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Notes on magnetic curves in 3D semi-Riemannian manifolds

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Abstract: A magnetic field is defined by the property that its divergence is zero in three-dimensional semi-Riemannian manifolds. Each magnetic field generates a magnetic flow whose trajectories are curves γ, called magnetic curves. In this paper, we investigate the effect of magnetic fields on the moving particle trajectories by variational approach to the magnetic flow associated with the Killing magnetic field on three-dimensional semi-Riemannian manifolds. We then investigate the trajectories of these magnetic fields and give some characterizations and examples of these curves.

Key words: Special curves, vector fields, flows, ordinary differential equations

1. Introduction

A charged particle moves along a regular curve in 3-dimensional space. The tangent, normal, and binormal vectors describe kinematic and geometric properties of the particle. These vectors affect the trajectory of the charged particle during motion in a magnetic field. The time dimension also affects its trajectory. Therefore, motion of the charged particle in a magnetic vector field should be investigated considering the time dimension. In this article, we investigate effects of magnetic fields on charged particle trajectories by variational approach to magnetic flow associated with the Killing magnetic field on a three-dimensional semi-Riemannian manifold M.

A divergence-free vector field defines a magnetic field in a three-dimensional semi-Riemannian manifold M. It is known that $V \in \chi(M^n)$ is Killing if and only if $\mathcal{L}_V g = 0$ or, equivalently, $\nabla V(p)$ is a skew-symmetric operator in $Tp(M^n)$, at each point $p \in M^n$. It is clear that any Killing vector field on (M^n, g) is divergence-free. In particular, if $n = 3$, then every Killing vector field defines a magnetic field that will be called a Killing magnetic field (for details, see [2]).

Lorentz force ϕ associated with the magnetic field V is defined by

$$\phi(\gamma') = V \times \gamma'$$

and trajectories γ called magnetic curves satisfy

$$\nabla_{\gamma'}\gamma' = \phi(\gamma') = V \times \gamma', \tag{1}$$

where ∇ is the Levi-Civita connection of the manifold M (in this article we call these curves T-magnetic curves to avoid confusion with other definitions). Using Eq. (1) we can study the magnetic field in a space that has

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nonzero sectional curvature C. This gives a more important and realistic approach than the classical approach. Furthermore, this equality and the Hall effect (which explains the dynamics of an electric current flow in $\mathbb{R}^3$ when exposed to a perpendicular magnetic field V) have some important applications in analytical chemistry, biochemistry, atmospheric science, geochemistry, cyclotrons, protons, cancer therapy, and velocity selectors. Solutions of the Lorentz force equation are Kirchhoff elastic rods. This provides an amazing connection between two apparently unrelated physical models and classical elastic theory. The Lorentz force is always perpendicular to both the velocity of the particle and the magnetic field created. When a charged particle moves in a static magnetic field, it traces a helical path and the axis of the helix is parallel to the magnetic field. The speed of the particle remains constant. Since the magnetic force is always perpendicular to the motion, the magnetic field does no work on an isolated charge. If the charged particle moves parallel to magnetic field, the Lorentzian force acting on the particle is zero. When the two vectors (velocity and the magnetic field) are perpendicular to each other, the Lorentz force is maximum (for details, see [4, 2, 3, 6, 7, 9, 8]).

When a charged particle moves along a curve γ in the magnetic field velocity (tangent vector), normal and binormal vectors are exposed to the magnetic field. The forces associated with the magnetic field for motion in the normal and binormal directions of the curve are then given by

$$\phi(N) = V \times N \text{ and } \phi(B) = V \times B$$

and the trajectories of charged particles are changed according to this equation. For example, when a charged particle moves in a static magnetic field in 3D Riemannian space the particle has a two different paths. If one of the tangent or binormal vectors is exposed to this field, it traces a circular helix path. On the other hand, if the normal vector is exposed the this field, it traces a slant helical path. Their axes are parallel to the magnetic field (see [5]).

2. Preliminaries

Let (M, g) be a 3-dimensional semi-Riemannian manifold with the standard flat metric g defined by

$$g(X, Y) = x_1 y_1 + x_2 y_2 - x_3 y_3 \tag{2}$$

for all $X = (x_1, x_2, x_3),\ Y = (y_1, y_2, y_3) \in \chi(M)$.

The Lorentz force of a magnetic field F on M is defined to be a skew-symmetric operator given by

$$g(\phi(X), Y) = F(X, Y) \tag{3}$$

for all $X,\ Y \in \chi(M)$.

The T-magnetic trajectories of F are curves γ on M that satisfy the Lorentz equation

$$\nabla_{\gamma'} \gamma' = \phi(\gamma'). \tag{4}$$

Furthermore, the cross product of two vector fields $X,\ Y \in \chi(M)$ is given by

$$X \times Y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_2 y_1 - x_1 y_2). \tag{5}$$

The mixed product of the vector fields $X,\ Y,\ Z \in \chi(M)$ is then defined by

$$g(X \times Y, Z) = dv_g(X, Y, Z), \tag{6}$$

where dv_g denotes a volume on M.

Let V be a Killing vector field and $F_V = \imath_V dv_g$ be the corresponding Killing magnetic force on M, where $\imath$ denotes the inner product. The Lorentz force of the F_V is then given as

$$\phi(X) = V \times X \tag{7}$$

for all $X \in \chi(M)$. Consequently, the T-magnetic trajectories γ determined by V are solutions of the Lorentz force equation written as

$$\nabla_{\gamma'}\gamma' = V \times \gamma'. \tag{8}$$

A unit speed curve γ is a T-magnetic trajectory of the magnetic field V if and only if V can be written along γ as

$$V(s) = f(s)T(s) + g(s)B(s), \tag{9}$$

where T and B are the tangent and binormal vectors of the curve γ, respectively (see [3]).

Lemma 1 *Let $\gamma : I \subset \mathbb{R} \to M$ be a nonnull immersed curve in a $3D$ semi-Riemannian manifold (M, g) with sectional curvature C and V be a vector field along the curve γ. $\Gamma : (-\varepsilon, \varepsilon) \times I \to M$ satisfy $\Gamma(s, 0) = \gamma(s)$, $\frac{\partial \Gamma}{\partial s}(s,t) = V(s)$. In this setting, we have the following functions:*
1. The speed function $v(s,t) = \left\| \frac{\partial \Gamma}{\partial s}(s,t) \right\|$,
2. The curvature function $\varkappa(s,t)$ of $\gamma_t(s)$,
3. The torsion function $\tau(s,t)$ of $\gamma_t(s)$.
The variations of these functions at $t = 0$ are:

$$V(v) = \left(\frac{\partial v}{\partial s}(s,t)\right)\bigg|_{t=0} = -\varepsilon_1 g(\nabla_T V, T), \tag{10}$$

$$V(\varkappa) = \left(\frac{\partial \varkappa}{\partial s}(s,t)\right)\bigg|_{t=0} = 2\varepsilon_2 g(\nabla_T^2 V, \nabla_T T) + 4\varepsilon_1 \varkappa^2 g(\nabla_T V, T) + 2\varepsilon_2 g(R(V,T)T, \nabla_T T), \tag{11}$$

$$V(\tau) = \left(\frac{\partial \tau}{\partial s}(s,t)\right)\bigg|_{t=0} = -2\varepsilon_2 g((1/\varkappa)\nabla_T^3 V - (\varkappa'/\varkappa^2)\nabla_T^2 V + \varepsilon_1(\varepsilon_2\varkappa + (C/\varkappa))\nabla_T V - \varepsilon_1 C(\varkappa'/\varkappa^2)V, \tau B), \tag{12}$$

where $\varepsilon_1 = g(T,T)$, $\varepsilon_2 = g(N,N)$, $\varepsilon_3 = g(B,B)$, and R and C are curvature tensor and sectional curvature of M, respectively [3, 7].

If $V(s)$ is the restriction to $\alpha(s)$ of a Killing vector field then the vector field V satisfies following condition [3]:

$$V(v) = V(\varkappa) = V(\tau) = 0. \tag{13}$$

Proposition 2 *Let γ be a unit speed spacelike or timelike space curve with $\tau(s)^2 - \varkappa(s)^2 \neq 0$. Then γ is a slant helix (which is defined by the property that the normal vector makes a constant angle with a fixed straight line) if and only if*

$$\frac{\varkappa^2}{(\varepsilon_3\varkappa^2 + \varepsilon_1\tau^2)}\left(\frac{\tau}{\varkappa}\right)'$$

is a constant function where $\varepsilon_1 = g(T,T)$, $\varepsilon_2 = g(N,N)$, and $\varepsilon_3 = g(B,B)$ (see [1]).

Proposition 3 *Let γ be a unit speed nonnull curve in semi-Riemannian manifold (M,g) with the Frenet apparatus $\{T,\ N,\ B,\ \varkappa,\ \tau\}$. The Serret-Frenet formula is then given by*

$$\begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2\varkappa & 0 \\ -\varepsilon_1\varkappa & 0 & -\varepsilon_3\tau \\ 0 & \varepsilon_2\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{14}$$

where $\varepsilon_1 = g(T,T)$, $\varepsilon_2 = g(N,N)$, and $\varepsilon_3 = g(B,B)$ [7].

3. Magnetic curves in 3D oriented semi-Riemannian manifolds

3.1. T-magnetic curves

In this section, we give some characterizations for T-magnetic curves in semi-Riemannian manifolds.

Proposition 4 *Let γ be a unit speed nonnull T-magnetic curve in semi-Riemannian manifold (M,g) with the Frenet apparatus $\{T,\ N,\ B,\ \varkappa,\ \tau\}$. The Lorentz force in the Frenet frame is then written as*

$$\begin{bmatrix} \phi(T) \\ \phi(N) \\ \phi(B) \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2\varkappa & 0 \\ -\varepsilon_1\varkappa & 0 & \varepsilon_3\varpi \\ 0 & -\varepsilon_2\varpi & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{15}$$

where ϖ is a certain function defined by $\varpi = g(\phi(N),B)$.

Proof Let γ be a unit speed T-magnetic curve in semi-Riemannian manifold (M,g) with the Frenet apparatus $\{T,\ N,\ B,\ \varkappa,\ \tau\}$. From the definition of the magnetic curve, we know that

$$\phi(T) = \varepsilon_2\varkappa N.$$

Since $\phi(N) \in span\{T,N,B\}$, we have

$$\phi(N) = \delta T + \lambda N + \mu B.$$

Then using the following equalities:

$$\begin{aligned} \delta &= \varepsilon_1 g(\phi(N),T) = -\varepsilon_1 g(\phi(T),N) = -\varepsilon_1\varkappa, \\ \lambda &= \varepsilon_2 g(\phi(N),T) = 0, \\ \mu &= \varepsilon_3 g(\phi(N),B) = \varepsilon_3\varpi, \end{aligned}$$

we get

$$\phi(N) = -\varepsilon_1\varkappa T + \varepsilon_3\varpi B.$$

Similarly, we can easily obtain

$$\phi(B) = -\varepsilon_2\varpi N.$$

□

Proposition 5 *Let γ be a unit speed nonnull curve in semi-Riemannian manifold (M,g). The curve γ is then a T-magnetic trajectory of a magnetic field V if and only if the vector field V can be written along the curve γ as*

$$V = \varepsilon_3(\varpi T + \varkappa B). \tag{16}$$

Proof Let γ be a unit speed T-magnetic trajectory of a magnetic field V. Using Proposition 4 and Eq.(7), we can easily see that

$$V = \varepsilon_3(\varpi T + \varkappa B).$$

Conversely, we assume that Eq. (16) holds. We then get $V \times T = \phi(T)$, so the curve γ is an N-magnetic trajectory of the magnetic vector field V. □

Theorem 6 *Let γ be a unit speed T-magnetic curve and V be a Killing vector field on a simply connected space form $(M(C), g)$. If the curve γ is one of the T-magnetic trajectories of $(M(C), g, V)$, then its curvature and torsion hold the following equations:*

$$\varkappa^2(\frac{\varpi}{2} + \tau) + \varepsilon_1 A = 0, \tag{17}$$

$$\varepsilon_3\varkappa'' - \varepsilon_2\varkappa\tau(\varpi + \tau) - \varepsilon_3 C\varkappa + \frac{\varkappa^3}{2} - B\varkappa = 0, \tag{18}$$

where C is curvature of the Riemannian space M and A, B are constant.

Proof Let V be a magnetic field in a semi-Riemannian $3D$ manifold M. Then V satisfies Eq. (16). Differentiating Eq. (16) with respect to s, we have

$$\nabla_T V = \varepsilon_3\varpi' T - \varepsilon_1\varkappa(\varpi + \tau)N + \varepsilon_3\varkappa' B, \tag{19}$$

and differentiation of Eq. (19) gives us

$$\nabla_T^2 V = \varkappa^2(\varpi + \tau)T - (\varepsilon_1\varpi\varkappa' + 2\varepsilon_1\varkappa'\tau + \varepsilon_1\varkappa\tau')N + (\varepsilon_3\varkappa'' - \varepsilon_2\varkappa\tau\varpi - \varepsilon_2\varkappa\tau^2)B. \tag{20}$$

Lemma 1 implies that $V(v) = 0$, so considering Eq. (19) we get

$$\varpi = const.$$

If Eq. (19) and Eq. (20) are then considered with $V(\varkappa) = 0$ in Lemma 1, we obtain

$$\varkappa^2(\frac{\varpi}{2} + \tau) + \varepsilon_1 A = 0. \tag{21}$$

Similarly, when Eq. (19) and Eq. (20) are considered with $V(\tau) = 0$ in Lemma 1, we can easily see that

$$\varepsilon_3\varkappa'' - \varepsilon_2\varkappa\tau(\varpi + \tau) - \varepsilon_3 C\varkappa + \frac{\varkappa^3}{2} - B\varkappa = 0. \tag{22}$$

□

Corollary 7 *Let γ be a unit speed T-magnetic curve and V be a Killing vector field on a simply connected space form $(M(C), g)$. If the function ϖ is equal to zero, then the curvature and torsion of the curve γ is given by*

$$\begin{aligned} \varepsilon_1\varkappa^2\tau &= -A, \\ 2\varepsilon_3\varkappa'' + \varkappa^3 - 2\varepsilon_2\varkappa\tau^2 - 2\varepsilon_3 C\varkappa - 2B\varkappa &= 0. \end{aligned}$$

We know that these two equations are just the Euler Lagrange equations for elasticae (for details, see [7]) in $M(C)$.

It is easily seen that a T-magnetic curve is a general helix in Lorentzian 3-space. Some characterizations and examples about it are detailed in [3, 6].

3.2. N-magnetic curves

In this section, we define a new kind of magnetic curve called N-magnetic curve in 3D semi-Riemannian manifold M. Moreover, we obtain some characterizations and examples of this curve. We draw the images of these curves using the program Mathematica.

Definition 8 *Let $\alpha : I \subset \mathbb{R} \to M$ be a nonnull curve in semi-Riemannian manifold (M, g) and F_V be a magnetic field on M. We call the curve α an N-magnetic curve if its normal vector field satisfies the Lorentz force equation; that is,*

$$\nabla_{\alpha'} N = \phi(N) = V \times N.$$

Proposition 9 *Let α be a unit speed nonnull N-magnetic curve in semi-Riemannian manifold (M, g) with the Frenet apparatus $\{T,\ N,\ B,\ \varkappa,\ \tau\}$. The Lorentz force in the Frenet frame can then be written as*

$$\begin{bmatrix} \phi(T) \\ \phi(N) \\ \phi(B) \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2 \varkappa & \varepsilon_3 \varpi_1 \\ -\varepsilon_1 \varkappa & 0 & -\varepsilon_3 \tau \\ -\varepsilon_1 \varpi_1 & \varepsilon_2 \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{23}$$

where ϖ_1 is a certain function defined by $\varpi_1 = g(\phi(T), B)$.

Proof Let α be a unit speed N-magnetic curve in semi-Riemannian manifold (M, g) with the Frenet apparatus $\{T,\ N,\ B,\ \varkappa,\ \tau\}$. Since $\phi(T) \in span\,\{T, N, B\}$, we have

$$\phi(T) = \lambda T + \eta N + \zeta B.$$

Then, using the following equalities:

$$\begin{aligned} \lambda &= \varepsilon_1 g(\phi(T), T) = 0, \\ \eta &= \varepsilon_2 g(\phi(T), N) = -\varepsilon_2 g(\phi(N), T) = -\varepsilon_2 g(\nabla_T N, T) = \varepsilon_2 \varkappa, \\ \zeta &= \varepsilon_3 g(\phi(T), B) = \varepsilon_3 \varpi_1, \end{aligned}$$

we get

$$\phi(T) = \varepsilon_2 \varkappa N + \varepsilon_3 \varpi_1 B.$$

Similarly, we can easily obtain that

$$\begin{aligned} \phi(N) &= -\varepsilon_1 \varkappa T - \varepsilon_3 \tau B, \\ \phi(B) &= -\varepsilon_1 \varpi_1 T + \varepsilon_2 \tau N. \end{aligned}$$

□

Proposition 10 *Let α be a unit speed nonnull curve in semi-Riemannian manifold (M, g). The curve α is then an N-magnetic trajectory of a magnetic field V if and only if the vector field V can be written along the curve α as*

$$V = -\varepsilon_3 \tau T - \varepsilon_3 \varpi_1 N + \varepsilon_3 \varkappa B. \tag{24}$$

Proof Let α be a unit speed N-magnetic trajectory of a magnetic field V. Using Proposition 9 and Eq. (7), we can easily obtain Eq. (24).

Conversely, we assume that Eq. (24) holds. Then we get $V \times N = \phi(N)$, and so the curve α is an N-magnetic trajectory of the magnetic vector field V. □

Theorem 11 ***(Main result)*** *Let α be a unit speed N-magnetic curve and V be a Killing vector field on a simply connected space form $(M(C), g)$. If the curve α is one of the N-magnetic trajectories of $(M(C), g, V)$ then its curvature and torsion satisfy the equation*

$$\varepsilon_3 \varkappa'' + \varepsilon_1 \frac{\tau'^2}{\varkappa} + 2\varepsilon_1 \tau (\frac{\tau'}{\varkappa})' - \varepsilon_2 C \varkappa + \frac{-\varkappa^3 + \varepsilon_2 \varkappa \tau^2}{2} = \eta, \tag{25}$$

where C is curvature of the Riemannian space M and η is a constant.

Proof Let V be a magnetic field in a semi-Riemannian $3D$ manifold M. Then V satisfy Eq. (24). Differentiating Eq. (24) with respect to s, we have

$$\nabla_T V = (-\varepsilon_3 \tau' - \varepsilon_2 \varpi_1 \varkappa) T - \varepsilon_3 \varpi_1' N + (\varepsilon_3 \varkappa' + \tau \varpi_1) B. \tag{26}$$

Eq. (14) and differentiation of Eq. (26) give us

$$\nabla_T^2 V = -\varepsilon_2 \varpi_1' \varkappa T + (-\varepsilon_3 \varpi_1'' - \varepsilon_1 \tau \varkappa' + \varepsilon_2 \tau^2 \varpi_1) N + (\varepsilon_3 \varkappa'' + \tau' \varpi_1 + 2\varpi_1' \tau) B. \tag{27}$$

Lemma 1 implies that $V(v) = 0$, so, considering Eq. (26), we get

$$\varpi_1 = \varepsilon_1 \frac{\tau'}{\varkappa}. \tag{28}$$

Then, if Eq.(26) and Eq.(27) are considered with $V(\varkappa) = 0$ in Lemma 1, we obtain

$$-\varepsilon_3 \varpi_1'' - \varepsilon_1 \tau \varkappa' + \varepsilon_2 \tau^2 \varpi_1 + \varepsilon_1 g(R(V, T)T, N) = 0.$$

In particular, since C is constant, $g(R(V, T)T, N) = Cg(V, N) = -\varepsilon_3 C \varpi_1$, we have

$$-\varepsilon_3 \varpi_1'' - \varepsilon_1 \tau \varkappa' + \varepsilon_2 \tau^2 \varpi_1 + \varepsilon_2 C \varpi_1 = 0. \tag{29}$$

Similarly, when Eq. (26) and Eq. (27) are considered with $V(\tau) = 0$ in Lemma 1, we can easily see that

$$[(1/\varkappa)(\varepsilon_3 \varkappa'' + \tau' \varpi_1 + 2\varpi_1' - \varepsilon_2 C \varkappa)]' + \left(\frac{-\varkappa^2 + \varepsilon_2 \tau^2}{2} \right)' = 0. \tag{30}$$

Finally, integration of Eq. (30) and Eq. (28) gives us

$$\varepsilon_3 \varkappa'' + \varepsilon_1 \frac{\tau'^2}{\varkappa} + 2\varepsilon_1 \tau (\frac{\tau'}{\varkappa})' - \varepsilon_2 C \varkappa + \frac{-\varkappa^3 + \varepsilon_2 \varkappa \tau^2}{2} = \eta,$$

where η is an arbitrary constant. □

Example 12 *Let M be a three-dimensional sphere and the curve α be a timelike N-magnetic curve and the constant function. If we get ,$\varkappa(s) = 1$, $\eta = 3/2$, and $\tau(s) = y(s)$ in Eq. (25), we obtain the following second-order nonlinear differential equation:*

$$2y(s)'^2 + 4y(s)y''(s) - y(s)^2 = 0. \tag{31}$$

Using the program Mathematica, we obtain the solution of Eq. (31):

$$y(s) = c_2 e^{\frac{2\sqrt{\frac{2}{3}}\log\left(e^{\sqrt{6}c_1}+e^{\sqrt{\frac{3}{2}}s}\right)-s}{\sqrt{6}}}.$$

Thus, the curves having the curvature $\varkappa(s) = 1$ and torsion $\tau(s) = c_2 e^{\frac{2\sqrt{\frac{2}{3}}\log\left(e^{\sqrt{6}c_1}+e^{\sqrt{\frac{3}{2}}s}\right)-s}{\sqrt{6}}}.$ are N-magnetic curves in a three-dimensional sphere.

Corollary 13 *Let V be a Killing vector field on $3D$ semi-Riemannian manifold (M, g). Then each trajectory of the magnetic field V makes an angle $\theta(s)$ with the normal vector field of the magnetic curve. For the angle $\theta(s)$ we have the following cases:*

Case 1. *If N and V are spacelike vectors and these vectors span a spacelike plane,*
$g(N,V) = \|N\|\,\|V\|\cos\theta$ *[11].*
Then using Eq. (24) and $g(N,N) = \varepsilon_2 = 1$, we obtain
$\cos\theta = \frac{g(N,V)}{\|N\|\|V\|} = \frac{g(N,-\varepsilon_3\tau T-\varepsilon_3\varpi_1 N+\varepsilon_3\varkappa B)}{\|V\|} = \frac{-\varepsilon_3\varpi_1}{\sqrt{\left|\varepsilon_1\tau^2+\varpi_1^2+\varepsilon_3\varkappa^2\right|}}.$

Case 2. *If N and V are spacelike vectors and these vectors span a timelike plane,*
$g(N,V) = \|N\|\,\|V\|\cosh\theta$ *[11].*
Then using Eq. (24) and $g(N,N) = \varepsilon_2 = 1$, we obtain
$\cosh\theta = \frac{g(N,V)}{\|N\|\|V\|} = \frac{g(N,-\varepsilon_3\tau T-\varepsilon_3\varpi_1 N+\varepsilon_3\varkappa B)}{\|V\|} = \frac{-\varepsilon_3\varpi_1}{\sqrt{\left|\varepsilon_1\tau^2+\varpi_1^2+\varepsilon_3\varkappa^2\right|}}.$

Case 3. *If N and V are timelike vectors in the same timecone,*
$g(N,V) = -\|N\|\,\|V\|\cosh\theta$ *[11].*
Then using Eq. (24) and $g(N,N) = \varepsilon_2 = -1$, we obtain
$\cosh\theta = \frac{g(N,V)}{\|N\|\|V\|} = \frac{g(N,-\varepsilon_3\tau T-\varepsilon_3\varpi_1 N+\varepsilon_3\varkappa B)}{-\|V\|} = \frac{\varepsilon_3\varpi_1}{\sqrt{\left|\varepsilon_1\tau^2-\varpi_1^2+\varepsilon_3\varkappa^2\right|}}.$

Case 4. *If N spacelike (resp. timelike) and V timelike (resp. spacelike) vectors are in the future timecone,*
$g(N,V) = \|N\|\,\|V\|\sinh\theta$ *[11].*
Then using Eq. (24) and $g(N,N) = \varepsilon_2$, we obtain $\sinh\theta = \frac{g(N,V)}{\|N\|\|V\|} = \frac{g(N,-\varepsilon_3\tau T-\varepsilon_3\varpi_1 N+\varepsilon_3\varkappa B)}{\varepsilon_2\|V\|} = \frac{\varepsilon_3\varpi_1}{\sqrt{\left|\varepsilon_1\tau^2+\varepsilon_2\varpi_1^2+\varepsilon_3\varkappa^2\right|}}.$

Corollary 14 *Let α be a unit speed nonnull N-magnetic curve in Lorentzian 3-space. If the function ϖ_1 is nonzero constant, then the curve α is a slant helix whose axis is the vector field.*

Proof We assume that α is an N-magnetic curve in Lorentzian 3-space with nonzero constant function ϖ_1, and then from Eq. (28) and Eq. (29), we get

$$\varepsilon_1\frac{\tau'}{\varkappa} = -\varepsilon_3\frac{\varkappa'}{\tau} = \varpi_1, \tag{32}$$

which implies that

$$\varepsilon_1\varkappa^2+\varepsilon_3\tau^2=const.$$

Furthermore, using Eq. (32) with the following equation for a different point of view, we get

$$-\tau\varkappa'+\varkappa\tau'=-\varepsilon_3\varpi_1(\varepsilon_3\varkappa^2+\varepsilon_1\tau^2)=const.$$

or

$$\frac{\varkappa^2}{(\varepsilon_3\varkappa^2+\varepsilon_1\tau^2)^{3/2}}\left(\frac{\tau}{\varkappa}\right)'=const.,$$

where ϖ_1 is a constant function. By Proposition 2 we obtain that α is a slant helix in Lorentzian 3-space. $\square$

Example 15 *We consider a spacelike N-magnetic curve α with timelike normal vector in Lorentzian space defined by*

$$\alpha(s)=\begin{pmatrix}\frac{\sqrt{3}-2}{2\sqrt{3}(\sqrt{3}+2)}\cos(\sqrt{3}+2)s+\frac{\sqrt{3}+2}{2\sqrt{3}(\sqrt{3}-2)}\cos(\sqrt{3}-2)s,\\ \frac{\sqrt{3}-2}{2\sqrt{3}(\sqrt{3}+2)}\sin(\sqrt{3}+2)s+\frac{\sqrt{3}+2}{2\sqrt{3}(\sqrt{3}-2)}\sin(\sqrt{3}-2)s,\\ -\frac{\cos 2s}{2\sqrt{3}}.\end{pmatrix}$$

The picture of the spacelike N-magnetic curve α is rendered in Figure 1. The curve α has the following curvature and torsion:

$$\varkappa(s)=\cos 2s,$$

$$\tau(s)=\sin 2s,$$

[10]. By using Corollary 14 we can easily see that $\varpi_1=2$, and so α is an N-magnetic curve.

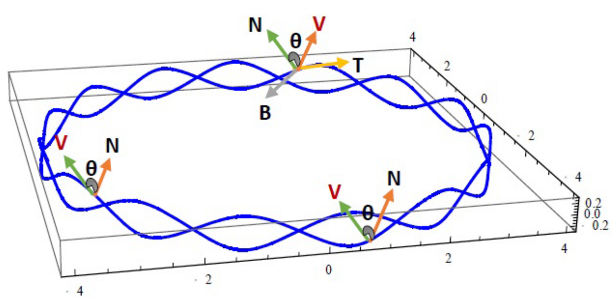

Figure 1. Spacelike N-magnetic cuve with $\varpi_1=2$.

Example 16 *We consider a spacelike N-magnetic curve α with spacelike normal vector in Lorentzian 3-space defined by*

$$\alpha(s)=(\frac{4\sinh 3s}{15},\frac{1}{40}\sinh 8s+\frac{2}{5}\sinh(-2s),\frac{1}{40}\cosh 8s-\frac{2}{5}\cosh(-2s)).$$

The picture of the spacelike N-magnetic curve α is rendered in Figure 2. The curve has the following curvature and torsion:

$$\varkappa(s)=4\cosh 3s,$$

$$\tau(s)=4\sinh 3s,$$

[10]. By using Corollary 14 we can easily see that $\varpi_1=3$, and so α is an N-magnetic curve.

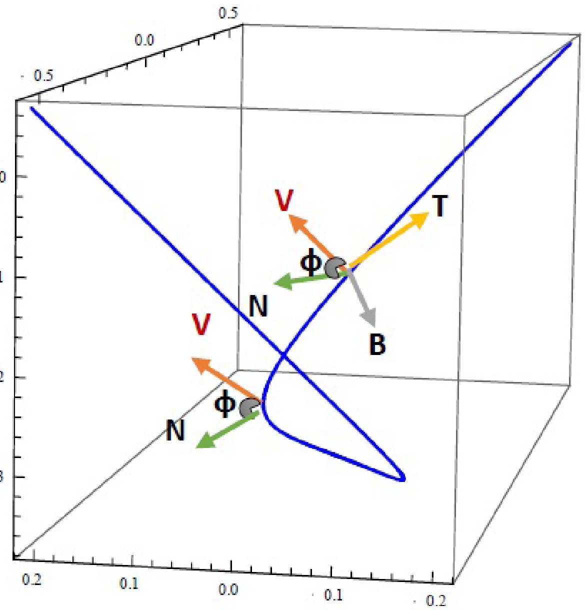

Figure 2. Spacelike N-magnetic curve with $\varpi_1 = 3$.

Example 17 *We consider a timelike N-magnetic curve α in Lorentzian 3-space defined by*

$$\alpha(s) = (\frac{\sqrt{3}}{2}\cosh 2s, \frac{1}{6}\cosh 3s + \frac{3}{2}\cosh(-s), \frac{1}{6}\sinh 3s + \frac{3}{2}\sinh(-s)).$$

The picture of the timelike N-magnetic curve α is rendered in Figure 3. The curve has the following curvature and torsion:

$$\varkappa(s) = \sqrt{3}\sinh 2s,$$

$$\tau(s) = \sqrt{3}\cosh 2s,$$

[10]. By using Corollary 14 we can easily see that $\varpi_1 = -2$, and so α is a N-magnetic curve.

Corollary 18 *Let α be a nonnull N-magnetic curve in Lorentzian 3-space. If the function ϖ_1 is zero then α is a circular helix whose axis is the vector field V.*

Proof It is obvious from Eq. (23). □

3.3. B-magnetic curves

In this section, we define a new kind of magnetic curve called B-magnetic curve in a 3D semi-Riemannian manifold. We obtain some characterizations and examples of this curve, and we draw the imagse of these curves using the program Mathematica.

Definition 19 *Let $\beta : I \subset \mathbb{R} \to M$ be a nonnull curve in 3D semi-Riemannian manifold (M, g) and F_V be a magnetic field on M. We call the curve β a B-magnetic curve if the binormal vector field of the curve satisfies the Lorentz force equation; that is,*

$$\nabla_{\beta'} B = \phi(B) = V \times B. \tag{33}$$

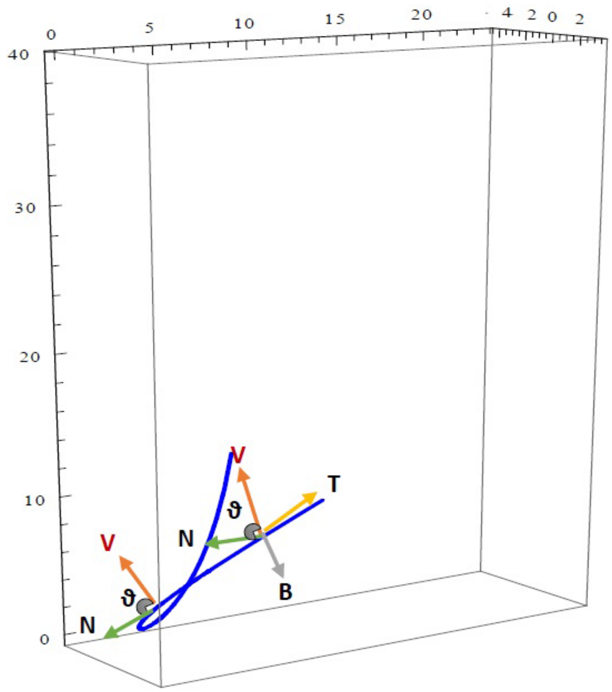

Figure 3. Timelike N-magnetic curve with $\varpi_1 = -2$.

Proposition 20 *Let β be a unit speed nonnull B-magnetic curve in $3D$ semi-Riemannian manifold (M, g) with the Serret–Frenet apparatus $\{T, N, B, \varkappa, \tau\}$. Then the Lorentz force in the Frenet frame written as*

$$\begin{bmatrix} \phi(T) \\ \phi(N) \\ \phi(B) \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2\varpi_2 & 0 \\ -\varepsilon_1\varpi_2 & 0 & -\varepsilon_3\tau \\ 0 & \varepsilon_2\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{34}$$

where ϖ_2 is a certain function defined by $\varpi_2 = g(\phi(T), N)$.

Proof Let β be a unit speed nonnull B-magnetic curve in $3D$ semi-Riemannian space with the Frenet apparatus $\{T, N, B, \varkappa, \tau\}$. Since we have

$$\phi(T) = \eta T + \rho N + \sigma B,$$

then using the following equalities:

$$\eta = \varepsilon_1 g(\phi(T), T) = 0,$$

$$\rho = \varepsilon_2 g(\phi(T), N) = \varepsilon_2\varpi_2,$$

$$\sigma = \varepsilon_3 g(\phi(T), B) = -\varepsilon_3 g(\phi(B), T) = -\varepsilon_3 g(\nabla_T B, T) = -\varepsilon_3 g(\varepsilon_3\tau N, T) = 0.$$

we get

$$\phi(T) = \varepsilon_2\varpi_2 N.$$

Similarly, we can easily obtain

$$\phi(T) = -\varepsilon_1\varpi_2 T - \varepsilon_3\tau B,$$

$$\phi(T) = \varepsilon_2\tau N.$$

□

Proposition 21 *Let β be a unit speed nonnull curve in semi-Riemannian manifold (M, g). The curve β is then the B-magnetic trajectory of a magnetic field V if and only if the magnetic vector field V can be written along the curve β as*

$$V = -\varepsilon_3 \tau T + \varepsilon_3 \varpi_2 B. \tag{35}$$

Proof Let β be a unit speed B-magnetic trajectory of a magnetic field V. Using Proposition 20 and Eq. (7), we obtain Eq. 35.

$$V = -\varepsilon_3 \tau T + \varepsilon_3 \varpi_2 B$$

Conversely, we assume that Eq. (35) holds. Then we get $\phi(B) = V \times B$, and so the curve β is a B-magnetic trajectory of the magnetic field V. □

Theorem 22 ***(main result)*** *Let β be a unit speed B-magnetic curve and V be a Killing vector field on a simply connected space form $(M(C), g)$. If the curve β is one of the B-magnetic trajectories of $(M(C), g, V)$, then its curvature and torsion satisfy the following equation:*

$$\varepsilon_3 (\varkappa')^2 + (\varepsilon_2 \tau^2 - \varepsilon_2 \frac{C}{2} - \eta)\varkappa^2 - (2a\varepsilon_2 \tau^2 + 3\varepsilon_2 C)\varkappa + \frac{\varkappa^4}{4} = \xi, \tag{36}$$

where C is curvature of the Riemannian space M and a, η, and ξ are constants.

Proof Let β be a magnetic field in a 3D semi-Riemannian manifold. Then V satisfies Eq. (35). Differentiating Eq. (35), we have

$$\nabla_T V = -\varepsilon_3 \tau T + \varepsilon_1 \tau(\varkappa - \varpi_2)N + \varepsilon_3 \varpi_2' B. \tag{37}$$

Lemma 1 implies that $V(v) = 0$, so Eq. (37) gives us

$$\tau' = 0. \tag{38}$$

If we differentiate Eq. (37) with respect to s,

$$\nabla_T^2 V = \varkappa\tau(\varpi_2 - \varkappa)T + \varepsilon_1 \tau(\varkappa' - 2\varpi_2')N + (\varepsilon_3 \varpi_2'' + \varepsilon_2 \tau^2 \varkappa - \varepsilon_2 \tau^2 \varpi_2)B. \tag{39}$$

Then, if Eq. (37) and Eq. (39) are considered with $V(\varkappa) = 0$ in Lemma 1, we obtain

$$[\varepsilon_1 \tau(\varkappa' - 2\varpi_2')]' + g(R(V, T)T, N) = 0.$$

In particular, since C is constant, $g(R(V, T)T, N) = Cg(V, N) = 0$ we have

$$\tau(\varkappa' - 2\varpi_2') = 0. \tag{40}$$

Similarly, if we combine Eq. (37) and Eq. (39) wih $V(\tau) = 0$ in Lemma 1 we get

$$[\frac{1}{\varkappa}(\varepsilon_3 \varpi_2'' + \varepsilon_2 \tau^2 \varkappa - \varepsilon_2 \varpi_2 \tau^2 - \varepsilon_2 C\varpi_2)]' + \left(\frac{\varkappa^2}{2}\right)' = 0, \ \tau = const. \tag{41}$$

If we integrate Eq. (41) we obtain

$$\varepsilon_3 \varpi_2'' + \varepsilon_2 \tau^2 \varkappa - \varepsilon_2 \varpi_2 \tau^2 - \varepsilon_2 C \varpi_2 + \frac{\varkappa^3}{2} - B\varkappa = 0,\ \tau = const. \tag{42}$$

Finally, if Eq. (42) is combined with Eq. (40) and is multiplied by $2\varkappa'$, we get

$$2\varepsilon_3 \varkappa'' \varkappa' + (2\varepsilon_3 \tau^2 - \varepsilon_2 C - 2B)\varkappa\varkappa' - (2a\varepsilon_2 \tau^2 + 3\varepsilon_2 C)\varkappa' + \varkappa^3 \varkappa' = 0,\ \tau = const., \tag{43}$$

whose integration is given

$$\varepsilon_3 (\varkappa')^2 + (\varepsilon_2 \tau^2 - \varepsilon_2 \frac{C}{2} - \eta)\varkappa^2 - (2a\varepsilon_2 \tau^2 + 3\varepsilon_2 C)\varkappa + \frac{\varkappa^4}{4} = \xi,\ \tau = const. \tag{44}$$

□

Example 23 *Let M be a three-dimensional sphere and β be a nonnull B-magnetic curve. If we get $\tau = a = 1$, $\eta = -2$, and $\xi = 5$ in Eq. (44), then we obtain the following first-order nonlinear ordinary differential equation:*

$$y'(t)^2 + 4y(t)^2 - 5y(t) + \frac{y(t)^4}{4} - 5 = 0.$$

We obtain Figures 4 and 5 using the program Mathematica.

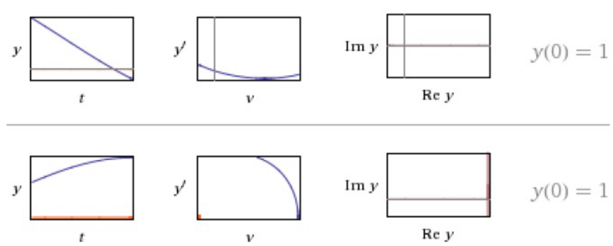

Figure 4. Plots of sample individual solutions.

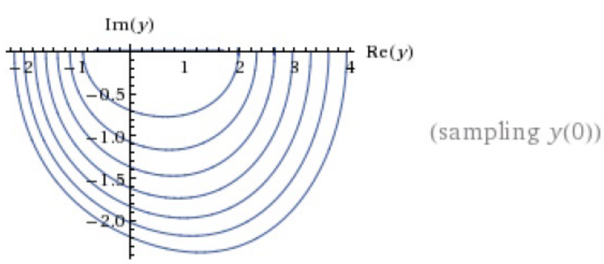

Figure 5. Sample solution family.

Corollary 24 *Let V be a Killing vector field on 3D semi-Riemannian manifold (M, g). Then each trajectory of the magnetic field V makes an angle $\theta(s)$ with the binormal vector field of the magnetic curve. For the angle $\theta(s)$ we have the following cases:*

Case 1. *If B and V are spacelike vectors and these vectors span a spacelike plane,*
$g(B, V) = \|B\| \, \|V\| \cos\theta$ [11].

Then using Eq. (35) and $g(B,B)=\varepsilon_3=1$, we obtain
$\cos\theta=\frac{g(B,V)}{\|B\|\|V\|}=\frac{g(B,-\tau T+\varpi_2 B)}{\|V\|}=\frac{\varpi_2}{\sqrt{|\varepsilon_1\tau^2+\varpi_2^2|}}$.

Case 2. *If B and V are spacelike vectors and these vectors span a timelike plane,*
$g(B,V)=\|B\|\,\|V\|\cosh\theta$ *[11].*
Then using Eq. (35) and $g(B,B)=\varepsilon_3=1$, we obtain
$\cosh\theta=\frac{g(B,V)}{\|B\|\|V\|}=\frac{g(B,-\tau T+\varpi_2 B)}{\|V\|}=\frac{\varpi_2}{\sqrt{|\varepsilon_1\tau^2+\varpi_2^2|}}$.

Case 3. *If B and V are timelike vectors in the same timecone,*
$g(B,V)=-\|B\|\,\|V\|\cosh\theta$ *[11].*
Then using Eq. (35) and $g(B,B)=\varepsilon_3=-1$, we obtain
$\cosh\theta=\frac{g(B,V)}{\|B\|\|V\|}=\frac{g(B,-\tau T+\varpi_2 B)}{-\|V\|}=\frac{\varpi_2}{\sqrt{|\tau^2-\varpi_2^2|}}$.

Case 4. *If N spacelike (resp. timelike) and V timelike (resp. spacelike) vectors are in the future timecone,*
$g(B,V)=\|B\|\,\|V\|\sinh\theta$ *[11].*
Then using Eq. (35) and $g(B,B)=\varepsilon_3$, we obtain
$\sinh\theta=\frac{g(B,V)}{\|B\|\|V\|}=\frac{g(B,-\varepsilon_3\tau T+\varepsilon_3\varpi_2 B)}{\varepsilon_3\|V\|}=\frac{\varpi_2}{\sqrt{|\varepsilon_1\tau^2+\varepsilon_3\varpi_2^2|}}$.

Corollary 25 *Let β be a nonnull B-magnetic curve in a Lorentzian 3-space with ϖ_2 constant; then β is a circular helix whose axis is the vector field V.*

Proof It is obvious from Eq. (36). □

Example 26 *We consider a timelike B-magnetic curve in Lorentzian 3-space defined by*

$$\beta(s)=(\cosh\frac{s}{\sqrt{2}},\frac{s}{\sqrt{2}},\sinh\frac{s}{\sqrt{2}})$$

[10]. The picture of the B-magnetic curve β is rendered in Figure 6.

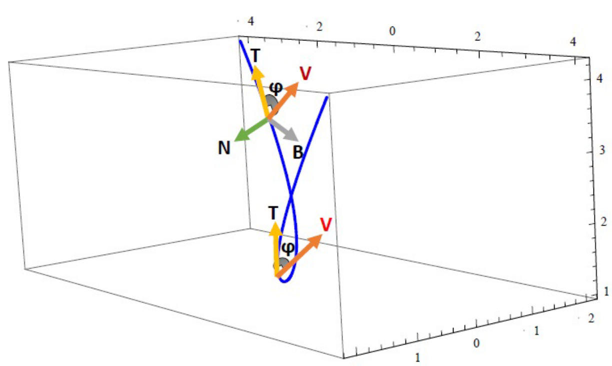

Figure 6. Timelike B-magnetic curve.

The curve β has the following curvature and torsion:

$$\varkappa(s)=\tau(s)=\frac{1}{2}.$$

Using Corollary 25 we can easily see that β is a B-magnetic curve.

Example 27 *We consider a timelike B-magnetic curve in Lorentzian 3-space defined by*

$$\beta(s) = (\frac{1}{\sqrt{2}}\sinh s, \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cosh s)$$

[10]. The picture of the B-magnetic curve β is rendered in Figure 7.

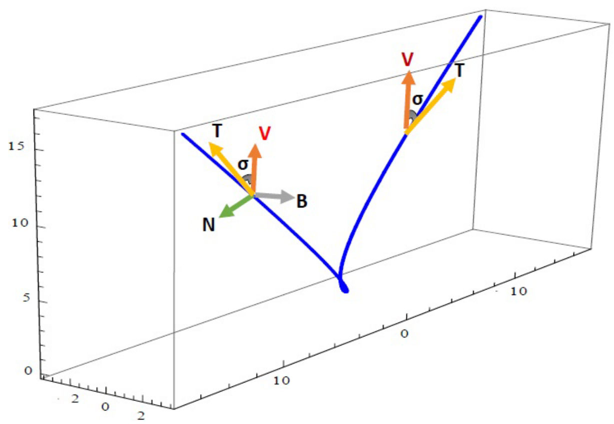

Figure 7. Spacelike B-magnetic curve.

The curve β has the following curvature and torsion:

$$\varkappa(s) = \tau(s) = \frac{1}{\sqrt{2}}.$$

Using Corollary 25 we can easily see that β is a B-magnetic curve.

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