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Stability in a job market with linearly increasing valuations and quota system

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Abstract: We consider a job market in which preferences of players are represented by linearly increasing valuations. The set of players is divided into two disjoint subsets: a set of workers and a set of firms. The set of workers is further divided into subsets, which represent different categories or classes in everyday life. We consider that firms have vacant posts for all such categories. Each worker wants a job for a category to which he/she belongs. Firms have freedom to hire more than one worker from any category. A worker can work in only one category for at most one firm. We prove the existence of a stable outcome for such a market. The college admission problem by Gale and Shapley is a special case of our model.

Key words: Stable marriage, pairwise stability, job allocation, linear valuation, quota system

1. Introduction

Most theoretical work on bipartite matching traces its history to the papers of Gale and Shapley [6] and Shapley and Shubik [11]. In bipartite matching, we have two disjoint sets, F and W. The main purpose is to match the elements of F to the elements of W. Matching between elements of the same set is forbidden in bipartite matching markets. For convenience, we use the term "player" for an element in $F \cup W$. If we match exactly one player of set F to exactly one player of the set W, then the matching is called "one-to-one" matching. If a group of players of one set is matched to one player of the other set, such a matching is called "many-to-one" matching. If there is freedom for players of both sets to be matched with as many players of the opposite set as they want, such matching is called "many-to-many" matching.

The motivation of the remarkable paper by Gale and Shapley [6] was to find a rational criterion for matching students with colleges that respected the preferences of both groups. They formulated the model without side payments. Each college has a complete preference list of those students whom the college is willing to admit as well as a quota giving an upper bound to the number of students that can be admitted. The original approach was to first consider a special case, in which each college could accept only one student. Due to the resemblance of this special case to the marriage of a man and a woman, this model is known as the "marriage model". In [6], Gale and Shapley proposed an algorithm for finding a "stable" matching, in which no man and no woman are matched to an unacceptable mate, and no man and no woman who are not matched to each other would both prefer to be. Gale and Shapely also discussed the criterion for stability of the college admission problem, that is, the many-to-one case.

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In contrast to [6], money plays an explicit role in the paper by Shapley and Shubik [11]. The model by Shapley and Shubik [11] is called the "assignment game". Players in this model can be thought of as buyers and sellers. Shapely and Shubik used linear programming theory to show the existence of stable assignments. They also showed that the core of the assignment game is nonempty.

The two main directions in which the marriage model and assignment game have developed in the literature involve models in which price setting is accomplished simultaneously with matching, and models of many-to-one and many-to-many matchings.

For the developments involving models with price setting, Eriksson and Karlander [3] provided a single market to unify these discrete and continuous models. However, their proof for stable outcomes and lattice properties of the core does not include the assignment game by Shapley and Shubik [11]. Sotomayor [13], unified these models and proved the nonemptiness of the core, which holds for both markets. She used combinatorial arguments to prove these results. A generalization of the hybrid models of [3] and [13] was provided by Farooq [4]. In [4], preferences of players are assumed to be linear functions of money, whereas money is treated as a continuous variable. He showed the existence of a stable outcome for his model by constructing an algorithm. Ali and Farooq [1] also considered a matching model with linear valuations with money as a discrete variable.

The paper by Kelso and Crawford [9] is an example of models that involve many-to-one matching. They introduced a gross substitute condition to show the nonemptiness of the core. Roth [10] considered a model of many-to-many matching of firms and workers. For this model, he showed the existence of firm-optimal and worker-optimal stable outcomes. He also proved that the stable outcome that is best for all the firms is the worst for all the workers and vice versa. For many-to-many matching, set-wise stability was proposed by Sotomayor [12]. She proved that set-wise stability is a stronger requirement than pairwise stability. She also showed that set-wise stability is a general concept of stability. Ali and Farooq [2] extended the model of Farooq [4] by considering many-to-one matching. In this model, firms are allowed to hire as many workers as these firms require. They presented an algorithm to show the existence of stable many-to-one matching for their model. Recently, Femenia et al. [5] introduced a matching model in which they presented the definition of stability with a quota restriction. They proved that the set of stable solutions may be empty under unrestricted institution preferences. They also showed that the existence of stable matching with a quota is guaranteed when there is a responsive restriction on the preferences of the institution. Karakaya and Koray [7] considered a two-sided many-to-one matching model with quota and budget constraints. The players in this model were divided into two sets: departments of the university and applicants. A department-proposing algorithm was presented in [7], which also showed that if the algorithm terminates, it yields a core stable matching. In continuation of [7], Karakaya [8] offered an applicant-proposing algorithm. Karakaya [8] also showed that the department-proposing and applicant-proposing algorithms are extensions of the Gale and Shapley algorithm [6]. In addition, if either of these two algorithms terminates, they produce a core stable matching. However, it was shown by Karakaya [8] that unfortunately these algorithms do not always stop.

In this paper, we consider a two-sided matching model in which firms have to select workers from different categories or classes of the set of workers. Factors like education, availabilities of resources, abilities, and age may be criteria for the division of the set of workers into different categories. An example of such a model is Quaid-i-Azam University, Islamabad, Pakistan. The university provides admission to students from all major regions of the country as per approved quota (for details, see *http://www.qau.edu.pk/admission-quota/*). No student can join more than one department. Each department has to select students from all the regions (if available) considering the quota of the region. We notice that in all of these models, i.e. the marriage model

[6], assignment games [11], hybrid models [1, 3, 4, 13], and models involving many-to-one and many-to-many matchings [2, 8, 7, 9, 10, 12], workers are matched with firms without considering different categories or classes of set of workers. The purpose of this paper is to take such real-life factors into account and propose a model in which the set of workers is divided into categories. Each firm can hire more than one worker and it has vacant posts for each category. Each firm requires a finite number of workers for any particular category. This finite number is called the quota of that category. Firms hire workers considering the category and quota of the category. Workers in this model can work in only one category, for at most one firm.

We organize this paper as follows. We describe our model and pairwise stability in our model in Section 2. In Section 3, we propose an algorithm to show the existence of a stable outcome. We also present important features of the algorithm in this section.

2. Model description and pairwise stability

We consider a matching market with two disjoint sets: the set of workers and set of firms. Each worker wants a job and each firm needs workers. A worker increases the revenue of the firm and in return the firm pays its worker a certain amount of money, called the *salary* of the worker. We divide the set of workers into different types or classes. We call these types or classes *categories* of workers. We consider factors like abilities, educations, experiences, ages, etc. as criteria for the division of the set of workers into these categories. Firms hire workers from each category to fulfill their quota, where the quota of a firm is the total number of workers that the firm can hire.

Now we describe our model mathematically. We use W and F to denote finite sets of workers and firms, respectively. $E = W \times F$ denotes all possible pairs of workers and firms. We divide the set of workers into different categories. Supposing that there are m categories, then we have m subsets of W. We denote these subsets by w^a, $a \in I$ where $I = \{1, 2, \cdots, m\}$ is an indexing set. Also, $\cup_{a \in I} w^a = W$. We also assume that for some $a, a' \in I$, $w^a \cap w^{a'} \neq \emptyset$, where $a \neq a'$. This means that all subsets of W may not be disjoint.

Each firm hires some workers from w^a, $a \in I$. The maximum number of workers that a firm j can hire from w^a is denoted by $\mu^a(j)$ for some $a \in I$. The total number of workers that firm j can hire is denoted by $\mu(j)$. Thus, $\sum_{a=1}^{m} \mu^a(j) = \mu(j)$. If $\sum_{a=1}^{m} \mu^a(j) < \mu(j)$, for $j \in F$, firm j is said to be unsaturated. Note here that each worker can work for only one category of at most one firm and firms cannot hire more workers than their quota for each category.

A worker and firm can negotiate between each other by increasing or decreasing the salary. We assume that salaries in this model are bounded; that is, for each $(i, j) \in E$, we have $\underline{\pi}_{ij}$ and $\overline{\pi}_{ij}$, which denote the lower and upper bound on the salary, respectively. Also, $\underline{\pi}$, $\overline{\pi} \in \mathbf{Z}^E$[1] with $\underline{\pi} \leq \overline{\pi}$ [2]. We denote the *salary vector* by $p = (p_{ij} \mid (i, j) \in E) \in \mathbf{Z}^E$. The salary vector is called *feasible* if $\underline{\pi} \leq p \leq \overline{\pi}$. We assume that each firm has a list of preferences of workers that it wants to hire and, similarly, each worker has a preferences list of firms for which he/she is willing to work. We represent the preferences of players by strictly increasing functions, called linearly increasing valuations[3]. For each $(i, j) \in E$, define $\nu_{ij}(x)$ and $\nu_{ji}(x)$ from $\mathbf{Z}$ into $\mathbf{R}$. $\nu_{ij}(x)$ represents the valuation of a worker i, at a salary x from a firm j, when worker i joins firm j (that is, is matched with j). Similarly, $\nu_{ji}(x)$ represents the valuation of the firm j when it hires worker i and pays

[1] $\mathbf{R}^A$ represents the $|A|$-dimensional Euclidean space with coordinate indexed by the elements of the set A and $\mathbf{Z}^A$ stands for the set of all integral vectors contained in $\mathbf{R}^A$.

[2] For any two vectors $x \in \mathbf{R}^E$ and $y \in \mathbf{R}^E$, we say that $x \leq y$ if $x_{ij} \leq y_{ij}$ for all $(i, j) \in E$.

[3] By valuation, we mean estimation of some asset or real property.

him or her salary x. Furthermore, we define these *linearly increasing valuations* $\nu_{ij}(\cdot)$ and $\nu_{ji}(\cdot)$, from $\mathbf{Z}$ into $\mathbf{R}$ by

$$\nu_{ij}(x) = \alpha_{ij}x + \beta_{ij}, \quad \nu_{ji}(x) = \alpha_{ji}x + \beta_{ji}, \tag{2.1}$$

where $x \in \mathbf{Z}$, α_{ij} and α_{ji} are given positive real numbers, and β_{ij} and β_{ji} are any given real numbers.

With the help of valuations defined in (2.1), players of both sets can compare the players of the opposite side. A worker i *prefers* firm j to firm j' at salary $x, x' \in \mathbf{Z}$, if $\nu_{ij}(x) > \nu_{ij'}(x')$. If $i \in W$ is *indifferent* between firms j and j' at salary $x, x' \in \mathbf{Z}$, then it can be written as $\nu_{ij}(x) = \nu_{ij'}(x')$. We can define the terms "prefer" and "indifferent" for a firm in the similar way.

If a worker is willing to join a firm, then the firm is *acceptable* to the worker. Similarly, if a firm agrees to hire a worker, then the worker is acceptable to the firm. By $\nu_{ij}(x) \geq 0$, we mean that j is *acceptable* to i at salary $x \in \mathbf{Z}$ and $\nu_{ji}(x) \geq 0$ means i is acceptable to j at salary x.

We define $S^a = (S^a_j \mid j \in F)$, for all $a \in I$, where S^a_j is defined as follows:

$$S^a_j = \{i \in w^a \mid i \text{ and } j \text{ are matched}\}. \tag{2.2}$$

If $S^a_j = \emptyset$ for some $a \in I$ and $j \in F$, we say there is no worker working for firm j in category w^a. We say that firm j *employs* worker i in category w^a if $i \in S^a_j$.

A set $X = \{(S^a_j, j) \mid a \in I, j \in F\}$ is called a *job allocation* if:

(i) $|S^a_j| \leq \mu^a(j)$ for all $a \in I$, $j \in F$;

(ii) $S^a_j \cap S^a_{j'} = \emptyset$ for all $j, j' \in F$ with $j \neq j'$.

Condition (i) implies a quota condition on each $j \in F$, whereas the second condition suggests that each worker can work for only one firm.

Fixing p_{ij}, we define $q \in \mathbf{R}^W$ as follows:

$$q_i = \begin{cases} \nu_{ij}(p_{ij}) & \text{if } i \in S^a_j \text{ for any } j \in F \\ 0 & \text{otherwise} \end{cases} \qquad (\forall i \in W). \tag{2.3}$$

We have $r = (r_j \mid j \in F) \in \mathbf{R}^F$ where $r_j \in \mathbf{R}^m$ for each $j \in F$, which is defined as

$$r^a_j = \begin{cases} \min\{\nu_{ji}(-p_{ij}) \mid i \in S^a_j\} & \text{if } |S^a_j| = \mu^a(j) \\ 0 & \text{otherwise} \end{cases} \qquad (\forall a \in I), \tag{2.4}$$

where the minimum over an empty set is defined to be 0.

A quadruple $(X; p, q, r)$ is said to be an outcome if X is a job allocation, p is a feasible salary vector, and q and r are defined by (2.3) and (2.4), respectively,

In the sequel, whenever we say that $S^a_j \in X$ (or $j \in X$), we always mean that $(S^a_j, j) \in X$. Also, by $(i, j) \in X$, we always mean that $i \in S^a_j$ for some $a \in I$.

An outcome $(X; p, q, r)$ is said to be *blocked* by a worker-firm pair that are not matched to one another but the worker prefers the firm to his/her current employer[4] and the firm prefers the worker to its current

[4]For convenience, we say that a player is self-matched, if it is unmatched.

workers or still has a vacancy to hire a worker. Mathematically, $(i,j) \in E$ is a blocking pair of the outcome $(X;p,q,r)$ if there exists $\alpha \in \mathbf{Z}$ with $\underline{\pi}_{ij} \leq \alpha \leq \overline{\pi}_{ij}$ such that $i \notin S_j^a$ and $\nu_{ij}(\alpha) > q_i$, $\nu_{ji}(-\alpha) > r_j^a$, for some $a \in I$.

2.1. Pairwise stability

Now we define the pairwise stability for outcome $(X;p,q,r)$ as follows:

(ps1) $\nu_{ij}(p_{ij}) \geq 0$ and $\nu_{ji}(-p_{ij}) \geq 0$ for all $(i,j) \in X$.

(ps2) $\nu_{ij}(\alpha) \leq q_i$ or $\nu_{ji}(-\alpha) \leq r_j^a$ for all $a \in I$, for all $\alpha \in \mathbf{Z}$ with $\underline{\pi}_{ij} \leq \alpha \leq \overline{\pi}_{ij}$ and for all $(i,j) \in E$.

Condition (ps1) reflects the mutual acceptability of matched pairs. Condition (ps2) requires that the outcome cannot be blocked by any pair.

3. Existence of a stable outcome in our model

In this section, we shall prove the existence of pairwise stability for our model described in Section 2. In order to show that a pairwise stable outcome always exists, we construct an algorithm. In this algorithm we initially set the maximum possible salary such that workers are acceptable to the firms and we find a set of mutually acceptable worker-firm pairs. Then we match workers to their most favorite firms. We modify the salary of worker-firm pairs if the worker is not matched to his/her most favorite firm. In each iteration, we modify the salary vector, preserving its feasibility, to obtain pairwise stability. If every worker from the set of mutually acceptable players is matched, then our algorithm terminates. At the end of this section, we prove that at termination our algorithm outputs stable matching.

For each $(i,j) \in E$ we define $p_{ij} \in \mathbf{Z}$ as follows:

$$p_{ij} := \begin{cases} \overline{\pi}_{ij} & \text{if } \nu_{ji}(-\overline{\pi}_{ij}) \geq 0 \text{ and} \\ \max\{\underline{\pi}_{ij}, \lfloor \frac{\beta_{ji}}{\alpha_{ji}} \rfloor\} & \text{otherwise} \end{cases} \qquad (\forall (i,j) \in E). \tag{3.1}$$

Note that p is a feasible salary vector. Also note that p_{ij}, defined by (3.1), is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]$ for which $\nu_{ji}(-p_{ij}) \geq 0$.

Next, define $\widetilde{E}$ by

$$\widetilde{E} = \{(i,j) \in E \mid \nu_{ij}(p_{ij}) \geq 0 \text{ and } \nu_{ji}(-p_{ji}) \geq 0\}. \tag{3.2}$$

$\widetilde{E}$ denotes the set of mutually acceptable players.

For all $i \in W$, define $\tilde{q}_i$ as follows:

$$\tilde{q}_i = \max\{\nu_{ij}(p_{ij}) \mid (i,j) \in \widetilde{E}\} \quad (\forall i \in W). \tag{3.3}$$

Also, define $\widetilde{E}_W^a$ and $\widetilde{E}_W$ as follows:

$$\widetilde{E}_W^a = \{(i,j) \in \widetilde{E} \mid i \in w^a, \nu_{ij}(p_{ij}) = \tilde{q}_i\} \quad (\forall a \in I), \tag{3.4}$$

and

$$\widetilde{E}_W = \cup_{a \in I} \widetilde{E}_W^a. \tag{3.5}$$

$\widetilde{E}^a_W$ denotes the set of worker-firm pairs in which the worker belongs to category "a" and the firm is the most preferred by the worker. Let $\widetilde{F}$ denote the set of matched firms in X; that is, $\widetilde{F} \subseteq F$ and it is defined by

$$\widetilde{F} = \{j \in F | j \text{ is matched in } X\}. \tag{3.6}$$

Now we find a job allocation $X = \{(S^a_j, j) \mid a \in I, j \in F\}$[5] in the bipartite graph $(W, F; \widetilde{E}_W)$ that satisfies the following:

$$X \text{ matches all members of } \widetilde{F}; \tag{3.7}$$

$$S^a_j \cap S^a_{j'} = \emptyset \text{ for all } S^a_j, S^a_{j'} \in X \text{ with } j \neq j' \text{ among the matching satisfying (3.7)}; \tag{3.8}$$

$$r^a_j, \text{ for each } a \in I \text{ and } j \in F, \text{ is maximum among the matchings satisfying (3.7) and (3.8)}; \tag{3.9}$$

$$\big||S^a_j| - \mu^a(j)\big|, \text{ for each } a \in I \text{ and } j \in F, \text{ is minimum among the matching satisfying (3.7), (3.8), and (3.9)}. \tag{3.10}$$

Define U^a, for each $a \in I$, by

$$U^a := \{(i, j) \in \widetilde{E}^a_W \mid i \notin S^a_j, a \in I\}. \tag{3.11}$$

U^a contains worker-firm pairs in which a worker from category "a" is not matched to his or her most favorite firm.

Lemma 3.1 *For each $a \in I$, if $(i, j) \in U^a$, then there exists $k \in W \setminus \{i\}$ such that $k \in S^a_j$ and $r^a_j \geq \nu_{ji}(-p_{ij})$, where r^a_j is defined by (2.4).*

Proof One can easily verify the assertion by considering (3.9) and (3.10). □

Lemma 3.2 *If $U^a = \emptyset$ for all $a \in I$, then X satisfies (ps1) and (ps2).*

Proof We know that if $(i, j) \in X$ then $(i, j) \in \widetilde{E}$, since $\nu_{ij}(p_{ij})$ and $\nu_{ji}(-p_{ij})$ are nonnegative for all $(i, j) \in \widetilde{E}$. Therefore, (ps1) holds.

Let $U^a = \emptyset$ for all $a \in I$. On the contrary, suppose that (ps2) does not hold true. This means that there exists $(i, j) \in E$ and $\alpha \in [\underline{\pi}_{ij}, \overline{\pi}_{ij}]$ such that $\nu_{ij}(\alpha) > q_i$ and $\nu_{ji}(-\alpha) > r^a_j$. Since $r^a_j \geq 0$, we have $p_{ij} \geq \alpha$ by (3.1). This implies that $\nu_{ij}(p_{ij}) \geq \nu_{ij}(\alpha) > q_i$. Since $U^a \subseteq \widetilde{E}^a_W$ and $U^a = \emptyset$ imply that $\nu_{ij}(p_{ij}) < q_i$ by (3.4), we have a contradiction. Thus, our assumption is wrong, and X satisfies (ps2). This completes the proof. □

(ps2) may not hold if $U^a \neq \emptyset$, for some $a \in I$. We suppose that $U^a \neq \emptyset$, for some $a \in I$. Now we modify the salary vector to obtain (ps2). In the modification procedure, we shall decrease the salary vector in such a way that the feasibility of the salary vector and (ps1) are preserved. We initially set salary vector p given by (3.1).

We know that, for some $a, a' \in I$, $U^a \cap U^{a'} \neq \emptyset$ with $a \neq a'$. Define r^*_j for $j \in F$ as follows:

$$r^*_j = \min\{r^a_j \mid a \in I \text{ s.t } (i, j) \in U^a\} \quad (\forall j \in F). \tag{3.12}$$

[5] If some $j \in F$ is not matched in X then we can always add a pair (S^a_j, j) in X with $S^a_j = \emptyset$, for all $a \in I$.

For each $(i,j) \in U$, for some $a \in I$, we find a modified salary $\tilde{p}$ such that $\nu_{ji}(-\tilde{p}_{ij}) \geq r_j^a$, for some $(i,j) \in U^a$. To calculate $\tilde{p}$, we find an integer n_{ij} for $(i,j) \in U^a$, $a \in I$, as follows:

$$n_{ij} := \max\{1, \lceil \frac{r_j^* - \nu_{ji}(-p_{ij})}{\alpha_{ji}} \rceil\} \quad (\forall (i,j) \in U^a, a \in I). \tag{3.13}$$

Define the modified salary $\tilde{p}$ by

$$\tilde{p}_{ij} := \begin{cases} \max\{\underline{\pi}_{ij}, p_{ij} - n_{ij}\} & \text{if } (i,j) \in U^a, \\ p_{ij} & \text{otherwise.} \end{cases} \tag{3.14}$$

$\tilde{p}$ is feasible since $\underline{\pi}_{ij} \leq \tilde{p}_{ij} \leq \overline{\pi}_{ij}$ for each $(i,j) \in E$.

Lemma 3.3 *For each $(i,j) \in U^a$, $a \in I$, n_{ij} is the minimum integer for which $\nu_{ji}(-(p_{ij} - n_{ij})) \geq r_j^*$ holds.*

Proof Let $(i,j) \in U^a$ for some $a \in I$. For $r_j^* = \nu_{ji}(-p_{ij})$ we have $n_{ij} = 1$, which is the minimum integer by (3.13) and $\nu_{ji}(-p_{ij} + 1) = \nu_{ji}(-p_{ij}) + \alpha_{ji} > r_j^*$. This proves the assertion.

Now consider $r_j^* > \nu_{ji}(-p_{ij})$ and, on the contrary, suppose that $\nu_{ji}(-p_{ij} + n_{ij} - 1) \geq r_j^*$. We have

$$\nu_{ji}(-p_{ij}) + \alpha_{ji}(n_{ij} - 1) \geq r_j^*.$$

After simplification, we have

$$n_{ij} - 1 \geq \frac{r_j^* - \nu_{ji}(-p_{ij})}{\alpha_{ji}}.$$

Definition (3.13) further implies that

$$\lceil \frac{r_j^* - \nu_{ji}(-p_{ij})}{\alpha_{ji}} \rceil - 1 \geq \frac{r_j^* - \nu_{ji}(-p_{ij})}{\alpha_{ji}}.$$

This, however, is not true. Hence, the assertion holds. This completes the proof. □

Next we define a subset L^a of U^a by

$$L^a = \{(i,j) \in U^a \mid p_{ij} - n_{ij} < \underline{\pi}_{ij}\} \quad (\forall a \in I). \tag{3.15}$$

Observe that $\tilde{p}_{ij} = \underline{\pi}_{ij}$ for all $(i,j) \in L^a$, $a \in I$.

In the following lemma we show the importance of the modified salary vector $\tilde{p}$.

Lemma 3.4 *Suppose $U^a \backslash L^a \neq \emptyset$, $a \in I$. There is at least one $(i,j) \in U^a \backslash L^a$, $a \in I$, for which $\nu_{ji}(-\tilde{p}_{ij}) \geq r_j^a$. Moreover, $\tilde{p}_{ij}$ is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]$ for which this inequality holds.*

Proof Let $(i,j) \in U^a \setminus L^a$ for some $a \in I$. Then, by (3.14), we have

$$\begin{aligned} \nu_{ji}(-\tilde{p}_{ij}) &= \nu_{ji}(-p_{ij} + n_{ij}) \\ &= \nu_{ji}(-p_{ij}) + \alpha_{ji}(n_{ij}). \end{aligned}$$

If $\nu_{ji}(-p_{ij}) = r_j^*$, then $n_{ij} = 1$ by (3.13). Thus, we have $\nu_{ji}(-\tilde{p}_{ij}) > r_j^*$ since $\alpha_{ij} > 0$. Otherwise,

$$\begin{aligned}\nu_{ji}(-\tilde{p}_{ij}) &= \nu_{ji}(-p_{ij}) + \alpha_{ji}(\lceil \frac{r_j^* - \nu_{ji}(-p_{ij})}{\alpha_{ji}} \rceil) \\ &\geq \nu_{ji}(-p_{ij}) + \alpha_{ji}(\frac{r_j^* - \nu_{ji}(-p_{ij})}{\alpha_{ji}}) \\ &= r_j^*.\end{aligned}$$

Thus, we have $\nu_{ji}(-\tilde{p}_{ij}) \geq r_j^*$ and $r_j^* = r_j^a$ for some $(i,j) \in U^a$; therefore, the first part of the assertion holds.

Next we prove that $\tilde{p}_{ij}$, for $(i,j) \in U^a \setminus L^a$, is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}] \in \mathbf{Z}$ such that $\nu_{ji}(-\tilde{p}_{ij}) \geq r_j^a$. Since $r_j^a \geq r_j^*$, therefore, $\nu_{ji}(-\tilde{p}_{ij}) \geq r_j^a \geq r_j^*$. By Lemma 3.3, n_{ij} is the minimum integer for which $\nu_{ji}(-\tilde{p}_{ij}) \geq r_j^*$. Thus, by (3.14), p_{ij} is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]$ such that $\nu_{ji}(-\tilde{p}_{ij}) \geq r_j^a$. □

Lemma 3.5 *For each* $(i,j) \in L^a$, $a \in I$, $\nu_{ji}(-\tilde{p}_{ij}) < r_j^a$.

Proof On the contrary, suppose that $\nu_{ji}(-\tilde{p}_{ij}) \geq r_j^a$ for any $(i,j) \in L^a$, $a \in I$. Since for all $(i,j) \in L^a$, $a \in I$, $\tilde{p}_{ij} = \underline{\pi}_{ij}$, let $\underline{\pi}_{ij} = p_{ij} - n'_{ij}$. We have $\nu_{ji}(-(p_{ij} - n'_{ij})) \geq r_j^a$. Note here $p_{ij} - n_{ij} < \underline{\pi}_{ij} = p_{ij} - n'_{ij}$ for $(i,j) \in L^a$. This implies that $n_{ij} > n'_{ij}$. This, however, contradicts the minimality of n_{ij} by Lemma 3.3. Hence, our assumption is wrong. □

Define a subset $\widetilde{E}_0^a$ of U^a by

$$\widetilde{E}_0^a = \{(i,j) \in U^a \mid \nu_{ij}(\tilde{p}_{ij}) < 0\} \quad (\forall a \in I). \tag{3.16}$$

Now we propose our algorithm.

Algorithm `Job_Allocation`

Step 0: Out $r = \mathbf{0}$ and $\widetilde{F} = \emptyset$. Initially define p, $\widetilde{E}$, $\tilde{q}$, $\widetilde{E}_W^a$, for all $a \in I$, and $\widetilde{E}_W$ by (3.1)−(3.5), respectively. Find a matching X in the bipartite graph $(W, F; \widetilde{E}_W)$ satisfying (3.7)−(3.10). For all $a \in I$, define S^a by (2.2), r^a by (2.4) and U^a by (3.11).

Step 1: If $U^a = \emptyset$, for all $a \in I$, then stop.

Step 2: For each pair $(i,j) \in U^a$, $a \in I$, calculate r^* by (3.12) and n_{ij} by (3.13), and find $\tilde{p}$ by (3.14). Define L^a and $\widetilde{E}_0^a$, for all $a \in I$, by (3.15) and (3.16), respectively.

Step 3: Put $p := \tilde{p}$ and modify $\widetilde{E}$ by

$$\widetilde{E} := \widetilde{E} \setminus \cup_{a \in I}\{L^a \cup \widetilde{E}_0^a\}. \tag{3.17}$$

Calculate $\tilde{q}$ by (3.3) and $\widetilde{E}_W^a$ by (3.4), and modify $\widetilde{E}_W$ by (3.5) for the updated $\widetilde{E}$ and p. Find a matching X in the bipartite graph $(W, F; \widetilde{E}_W)$ satisfying (3.7)−(3.10). Define S^a by (2.2) and r^a by (2.4), and update U^a by (3.11), for all $a \in I$. Go to Step 1.

For the sake of convenience we will use the following terminology in the rest of the work. We will use $(old)*$ and $(new)*$ for sets/vectors/integers before and after update in some iteration of the `Job_Allocation`, respectively.

Lemma 3.6 *In each iteration of* `Job_Allocation`*, there exists a matching in* $(W, F; \widetilde{E}_W)$ *satisfying* (3.7)−(3.10) *at Step 3.*

Proof Let $i \in (old)S_j^a$ after the tth iteration, $(t \geq 1)$. This means that $(i,j) \in (old)(\widetilde{E}_W^a \setminus U^a)$. In the $(t+1)$th iteration, we have $(old)p_{ij} = (new)p_{ij}$ by (3.14) at Step 2. It implies $(old)q_i = (new)q_i$. Thus, $(i,j) \in (new)\widetilde{E}_W^a$ and $(old)(\widetilde{E}_W^a \setminus U^a) \subseteq (new)\widetilde{E}_W^a$ at Step 3. Since $(old)S_j^a$, for each $j \in F$ and $a \in I$, satisfies (3.10) at Step 3 in the tth iteration, one can therefore find a matching in $(P, Q; (new)\widetilde{E}_P)$ at Step 3 satisfying (3.7)−(3.10). □

The next lemma shows important features of the `Job_Allocation`.

Lemma 3.7 *In each iteration of the* `Job_Allocation`*, the following hold:*

(i) *For some $a \in I$, if $U^a \setminus \{L^a, \widetilde{E}_0^a\} \neq \emptyset$ at Step 2, then p_{ij} decreases at Step 3, for all $(i,j) \in U^a \setminus \{L^a, \widetilde{E}_0^a\}$.*

(ii) *If $L^a \neq \emptyset$ or $\widetilde{E}_0^a \neq \emptyset$, for some $a \in I$, then $\widetilde{E}$ decreases.*

(iii) *For $j \in F$, $|S_j^a|$ increases or remains the same and $|S_j^a| \leq \mu^a(j)$ for all $a \in I$.*

(iv) *$\widetilde{E}$ decreases or remains the same.*

(v) *r_j^a increases or remains the same, for each $j \in F$ and $a \in I$.*

Proof (i) Initially, p is defined at Step 0. For some $a \in I$, if $U^a \neq \emptyset$, then in each iteration at Step 2, we find $\tilde{p}$ by (3.14). If $U^a \setminus \{L^a, \widetilde{E}_0^a\} \neq \emptyset$ then we have $\tilde{p}_{ij} = p_{ij} - n_{ij}$, for all $(i,j) \in U^a \setminus \{L^a, \widetilde{E}_0^a\}$, where n_{ij} is a positive integer. This proves the assertion.

(ii) Initially, $\widetilde{E}$ is defined by (3.2) at Step 0 and it is modified by (3.17) at Step 3 in each iteration. If $L^a \neq \emptyset$ or $\widetilde{E}_0^a \neq \emptyset$, for some $a \in I$, by (3.17), $(new)\widetilde{E}$ decreases at Step 3.

(iii) First we show that $|S_j^a| \leq \mu^a(j)$ for all $j \in F$, $a \in I$. On the contrary, suppose that there exist $j \in F$ and $a \in I$ such that $|S_j^a| > \mu^a(j)$ for a matching X at Step 3. Let $||S_j^a| - \mu^a(j)| = l$, where $l > 0$, be the minimum by (3.10). We can find $S_j'^a \subseteq S_j^a$ such that $|S_j'^a| = \mu^a(j)$. Thus, we have

$$\big||S_j'^a| - \mu^a(j)\big| = 0 < l,$$

which contradicts the minimality of l. Thus, our assumption is wrong. Therefore, for all $j \in F$, $a \in I$, we have

$$|S_j^a| \leq \mu^a(j). \tag{3.18}$$

Now we show that $|S_j^a|$ increases or remains the same for all $j \in F$ and $a \in I$. First, we consider the case $|S_j^a| < \mu^a(j)$, for any $j \in F$ and $a \in I$. Suppose $(i,j) \in \widetilde{E}_W^a$ with $(i,j) \notin X$ at Step 3. After the execution of

Step 3, we have a matching X that satisfies (3.7)$-$(3.10) and $i \in S_j^a$ due to (3.9) and (3.10). Therefore, $|S_j^a|$ increases. For any $j \in F$ and $a \in I$ with $|S_j^a| = \mu^a(j)$, $|S_j^a|$ remains the same by (3.10).

(iv) By (ii), $\widetilde{E}$ decreases if $L^a \neq \emptyset$ or $\widetilde{E}_0^a \neq \emptyset$, for some $a \in I$. If $L^a = \widetilde{E}_0^a = \emptyset$ with $U^a \neq \emptyset$, for all $a \in I$, in this case $\widetilde{E}$ remains the same in each iteration.

(v) We have a matching X satisfying (3.7)$-$(3.10) and we define r by (2.4). On the contrary, suppose that there exist $j \in F$ and $a \in I$ such that $(new)r_j^a < (old)r_j^a$. This contradicts that X satisfies (3.9). Thus, our assumption is wrong. $\square$

Lemma 3.8 *In each iteration of the* `Job_Allocation` *at Step 3, if* $\nu_{ji}(-p_{ij}) > r_j^a$, $a \in I$, *for some* $(i,j) \in E$, *then* p_{ij} *is the maximum integer in* $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]$ *for which this inequality holds.*

Proof Let $(i,j) \notin U^a$, $a \in I$. In the first iteration of the `Job_Allocation`, we have $(new)p_{ij} = (old)p_{ij}$ defined by (3.1). We know that $(old)p_{ij}$ is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]$ for which $\nu_{ji}(-(old)p_{ij}) \geq 0$; therefore, $(new)p_{ij}$ is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}] \in \mathbf{Z}$ for which $\nu_{ji}(-(new)p_{ij}) > (new)r_j^a$ since $(new)r_j^a \geq 0$.

We suppose that the assertion holds for the tth iteration, $t \geq 2$. We show that the assertion holds for the $(t+1)$th iteration. By (3.14), $(new)p_{ij} = (old)p_{ij}$ for $(i,j) \notin U^a$, $a \in I$. Therefore,

$$\nu_{ji}(-(old)p_{ij}) = \nu_{ji}(-(new)p_{ij}) > (new)r_j^a \geq (old)r_j^a, \tag{3.19}$$

where the last inequality holds by Lemma 3.7(vi). Thus, $\nu_{ji}(-(old)p_{ij}) > (old)r_j^a$. By induction hypothesis, $(old)p_{ij}$ is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]$ for which this inequality holds. By (3.19), $(new)p_{ij}$ is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]$ such that $\nu_{ji}(-(new)p_{ij}) > (new)r_j^a$.

Next we consider the case $(i,j) \in U^a$, $a \in I$. By assumption, $\nu_{ji}(-(new)p_{ij}) > (new)r_j^a \geq (old){r^a}_j$, and it means that $(i,j) \in U^a \setminus L^a$. Thus, $(new)p_{ij}$ is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]$ such that $\nu_{ij}(-(new)p_{ij}) > (new)r_j^a$, by Lemma 3.4. This completes the proof. $\square$

Theorem 3.9 *If* `Job_Allocation` *terminates, then* $(X; p, q, r)$ *satisfies (ps1) and (ps2).*

Proof Initially we define $\widetilde{E}$ by (3.2) and in each iteration we modify it by (3.17) at Step 3. Since $\nu_{ij}(p_{ij})$ and $\nu_{ji}(-p_{ij})$ are nonnegative for all $(i,j) \in \widetilde{E}$, therefore $\nu_{ij}(p_{ij}) \geq 0$ and $\nu_{ji}(-p_{ij}) \geq 0$ for all $(i,j) \in X$ at termination. Thus, (ps1) satisfies at termination.

On the contrary to (ps2), suppose that there exists $(i,j) \in E$ and $\alpha \in [\underline{\pi}_{ij}, \overline{\pi}_{ij}] \in \mathbf{Z}$ such that $\nu_{ij}(\alpha) > q_i$, $\nu_{ji}(-\alpha) > r_j^a$ for some $a \in I$. If $p_{ij} < \alpha$, then $\nu_{ji}(-p_{ij}) > \nu_{ji}(-\alpha) > r_j^a$ for some $a \in I$. By Lemma 3.8, this is not true. If $p_{ij} \geq \alpha$ then

$$\nu_{ij}(p_{ij}) \geq \nu_{ij}(\alpha) > q_i.$$

However, $U^a = \emptyset$, for each $a \in I$, at termination. Then, by (3.4) and (3.11), $\nu_{ij}(p_{ij}) < q_i$, a contradiction. Hence, (ps2) also holds at termination. $\square$

Theorem 3.10 `Job_Allocation` *terminates after a finite number of iterations.*

Proof Consider the case when $L^a = \widetilde{E}_0^a = \emptyset$, for all $a \in I$. By Lemma 3.7(i), p_{ij} decreases for each $(i,j) \in U^a \setminus \{L^a, \widetilde{E}_0^a\}$, for all $a \in I$. However, p is discrete and $p_{ij} \geq \max\{\underline{\pi}_{ij}, \frac{\beta_{ji}}{\alpha_{ji}}\}$ for each $(i,j) \in U^a$, $a \in I$. Therefore, p_{ij} can be decreased only by a finite number of times.

Next we consider the case when $L^a \neq \emptyset$, or $\widetilde{E}_0^a \neq \emptyset$. In either case, $\widetilde{E}$ reduces at Step 3. By Lemma 3.7(iv), in each iteration of the `Job_Allocation`, $\widetilde{E}$ decreases or remains the same, and therefore this case is possible at most $|E|$ times. This completes the proof. $\square$

4. Concluding remarks

We have presented a job market in which we have taken different categories of the workers into account. Each firm employs worker according to its eligibility criteria. The preferences of players are represented by increasing utility functions. Here we list some important points about the model and `Job_Allocation`:

1. The existence of a pairwise stable outcome is guaranteed in our model. The salary in this model is a discrete variable. The marriage model [6] and the Ali and Farooq model [1] are special cases of this model. A possible extension of the model is to consider salary as a continuous variable.

2. By Lemma 3.8, we have that p_{ij} is the maximum integer in $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]$ for which $\nu_{ji}(-p_{ij}) > r_j^a$, $a \in I$, for some $(i,j) \in E$. Since valuations are linearly increasing, the matching will be worker-optimal. It is possible to define an algorithm by starting from the lowest possible salary and then increasing it accordingly. This type of algorithm will yield firm-optimal stable outcomes.

3. The complexity of the `Job_Allocation` is not polynomial as it depends on the length of $[\underline{\pi}_{ij}, \overline{\pi}_{ij}]$ for each $(i,j) \in E$.

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