{v}\{\sum_{k=1}^{p} C_k\frac{w_j'}{w_j-a_k}\}/\prod_{j=1}^{v}w_j'. \tag{4.2}$$

$C_k = [(a_1-a_k)(a_2-a_k)\cdots(a_{k-1}-a_k)(a_{k+1}-a_k)\cdots(a_p-a_k)]^{-1}$, $w'(z)$ is the derivative of $w(z)$ and satisfies the following equation:

$$\phi(z,w') = B_v(z)(w')^v + B_{v-1}(z)(w')^{v-1} + \cdots + B_0(z) = 0. \tag{4.3}$$

Let $z = re^{i\theta}, z = \frac{1}{r}e^{i\theta}, z = e^{i\theta}$, using Jensen theorem for meromorphic function on annuli and by (4.1), we have

$$\begin{aligned}
I =&\frac{1}{2\pi}\int_0^{2\pi}\log|\prod_{k=1}^{p}\prod_{j=1}^{v}\frac{1}{w_j(re^{i\theta})-a_k}|d\theta + \frac{1}{2\pi}\int_0^{2\pi}\log|\prod_{k=1}^{p}\prod_{j=1}^{v}\frac{1}{w_j(\frac{1}{r}e^{i\theta})-a_k}|d\theta\\
&-2\times\frac{1}{2\pi}\int_0^{2\pi}\log|\prod_{k=1}^{p}\prod_{j=1}^{v}\frac{1}{w_j(e^{i\theta})-a_k}|d\theta\\
=&\frac{1}{2\pi}\int_0^{2\pi}\log|\prod_{k=1}^{p}\frac{A_v(re^{i\theta})}{\psi(re^{i\theta},a_k)}|d\theta + \frac{1}{2\pi}\int_0^{2\pi}\log|\prod_{k=1}^{p}\frac{A_v(\frac{1}{r}e^{i\theta})}{\psi(\frac{1}{r}e^{i\theta},a_k)}|d\theta\\
&-2\times\frac{1}{2\pi}\int_0^{2\pi}\log|\prod_{k=1}^{p}\frac{A_v(e^{i\theta})}{\psi(e^{i\theta},a_k)}|d\theta\\
=&\sum_{k=1}^{p}\frac{1}{2\pi}\int_0^{2\pi}\log|\frac{A_v(re^{i\theta})}{\psi(re^{i\theta},a_k)}|d\theta + \sum_{k=1}^{p}\frac{1}{2\pi}\int_0^{2\pi}\log|\frac{A_v(\frac{1}{r}e^{i\theta})}{\psi(\frac{1}{r}e^{i\theta},a_k)}|d\theta\\
&-2\times\sum_{k=1}^{p}\frac{1}{2\pi}\int_0^{2\pi}\log|\frac{A_v(e^{i\theta})}{\psi(e^{i\theta},a_k)}|d\theta\\
=&\{\sum_{k=1}^{p}\frac{1}{2\pi}\int_0^{2\pi}\log|A_v(re^{i\theta})|d\theta + \sum_{k=1}^{p}\frac{1}{2\pi}\int_0^{2\pi}\log|A_v(\frac{1}{r}e^{i\theta})|d\theta - 2\times\sum_{k=1}^{p}\frac{1}{2\pi}\int_0^{2\pi}\log|A_v(e^{i\theta})|d\theta\}\\
&-\{\sum_{k=1}^{p}\frac{1}{2\pi}\int_0^{2\pi}\log|\psi(re^{i\theta},a_k)|d\theta + \sum_{k=1}^{p}\frac{1}{2\pi}\int_0^{2\pi}\log|\psi(\frac{1}{r}e^{i\theta},a_k)|d\theta\\
&-2\times\sum_{k=1}^{p}\frac{1}{2\pi}\int_0^{2\pi}\log|\psi(e^{i\theta},a_k)|d\theta\}
\end{aligned}$$

$$=[pm_0(r,A_v)-pm_0(r,\frac{1}{A_v})]-[\sum_{k=1}^{p}m_0(r,\psi(z,a_k))-\sum_{k=1}^{p}m_0(r,\frac{1}{\psi(z,a_k)})]$$
$$=pN_0(r,\frac{1}{A_v})-\sum_{k=1}^{p}N_0(r,\frac{1}{\psi(z,a_k)}). \tag{4.4}$$

By (4.2) we have

$$I=\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log|\prod_{k=1}^{p}\frac{1}{(w_j(re^{i\theta})-a_k)}|d\theta+\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log|\prod_{k=1}^{p}\frac{1}{(w_j(\frac{1}{r}e^{i\theta})-a_k)}|d\theta$$
$$-2\times\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log|\prod_{k=1}^{p}\frac{1}{(w_j(e^{i\theta})-a_k)}|d\theta$$
$$=\sum_{j=1}^{v}\{\frac{1}{2\pi}\int_0^{2\pi}\log^+|\prod_{k=1}^{p}\frac{1}{(w_j(re^{i\theta})-a_k)}|d\theta-\frac{1}{2\pi}\int_0^{2\pi}\log^+|\prod_{k=1}^{p}(w_j(re^{i\theta})-a_k)|d\theta\}$$
$$+\sum_{j=1}^{v}\{\frac{1}{2\pi}\int_0^{2\pi}\log^+|\prod_{k=1}^{p}\frac{1}{(w_j(\frac{1}{r}e^{i\theta})-a_k)}|d\theta-\frac{1}{2\pi}\int_0^{2\pi}\log^+|\prod_{k=1}^{p}(w_j(\frac{1}{r}e^{i\theta})-a_k)|d\theta\}$$
$$-2\times\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log|\prod_{k=1}^{p}\frac{w_j'(e^{i\theta})}{(w_j(e^{i\theta})-a_k)}\cdot\frac{1}{w_j'(e^{i\theta})}|d\theta$$
$$=\sum_{j=1}^{v}\{\frac{1}{2\pi}\int_0^{2\pi}\log^+|\sum_{k=1}^{p}C_k\frac{w_j'(re^{i\theta})}{(w_j(re^{i\theta})-a_k)}\cdot\frac{1}{w_j'(re^{i\theta})}|d\theta-\frac{1}{2\pi}\int_0^{2\pi}\log^+|\prod_{k=1}^{p}(w_j(re^{i\theta})-a_k)|d\theta\}$$
$$+\sum_{j=1}^{v}\{\frac{1}{2\pi}\int_0^{2\pi}\log^+|\sum_{k=1}^{p}C_k\frac{w_j'(\frac{1}{r}e^{i\theta})}{(w_j(\frac{1}{r}e^{i\theta})-a_k)}\cdot\frac{1}{w_j'(\frac{1}{r}e^{i\theta})}|d\theta-\frac{1}{2\pi}\int_0^{2\pi}\log^+|\prod_{k=1}^{p}(w_j(\frac{1}{r}e^{i\theta})-a_k)|d\theta\}$$
$$-2\times\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log|\prod_{k=1}^{p}\frac{w_j'(e^{i\theta})}{(w_j(e^{i\theta})-a_k)}|d\theta-2\times\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log|\frac{1}{w_j'(e^{i\theta})}|d\theta$$
$$\leq\{\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log^+|\frac{1}{w_j'(re^{i\theta})}|d\theta+\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log^+|\frac{1}{w_j'(\frac{1}{r}e^{i\theta})}|d\theta$$
$$-2\times\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log^+|\frac{1}{w_j'(e^{i\theta})}|d\theta\}$$
$$+\{\sum_{j=1}^{v}\sum_{k=1}^{p}\frac{1}{2\pi}\int_0^{2\pi}\log^+|\frac{w_j'(re^{i\theta})}{w_j(re^{i\theta})-a_k}|d\theta+\sum_{j=1}^{v}\sum_{k=1}^{p}\frac{1}{2\pi}\int_0^{2\pi}\log^+|\frac{w_j'(\frac{1}{r}e^{i\theta})}{w_j(\frac{1}{r}e^{i\theta})-a_k}|d\theta$$
$$-2\times\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log^+|\prod_{k=1}^{p}\frac{w_j'(e^{i\theta})}{(w_j(e^{i\theta})-a_k)}|d\theta\}$$

$$+\{-\sum_{j=1}^{v}\frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}|\prod_{k=1}^{p}(w_j(re^{i\theta})-a_k)|d\theta \ -\sum_{j=1}^{v}\frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}|\prod_{k=1}^{p}(w_j(\frac{1}{r}e^{i\theta})-a_k)|d\theta$$
$$+2\times\sum_{j=1}^{v}\frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}|\prod_{k=1}^{p}\frac{w_j(e^{i\theta})-a_k}{w_j'(e^{i\theta})}|d\theta+2\times\sum_{j=1}^{v}\frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}|w_j'(e^{i\theta})|d\theta\}$$
$$+O(1). \tag{4.5}$$

Among them,

$$W'(z)=w_1'(z)w_2'(z)\cdots w_v'(z)=(-1)^v\frac{\phi(z,0)}{B_v(z)}=(-1)^v\frac{B_0(z)}{B_v(z)}.$$

So we have

$$\sum_{j=1}^{v}\frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}|\frac{1}{w_j'(re^{i\theta})}|d\theta+\sum_{j=1}^{v}\frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}|\frac{1}{w_j'(\frac{1}{r}e^{i\theta})}|d\theta-2\times\sum_{j=1}^{v}\frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}|\frac{1}{w_j'(e^{i\theta})}|d\theta$$
$$=[\frac{1}{2\pi}\int_{0}^{2\pi}\log|\prod_{j=1}^{v}\frac{1}{w_j'(re^{i\theta})}|d\theta+\sum_{j=1}^{v}\frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}|w_j'(re^{i\theta})|d\theta]$$
$$+[\frac{1}{2\pi}\int_{0}^{2\pi}\log|\prod_{j=1}^{v}\frac{1}{w_j'(\frac{1}{r}e^{i\theta})}|d\theta+\sum_{j=1}^{v}\frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}|w_j'(\frac{1}{r}e^{i\theta})|d\theta]$$
$$-[2\times\frac{1}{2\pi}\int_{0}^{2\pi}\log|\prod_{j=1}^{v}\frac{1}{w_j'(e^{i\theta})}|d\theta+2\times\sum_{j=1}^{v}\frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}|w_j'(e^{i\theta})|d\theta]$$
$$=[\frac{1}{2\pi}\int_{0}^{2\pi}\log|\prod_{j=1}^{v}\frac{1}{w_j'(re^{i\theta})}|d\theta+\frac{1}{2\pi}\int_{0}^{2\pi}\log|\prod_{j=1}^{v}\frac{1}{w_j'(\frac{1}{r}e^{i\theta})}|d\theta-2\times\frac{1}{2\pi}\int_{0}^{2\pi}\log|\prod_{j=1}^{v}\frac{1}{w_j'(e^{i\theta})}|d\theta]$$
$$+[\sum_{j=1}^{v}\frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}|w_j'(re^{i\theta})|d\theta+\sum_{j=1}^{v}\frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}|w_j'(\frac{1}{r}e^{i\theta})|d\theta$$
$$-2\times\sum_{j=1}^{v}\frac{1}{2\pi}\int_{0}^{2\pi}\log^{+}|w_j'(e^{i\theta})|d\theta]$$
$$=N_0(r,\frac{1}{B_v})-N_0(r,\frac{1}{B_0})+vm_0(r,W')$$
$$=vN_0(r,W')-vN_0(r,\frac{1}{W'})+vm_0(r,W'). \tag{4.6}$$

Let $a=\max_{1\leq k\leq p}\{|a_k|\}$, $\quad F_j(z)=(w_j(z)-a_1)\cdots(w_j(z)-a_p)$,

$$|w_j(z)|^p\leq\begin{cases}2^p|F_j(z)| & z\in\mathbb{A}(\frac{1}{R_0},R_0)\text{ and }|w_j(z)|>2a,\\(2a)^p & \text{others.}\end{cases}$$

So we have

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+|F_j(re^{i\theta})|d\theta \geq p\cdot\frac{1}{2\pi}\int_0^{2\pi}\log^+|w_j(re^{i\theta})|d\theta - p\log^+ a - 2p\log 2. \tag{4.7}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+|F_j(\frac{1}{r}e^{i\theta})|d\theta \geq p\cdot\frac{1}{2\pi}\int_0^{2\pi}\log^+|w_j(\frac{1}{r}e^{i\theta})|d\theta - p\log^+ a - 2p\log 2. \tag{4.8}$$

$$2\times\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log^+|\prod_{k=1}^{p}\frac{w_j(e^{i\theta})-a_k}{w_j'(e^{i\theta})}|d\theta + 2\times\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log^+|w_j'(e^{i\theta})|d\theta$$
$$\leq 2\times\sum_{j=1}^{v}\sum_{k=1}^{p}\frac{1}{2\pi}\int_0^{2\pi}\log^+|w_j(e^{i\theta})|d\theta + O(1). \tag{4.9}$$

By (4.7), (4.8), (4.9) we have

$$\{-\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log^+|\prod_{k=1}^{p}(w_j(re^{i\theta})-a_k)|d\theta \ -\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log^+|\prod_{k=1}^{p}(w_j(\frac{1}{r}e^{i\theta})-a_k)|d\theta$$
$$+2\times\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log^+|\prod_{k=1}^{p}\frac{w_j(e^{i\theta})-a_k}{w_j'(e^{i\theta})}|d\theta + 2\times\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log^+|w_j'(e^{i\theta})|d\theta\}$$
$$\leq -pvm_0(r,W) + O(1). \tag{4.10}$$

Combining (4.4), (4.5), (4.6), (4.10) and using the First Fundamental Theorem for algebroid function on annuli we have

$$pT_0(r,W) \leq T_0(r,W') + \sum_{k=1}^{p}N_0(r,\frac{1}{W-a_k}) - N_0(r,\frac{1}{W'}) + Q_0(r,W). \tag{4.11}$$

$$Q_0(r,W) = \sum_{k=1}^{p}m_0(r,\frac{W'}{W-a_k}) + O(1).$$

And because

$$m_0(r,W') = m_0(r,\frac{W'}{W}\cdot W) \leq m_0(r,\frac{W'}{W}) + m_0(r,W) + O(1).$$

Then

$$T_0(r,W') = m_0(r,W') + N_0(r,W') \leq m_0(r,\frac{W'}{W}) + m_0(r,W) + N_0(r,W') + O(1)$$
$$\leq T_0(r,W) - N_0(r,W) + N_0(r,W') + m_0(r,\frac{W'}{W}) + O(1).$$

By (4.11) we have

$$(p-1)T_0(r,W) \le N_0(r,W) + \sum_{k=1}^{p} N_0(r, \frac{1}{W-a_k}) - N_1(r) + Q_1(r,W). \tag{4.12}$$

$$N_1(r) = 2N_0(r,W) - N_0(r,W') + N_0(r, \frac{1}{W'}).$$

$$Q_1(r,W) = \sum_{k=0}^{p} m_0(r, \frac{W'}{W-a_k}) + O(1), a_0 = 0.$$

because $N_0(r,W) \le T_0(r,W) + O(1)$, and so (4.12) can be rewritten as the following:

$$(p-2)T_0(r,W) \le \sum_{k=1}^{p} N_0(r, \frac{1}{W-a_k}) - N_1(r) + Q_1(r,W). \tag{4.13}$$

We further estimate $N_1(r)$ of (4.12). Firstly, we have

$$N_1(r) = N_1(r,W) - N_x(r,W). \tag{4.14}$$

$N_1(r,W)$ denotes the density index of all multiple values including finite or infinite, every τ multiple value counts $\tau-1$, $N_x(r,W)$ denotes the density index of branch point. In fact, by (1.2), if there are λ branches that take $a \in \mathbb{C}$ as the value and multiplicity is τ, we expand $W(z)-a$ in a fractional power series, by expansion $W(z)-a = (z-z_0)^{\frac{\tau}{\lambda}}\widehat{W}_0(z)$, we have $W'(z) = (z-z_0)^{\frac{\tau-\lambda}{\lambda}}\widehat{W}_1(z)$. By the expansion of $W'(z)$ we know: z_0 is $\tau-\lambda$ multiple zero of $W'(z)$ when $\tau-\lambda>0$; z_0 is $\lambda-\tau$ multiple pole of $W'(z)$ when $\tau-\lambda<0$.
By (1.2), (1.3) and $N_1(r) = 2N_0(r,W) - N_0(r,W') + N_0(r,\frac{1}{W'})$, we have

$$\begin{aligned}
n_1(r) =& 2n_0(r,W) - n_0(r,W') + n_0(r,\frac{1}{W'}) \\
=& 2\sum_{W=\infty} \tau - \{\sum_{W=\infty} (\tau+\lambda) + \sum_{W\neq\infty} (\lambda-\tau)^+\} + \sum_{W\neq\infty} (\tau-\lambda)^+ \\
=& \sum_{W=\infty} [(\tau+1)+(\tau-1)] - \{\sum_{W=\infty} [(\tau+1)+(\lambda-1)] \\
&+ \sum_{\substack{W\neq\infty \\ \lambda-\tau>0}} [(\lambda-1)-(\tau-1)]\} + \sum_{\substack{W\neq\infty \\ \tau-\lambda\ge 0}} [(\tau-1)-(\lambda-1)] \\
=& \sum(\tau-1) - \sum(\lambda-1) \\
=& n_1(r,W) - n_x(r,W).
\end{aligned}$$

So we have (4.14). By substituting (4.14) into (4.13) we have

$$(p-2)T_0(r,W) \le \sum_{k=1}^{p} N_0(r, \frac{1}{W-a_k}) - N_1(r,W) + N_x(r,W) + Q_1(r,W). \tag{4.15}$$

By (4.15) and Lemma 3, we have (3.1).

The remainder of the second fundamental theorem is the following formula:

$$S_0(r,W)=O(\log T_0(r,W))+O(\log r),$$

outside a set of finite linear measure, if $r\longrightarrow R_0=+\infty$, while

$$S_0(r,W)=O(\log^+ T_0(r,W))+O(\log\frac{1}{R_0-r}),$$

outside a set E of r such that $\int_E\frac{dr}{R_0-r}<+\infty$, when $r\longrightarrow R_0<+\infty$.

We notice that the following formula is true:

$$\sum_{k=1}^{p}N_0(r,\frac{1}{W-a_k})-N_1(r,W)\leq\sum_{k=1}^{p}\overline{N}_0(r,\frac{1}{W-a_k}).$$

$\overline{N}_0(r,\frac{1}{W-a_k})$ is reduced counting function of zeros (ignoring multiplicity). Then the second fundamental theorem can be rewritten as the following:

$$(p-2v)T_0(r,W)\leq\sum_{k=1}^{p}\overline{N}_0(r,\frac{1}{W-a_k})+S_0(r,W).$$

□

Proof of Theorem 3

By (1.1) we have

$$\psi(z,e^{i\alpha})=A_v(z)(e^{i\alpha}-w_1(z))(e^{i\alpha}-w_2(z))\cdots(e^{i\alpha}-w_v(z)). \tag{4.16}$$

We integrate (4.16) on α from 0 to 2π and by Lemma 4, then we have

$$\log^+|a|=\frac{1}{2\pi}\int_0^{2\pi}\log|e^{i\varphi}-a|d\varphi.$$

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\log|\psi(z,e^{i\alpha})|d\alpha&=\frac{1}{2\pi}\int_0^{2\pi}\log|A_v(z)(e^{i\alpha}-w_1(z))(e^{i\alpha}-w_2(z))\cdots(e^{i\alpha}-w_v(z))|d\alpha\\&=\frac{1}{2\pi}\int_0^{2\pi}\log|A_v(z)|d\alpha+\frac{1}{2\pi}\sum_{j=1}^{v}\int_0^{2\pi}\log|(e^{i\alpha}-w_j(z))|d\alpha\\&=log|A_v(z)|+\sum_{j=1}^{v}\log^+|w_j(z)|.\end{aligned} \tag{4.17}$$

By substituting $z = re^{i\theta}, z = \frac{1}{r}e^{i\theta}, z = e^{i\theta}$ into (4.17) respectively, integrating on θ and exchanging integral order, using Jensen theorem for meromorphic function on annuli, we have

$$\frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2\pi}\int_0^{2\pi}\log|\psi(re^{i\theta},e^{i\alpha})|d\theta]d\alpha$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\log|A_v(re^{i\theta})|d\theta + \frac{1}{2\pi}\int_0^{2\pi}(\sum_{j=1}^{v}\log^+|w_j(re^{i\theta})|)d\theta. \tag{4.18}$$

$$\frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2\pi}\int_0^{2\pi}\log|\psi(\frac{1}{r}e^{i\theta},e^{i\alpha})|d\theta]d\alpha$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\log|A_v(\frac{1}{r}e^{i\theta})|d\theta + \frac{1}{2\pi}\int_0^{2\pi}(\sum_{j=1}^{v}\log^+|w_j(\frac{1}{r}e^{i\theta})|)d\theta. \tag{4.19}$$

$$\frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2\pi}\int_0^{2\pi}\log|\psi(e^{i\theta},e^{i\alpha})|d\theta]d\alpha$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\log|A_v(e^{i\theta})|d\theta + \frac{1}{2\pi}\int_0^{2\pi}(\sum_{j=1}^{v}\log^+|w_j(e^{i\theta})|)d\theta. \tag{4.20}$$

By $(4.18) + (4.19) - 2\times(4.20)$, we have

$$\frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2\pi}\int_0^{2\pi}\log|\psi(re^{i\theta},e^{i\alpha})|d\theta]d\alpha + \frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2\pi}\int_0^{2\pi}\log|\psi(\frac{1}{r}e^{i\theta},e^{i\alpha})|d\theta]d\alpha$$
$$- 2\times\frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2\pi}\int_0^{2\pi}\log|\psi(e^{i\theta},e^{i\alpha})|d\theta]d\alpha$$
$$=\frac{1}{2\pi}\int_0^{2\pi}[m_0(r,\psi(z,e^{i\alpha})) - m_0(r,\frac{1}{\psi(z,e^{i\alpha})})]d\alpha$$
$$=\frac{1}{2\pi}\int_0^{2\pi}N_0(r,\frac{1}{\psi(z,e^{i\alpha})})d\alpha. \tag{4.21}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\log|A_v(re^{i\theta})|d\theta + \frac{1}{2\pi}\int_0^{2\pi}\log|A_v(\frac{1}{r}e^{i\theta})|d\theta - 2\times\frac{1}{2\pi}\int_0^{2\pi}\log|A_v(e^{i\theta})|d\theta$$
$$+ \frac{1}{2\pi}\int_0^{2\pi}(\sum_{j=1}^{v}\log^+|w_j(re^{i\theta})|)d\theta + \frac{1}{2\pi}\int_0^{2\pi}(\sum_{j=1}^{v}\log^+|w_j(\frac{1}{r}e^{i\theta})|)d\theta$$
$$- 2\times\frac{1}{2\pi}\int_0^{2\pi}(\sum_{j=1}^{v}\log^+|w_j(e^{i\theta})|)d\theta$$
$$=m_0(r,A_v) - m_0(r,\frac{1}{A_v}) + \sum_{j=1}^{v}m_0(r,w_j(z))$$
$$=N_0(r,\frac{1}{A_v}) + \sum_{j=1}^{v}m_0(r,w_j(z))$$

$$
\begin{aligned}
&= vN_0(r, W) + vm_0(r, W)\\
&= vT_0(r, W). \qquad (4.22)
\end{aligned}
$$

By (4.21) and (4.22), (3.2) is true. □

Proof of Theorem 4

On the one hand, by (1.1) and Remark 1 we have

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \log|\psi(z, e^{i\alpha})|d\alpha\\
&= \frac{1}{2\pi}\int_0^{2\pi} \log|A_v(z)e^{i\alpha v} + A_{v-1}(z)e^{i\alpha(v-1)} + \cdots + A_0(z)|d\alpha\\
&\leq \log(\sum_{j=0}^{v}|A_j(z)|) \leq \log A(z) + \log(v+1). \qquad (4.23)
\end{aligned}
$$

By substituting $z = re^{i\theta}, z = \frac{1}{r}e^{i\theta}, z = e^{i\theta}$ into (4.23). Integrating on θ and exchanging integral order, we have

$$
\frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2\pi}\int_0^{2\pi} \log|\psi(re^{i\theta}, e^{i\alpha})|d\theta]d\alpha \leq \frac{1}{2\pi}\int_0^{2\pi} \log A(re^{i\theta})d\theta + \log(v+1). \qquad (4.24)
$$

$$
\frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2\pi}\int_0^{2\pi} \log|\psi(\frac{1}{r}e^{i\theta}, e^{i\alpha})|d\theta]d\alpha \leq \frac{1}{2\pi}\int_0^{2\pi} \log A(\frac{1}{r}e^{i\theta})d\theta + \log(v+1). \qquad (4.25)
$$

$$
\frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2\pi}\int_0^{2\pi} \log|\psi(e^{i\theta}, e^{i\alpha})|d\theta]d\alpha \leq \frac{1}{2\pi}\int_0^{2\pi} \log A(e^{i\theta})d\theta + \log(v+1) = O(1). \qquad (4.26)
$$

By $(4.24) + (4.25) - 2 \times (4.26)$, using Theorem 3 and Jensen theorem for meromorphic function on annuli, we have

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2\pi}\int_0^{2\pi} \log|\psi(re^{i\theta}, e^{i\alpha})|d\theta]d\alpha + \frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2\pi}\int_0^{2\pi} \log|\psi(\frac{1}{r}e^{i\theta}, e^{i\alpha})|d\theta]d\alpha\\
&- 2 \times \frac{1}{2\pi}\int_0^{2\pi}[\frac{1}{2\pi}\int_0^{2\pi} \log|\psi(e^{i\theta}, e^{i\alpha})|d\theta]d\alpha\\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \log A(re^{i\theta})d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log A(\frac{1}{r}e^{i\theta})d\theta - 2 \times \frac{1}{2\pi}\int_0^{2\pi} \log A(e^{i\theta})d\theta + O(1).
\end{aligned}
$$

So we have

$$
\frac{1}{2\pi v}\int_0^{2\pi}[N_0(r, \frac{1}{\psi(z, e^{i\alpha})}) - N_0(r, \psi(z, e^{i\alpha}))]d\alpha \leq \mu(r, A) + O(1).
$$

Because $N_0(r, \psi(z, e^{i\alpha})) = 0$,

therefore

$$
\frac{1}{2\pi v}\int_0^{2\pi} N_0(r, \frac{1}{\psi(z, e^{i\alpha})})d\alpha \leq \mu(r, A) + O(1).
$$

So we have

$$T_0(r,W) \le \mu(r,A) + O(1).$$

On the other hand, there is the following formula by Viete theorem of algebraic equation:

$$\frac{A_j(z)}{A_v(z)} = \sum (-1)^{\alpha} w_{\alpha_1}(z) w_{\alpha_2}(z) \cdots w_{\alpha_{v-j}}(z). \tag{4.27}$$

where $(\alpha_1, \alpha_2, \cdots, \alpha_{v-j})$ is the combination of taking $v-j$ numbers from $(1,2,\cdots,v)$, $(-1)^{\alpha}$ is 1 or -1, which depends on $\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots \alpha_{v-j} \\ 1 & 2 & \cdots v-j \end{pmatrix}$ being even permutation or odd permutation. Now every $z \in \mathbb{A}(\frac{1}{R_0}, R_0)(1 < R_0 \le +\infty)$, let $|w_{\beta_1}(z) w_{\beta_2}(z) \cdots w_{\beta_{v-j}}(z)| = \max_{\alpha_1, \cdots \alpha_{v-j}} |w_{\alpha_1}(z) w_{\alpha_2}(z) \cdots w_{\alpha_{v-j}}(z)|$, by (4.27), we have

$$\begin{aligned} \log|A_j(z)| &\le \log|A_v(z)| + \log|w_{\beta_1}(z) w_{\beta_2}(z) \cdots w_{\beta_{v-j}}(z)| + \log C_v^j \\ &\le \log|A_v(z)| + \sum_{j=1}^{v} \log^+ |w_j(z)| + O(1). \end{aligned} \tag{4.28}$$

The right-hand side of (4.28) has nothing to do with number j, so any $z \in \mathbb{A}(\frac{1}{R_0}, R_0)(1 < R_0 \le +\infty)$

$$\log A(z) \le \log|A_v(z)| + \sum_{j=1}^{v} \log^+ |w_j(z)| + O(1). \tag{4.29}$$

By substituting $z = re^{i\theta}, z = \frac{1}{r}e^{i\theta}, z = e^{i\theta}$ into (4.29) respectively, integrating on θ and using Jensen theorem for meromorphic function on annuli, we have

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} \log A(re^{i\theta})d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log A(\frac{1}{r}e^{i\theta})d\theta - 2 \times \frac{1}{2\pi}\int_0^{2\pi} \log A(e^{i\theta})d\theta \\ \le&[\frac{1}{2\pi}\int_0^{2\pi} \log A_v(re^{i\theta})d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log A_v(\frac{1}{r}e^{i\theta})d\theta - 2 \times \frac{1}{2\pi}\int_0^{2\pi} \log A_v(e^{i\theta})d\theta] \\ &+ [\sum_{j=1}^{v} \frac{1}{2\pi}\int_0^{2\pi} \log^+ |w_j(re^{i\theta})|d\theta + \sum_{j=1}^{v} \frac{1}{2\pi}\int_0^{2\pi} \log^+ |w_j(\frac{1}{r}e^{i\theta})|d\theta \\ &+ \sum_{j=1}^{v} \frac{1}{2\pi}\int_0^{2\pi} \log^+ |w_j(e^{i\theta})|d\theta] + O(1). \end{aligned}$$

$$\mu(r,A) \le \frac{1}{v} N_0(r, \frac{1}{A_v}) + m_0(r,W) + O(1).$$

Then we have

$$\mu(r,A) \le T_0(r,W) + O(1).$$

So, based on above two aspects, Theorem 4 is true. □

Proof of Theorem 5

Let $A_{jk}(z) = \max\{|A_j(z)|, |A_k(z)|\}$, $f_{jk}(z) = \frac{A_j(z)}{A_k(z)}$. Using Jensen theorem for meromorphic function on annuli and Remark 1, we have

$$\begin{aligned}
&v\mu(r,A) + O(1) \geq v\mu(r, A_{jk}) \\
=&\frac{1}{2\pi}\int_0^{2\pi} \log A_{jk}(re^{i\theta})d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log A_{jk}(\frac{1}{r}e^{i\theta})d\theta - 2\times\frac{1}{2\pi}\int_0^{2\pi}\log A_{jk}(e^{i\theta})d\theta \\
=&[\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{|A_j(re^{i\theta})|}{|A_k(re^{i\theta})|}d\theta + \frac{1}{2\pi}\int_0^{2\pi}\log|A_k(re^{i\theta})|d\theta] + [\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{|A_j(\frac{1}{r}e^{i\theta})|}{|A_k(\frac{1}{r}e^{i\theta})|}d\theta \\
&+ \frac{1}{2\pi}\int_0^{2\pi}\log|A_k(\frac{1}{r}e^{i\theta})|d\theta] - [2\times\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{|A_j(e^{i\theta})|}{|A_k(e^{i\theta})|}d\theta + 2\times\frac{1}{2\pi}\int_0^{2\pi}\log|A_k(e^{i\theta})|d\theta] \\
=&m_0(r,\frac{A_j}{A_k}) + N_0(r,\frac{1}{A_k}) \\
\geq&T_0(r,\frac{A_j}{A_k}) \\
=&T_0(r,f_{jk}).
\end{aligned} \tag{4.30}$$

By Theorem 4 and (4.30), we have

$$T_0(r,f_{jk}) \leq v\mu(r,A) + O(1) \leq vT_0(r,W) + O(1).$$

By the above formula, for all meromorphic functions $f_{jk}(z)$ $(0 \leq j,k \leq v)$ that satisfy the following conditions

$$\varliminf_{r\to\infty}\frac{T_0(r,f_{jk})}{\log r} < \infty \qquad (R_0 = +\infty), \qquad \varliminf_{r\to R_0^-}\frac{T_0(r,f_{jk})}{\log\frac{1}{(R_0-r)}} < \infty \qquad (R_0 < +\infty).$$

By references (2) and (17), all functions $f_{jk}(z)$ are rational functions, because $A_0(z), A_1(z)$ $\cdots A_v(z)$ can not have nonconstant common factor, so all $A_j(z) (j = 1, 2 \cdots v)$ must be polynomials. Then $W(z)$ degenerates an algebraic function. □

Proof of Theorem 6

Firstly, using the second fundamental theorem for algebroid functions on annuli on $W(z)$ $\widehat{W}(z)$ and $a_j (j = 1, 2, \cdots 4v+1)$. Let $p = 4v+1$; we have

$$(p-2v)T_0(r,W) \leq \sum_{k=1}^{p}\overline{N}_0(r,\frac{1}{W-a_k}) + S_0(r,W). \tag{4.31}$$

$$(p-2\mu)T_0(r,\widehat{W}) \leq \sum_{k=1}^{p}\overline{N}_0(r,\frac{1}{\widehat{W}-a_k}) + S_0(r,\widehat{W}). \tag{4.32}$$

By (4.31)+(4.32), Remark 2 and $\mu \leq v$, we have

$$\begin{aligned}
(p-2v)[T_0(r,W) + T_0(r,\widehat{W})] \leq& \sum_{k=1}^{p}\overline{N}_{12}(r,a_j) + 2\sum_{k=1}^{p}\overline{N}_{\triangle}(r,a_j) \\
&+ O(\log[rT_0(r,W)T_0(r,\widehat{W})])
\end{aligned} \tag{4.33}$$

or

$$(p-2v)[T_0(r,W)+T_0(r,\widehat{W})] \leq \sum_{k=1}^{p} \overline{N}_{12}(r,a_j) + 2\sum_{k=1}^{p} \overline{N}_{\triangle}(r,a_j) + O(\log[\frac{1}{R_0-r}T_0(r,W)T_0(r,\widehat{W})]). \tag{4.34}$$

If $W(z) \not\equiv \widehat{W}(z)$, then we have

$$\sum \overline{n}_{\triangle}(r,a) \leq n_0(r, \frac{1}{R(\varphi,\psi)}).$$

$R(\varphi,\psi)$ denotes the resultant of $\varphi(z,W)$ and $\psi(z,W)$; it can be written as the following:

$$R(\varphi,\psi) = [A_v(z)]^{\mu}[B_{\mu}(z)]^{v} \prod_{\substack{1\leq j\leq v \\ 1\leq k\leq \mu}} [w_j(z)+\widehat{w}_k(z)].$$

It can be written in another form:

$$R(\varphi,\psi) = \begin{vmatrix} A_v(z) & A_{v-1}(z) & \cdots & \cdots & A_0(z) & 0 & \cdots & 0 \\ 0 & A_v(z) & A_{v-1}(z) & \cdots & A_1(z) & A_0(z) & \cdots & 0 \\ \vdots & \vdots & & \vdots & & & & \\ 0 & 0 & 0 & A_v(z) & A_{v-1}(z) & \cdots & \cdots & A_0(z) \\ B_{\mu}(z) & B_{\mu-1}(z) & \cdots & \cdots & B_0(z) & 0 & \cdots & 0 \\ 0 & B_{\mu}(z) & B_{\mu-1}(z) & \cdots & B_1(z) & B_0(z) & \cdots & 0 \\ \vdots & \vdots & & \vdots & & & & \\ 0 & 0 & 0 & B_{\mu}(z) & B_{\mu-1}(z) & \cdots & \cdots & B_0(z) \end{vmatrix}.$$

Thus we know that $R(\varphi,\psi)$ is a holomorphic function, and using Jensen theorem for meromorphic function on annuli, we have

$$\begin{aligned} & N_0(r, \frac{1}{R(\varphi,\psi)}) \\ =& \frac{1}{2\pi}\int_0^{2\pi} \log|R[\psi(re^{i\theta},W),\varphi(re^{i\theta},\widehat{W})]|d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log|R[\psi(\frac{1}{r}e^{i\theta},W),\varphi(\frac{1}{r}e^{i\theta},\widehat{W})]|d\theta \\ & - 2\times\frac{1}{2\pi}\int_0^{2\pi} \log|R[\psi(e^{i\theta},W),\varphi(e^{i\theta},\widehat{W})]|d\theta \\ =& \frac{\mu}{2\pi}\int_0^{2\pi} \log|A_v(re^{i\theta})|d\theta + \frac{v}{2\pi}\int_0^{2\pi} \log|B_{\mu}(re^{i\theta})|d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log|\prod_{\substack{1\leq j\leq v \\ 1\leq k\leq \mu}} [w_j(re^{i\theta})+\widehat{w}_k(re^{i\theta})]|d\theta \end{aligned}$$

$$+\frac{\mu}{2\pi}\int_0^{2\pi}\log|A_v(\frac{1}{r}e^{i\theta})|d\theta+\frac{v}{2\pi}\int_0^{2\pi}\log|B_\mu(\frac{1}{r}e^{i\theta})|d\theta$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\log|\prod_{\substack{1\le j\le v\\1\le k\le\mu}}[w_j(\frac{1}{r}e^{i\theta})+\widehat{w}_k(\frac{1}{r}e^{i\theta})]|d\theta-2\times\frac{\mu}{2\pi}\int_0^{2\pi}\log|A_v(e^{i\theta})|d\theta$$

$$-2\times\frac{v}{2\pi}\int_0^{2\pi}\log|B_\mu(e^{i\theta})|d\theta-2\times\frac{1}{2\pi}\int_0^{2\pi}\log|\prod_{\substack{1\le j\le v\\1\le k\le\mu}}[w_j(e^{i\theta})+\widehat{w}_k(e^{i\theta})]|d\theta$$

$$=[\frac{\mu}{2\pi}\int_0^{2\pi}\log|A_v(re^{i\theta})|d\theta+\frac{\mu}{2\pi}\int_0^{2\pi}\log|A_v(\frac{1}{r}e^{i\theta})|d\theta-2\times\frac{\mu}{2\pi}\int_0^{2\pi}\log|A_v(e^{i\theta})|d\theta]$$

$$+[\frac{v}{2\pi}\int_0^{2\pi}\log|B_\mu(re^{i\theta})|d\theta+\frac{v}{2\pi}\int_0^{2\pi}\log|B_\mu(\frac{1}{r}e^{i\theta})|d\theta-2\times\frac{v}{2\pi}\int_0^{2\pi}\log|B_\mu(e^{i\theta})|d\theta]$$

$$+[\frac{1}{2\pi}\int_0^{2\pi}\log|\prod_{\substack{1\le j\le v\\1\le k\le\mu}}[w_j(re^{i\theta})+\widehat{w}_k(re^{i\theta})]|d\theta+\frac{1}{2\pi}\int_0^{2\pi}\log|\prod_{\substack{1\le j\le v\\1\le k\le\mu}}[w_j(\frac{1}{r}e^{i\theta})+\widehat{w}_k(\frac{1}{r}e^{i\theta})]|d\theta$$

$$-2\times\frac{1}{2\pi}\int_0^{2\pi}\log|\prod_{\substack{1\le j\le v\\1\le k\le\mu}}[w_j(e^{i\theta})+\widehat{w}_k(e^{i\theta})]|d\theta]$$

$$\le\mu[m_0(r,A_v)-m_0(r,\frac{1}{A_v})]+v[m_0(r,B_\mu)-m_0(r,\frac{1}{B_\mu})]+\mu v[m_0(r,W)+m_0(r,\widehat{W})]+O(1)$$

$$=\mu v[T_0(r,W)+T_0(r,\widehat{W})]+O(1).$$

Then we have

$$\sum\overline{N}_\triangle(r,a)\le\frac{2\mu v}{\mu+v}\{T_0(r,W)+T_0(r,\widehat{W})\}+O(1)$$
$$\le v\{T_0(r,W)+T_0(r,\widehat{W})\}+O(1). \tag{4.35}$$

By substituting (4.35) into (4.33) and (4.34) we have

$$(p-4v)[T_0(r,W)+T_0(r,\widehat{W})]\le\sum_{k=1}^{p}\overline{N}_{12}(r,a_j)+O(\log rT_0(r,W)T_0(r,\widehat{W})). \tag{4.36}$$

or

$$(p-4v)[T_0(r,W)+T_0(r,\widehat{W})]\le\sum_{k=1}^{p}\overline{N}_{12}(r,a_j)+O(\log\frac{1}{R_0-r}T_0(r,W)T_0(r,\widehat{W})). \tag{4.37}$$

By the conditions of Theorem 6, we know that $W(z)$ and $\widehat{W}(z)$ take the same values ignoring multiplicity about $4v+1$ distinct a_j, and at the same time we have $\overline{N}_{12}(r,a_j)=0$. So (4.36) and (4.37) can be written as the following:

$$T_0(r,W)+T_0(r,\widehat{W})<O(\log[rT_0(r,W)T_0(r,\widehat{W})]). \tag{4.38}$$

$$T_0(r,W)+T_0(r,\widehat{W})<O(\log[\frac{1}{R_0-r}T_0(r,W)T_0(r,\widehat{W})]). \tag{4.39}$$

By Theorem 5 we know that (4.38) or (4.39) are not true. So it must be $W(z)\equiv\widehat{W}(z)$. □

Proof of Theorem 7

By Theorem 2 we have the conclusion of Theorem 7. □

References

[1] Axler S. Harmonic functions from a complex analysis viewpoint. Amer Math Monthly 1986; 93: 246–258.

[2] Khrystiyanyn AYa, Kondratyuk AA. On the Nevanlinna theory for meromorphic functions on annuli (I). Matematychni Studii 2005; 23: 19–30.

[3] Khrystiyanyn AYa, Kondratyuk AA. On the Nevanlinna theory for meromorphic functions on annuli (II). Matematychni Studii 2005; 24: 57–68.

[4] Hayman WK, Kennedy P. Subharmonic functions. London, UK: Academic Press, 1980.

[5] Hayman WK. Meromorphic functions. Oxford, UK: Oxford University Press, 1964.

[6] Yi H, Yang C. Uniqueness theory of meromorphic function. Beijing, China: Science Press, 1995.

[7] Yi H. On the multiple values and uniqueness of algebroid functions. Engineering Math 1991; 8: 1–8.

[8] Laine I. Nevanlinna theory and complex differential equations. Berlin, Germany: Walter de Gruyter, 1993.

[9] Nevanlinna R. Eindeutige analytische Funktionen. Berlin and New York: Springer-Verlag, 1953.

[10] Niino K, Ozawa M. Deficiencies of an entire algebroid function. Kodai Math 1970; 22: 98–113.

[11] Zhang Q. Uniqueness of algebroid functions. Math Pract Theory 2013; 43: 183–187.

[12] Rubel LA. Entire and meromorphic functions. New York, NY, USA: Springer, 1996.

[13] Toda N. On algebroid functions. Sugaku 1972; 24: 200–209.

[14] Tsuji M. Potential Theory in Modern Function Theory. Tokyo, Japan: Maruzen, 1959.

[15] He Y. On the algebroid functions and their derivatives (I). Acta Mathematica Sinica 1965; 15: 281–295.

[16] He Y. On the algebroid functions and their derivatives (II). Acta Mathematica Sinica 1965; 15: 500–510.

[17] He Y. On the multiple values of algebroid functions. Acta Mathematica Sinica 1979; 22: 733–742.

[18] He Y, Gao S. On algebroid functions taking the same values at the same points. Kodai Math 1986; 9: 256–265.

[19] He Y, Xiao X. Algebroid functions and ordinary differential equations in the complex domain. Beijing, China: Science Press, 1988.

[20] Xuan Z, Gao Z. Uniqueness theorems for algebroid functions. Complex Var Elliptic Equ 2006; 51: 701–712.