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Rational Schubert polynomials

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Abstract: We define and study the rational Schubert, rational Grothendieck, rational key polynomials in an effort to understand Molev's dual Schur functions from the viewpoint of Lascoux.

Key words: Rational Schubert polynomials, Schubert calculus

1. Introduction

In this work, we introduce a new set of combinatorially defined nonsymmetric functions whose symmetrizations are Molev's dual Schur functions [12]. Molev described some properties of dual Schur functions including a combinatorial presentation and an expansion formula in terms of the ordinary Schur functions and a multiplication rule for the dual Schur functions.

Schur functions are an old subject and much is known about them. They are studied in relation to many different subjects from a number of different points of view. We follow the Lascoux–Schützenberger approach, viewing Schür functions as (symmetric) special cases of Schubert polynomials. From this point of view, it is natural to ask how one can define a larger set of nonsymmetric functions, which will include Molev's dual Schur functions as their symmetric counterparts. This theme is the main focus of our work.

On the algebraic geometry side, we obtain a duality formula for the Schubert classes in Grassmannians in terms of rational Schubert (key) polynomials (Proposition 16).

We would also like to point out that a dominant rational Schubert polynomial can be described as a configuration of lines as in [5]: in this work, Fomin and Krillov gave a geometric interpretation of Schubert polynomials in terms of intersection points of line segments. In this context, a dominant rational Schubert polynomial geometrically corresponds to a configuration given by the Figure.

In Section 2, we review the Schubert polynomials of Lascoux and Schützenberger. In Section 3, we define and study the basics of rational Schubert polynomials. Using these properties, we express Cauchy kernel $K_n(\mathbf{z};\mathbf{x}) := \frac{1}{\prod_{i+j\leq n+1}(z_j-x_i)}$ in terms of usual Schubert polynomials and rational Schubert polynomials. In Section 4, we discuss the relation between the dual Schur polynomials and the rational Schubert polynomials;

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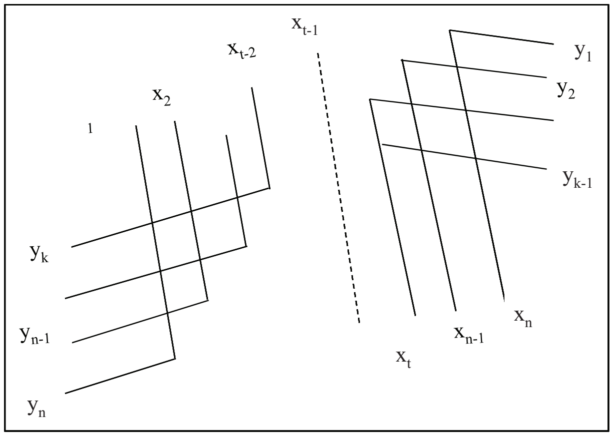


Figure

see [12]. In Section 5, we introduce rational Grothendieck and rational key polynomials in analogy with rational Schubert polynomials.

2. Preliminaries

Schubert varieties are indexed by combinatorial objects such as partitions, permutations, and Weyl group elements. Schubert varieties are useful for studying the cohomology ring of the flag manifold. It can be seen that the set of Schubert varieties forms a basis for $H^*(Fl_n)$ over $\mathbb{Z}$; see [1, 7]. The product of two basis elements can be calculated by using Schubert polynomials. The aim of a theory of Schubert polynomials is to produce explicit representatives for Schubert classes in the cohomology ring of a flag variety. Schubert varieties have many applications in discrete geometry, computer graphics, and computer vision.

Schubert polynomials were introduced in 1982 and extensively developed by Lascoux and Schützenberger; a less combinatorial version was considered by Bernstein et al. [1]. New developments of the theory were given by others [2, 3, 8, 10, 11, 4].

Let $\mathbf{x} := \{x_1, ...x_n\}$ be a totally ordered set of variables. We denote $s_i,\ i = 1, ..., n-1$ as the elementary transposition seen as the operator on $\mathbb{Z}[\mathbf{x}]$ that interchanges x_i and x_{i+1} and fixes all other variables. These operators satisfy the following braid relations:

$$s_i s_j = s_j s_i \ , |i-j| > 1 \quad \text{and} \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.$$

The operator ∂_i is defined by $f\partial_i := \frac{f - fs_i}{x_i - x_{i+1}},\ f \in \mathbb{Z}[\mathbf{x}]$. This operator was introduced by Newton and it is called the Newton divided difference. Similarly, we define operators $\pi_i = x_i \partial_i,\ \hat{\pi}_i = \pi_i - 1$. Here we describe the action of these operators on the set $\{1, x_{i+1}\}$:

	s_i	∂_i	π_i	$\hat{\pi}_i$
1	1	0	1	0
x_{i+1}	x_i	-1	0	$-x_{i+1}$

We have also Leibnitz formulas:

$$(fg)\partial_i = f(g\partial_i) + (f\partial_i)(gs_i) = g(f\partial_i) + (g\partial_i)(fs_i)$$

$$(fg)\pi_i = f(g\pi_i) + (f\pi_i)(gs_i) - gs_i f = g(f\pi_i) + (g\pi_i)(fs_i) - fs_i.$$

In particular, when we take $g = x_i$, the commutation relations may be seen as $x_i\partial_i = \partial_i x_{i+1} + 1$, $\hat{\pi}_i = \partial_i x_{i+1}$.

Schubert polynomials are sometimes indexed by sequences u by taking the code of permutation μ, and Y_u denotes the corresponding polynomial.

Let $\mathbf{x} := \{x_1, ..., x_n\}$ and $\mathbf{y} := \{y_1, ..., y_n, ...\}$ be two set of indeterminates. Given $\lambda \in N^n$ a partition (i.e. $\lambda_1 \geq ... \geq \lambda_n \geq 0$), the *dominant Schubert polynomial* is defined by

$$Y_\lambda(\mathbf{x};\mathbf{y}) = \prod_{i=1,...,n;j=1,...,\lambda_i} (x_i - y_j)$$

and we define Schubert polynomials to be all the nonzero images of the dominant Schubert polynomials under products of ∂_is. Sometimes, these polynomials are known as *double Schubert polynomials.* The operator ∂_i acts on the indices: $Y_{...,\lambda_{i+1},\lambda_i-1,...}(\mathbf{x};\mathbf{y}) = Y_\lambda(\mathbf{x};\mathbf{y})\partial_i$, $\lambda_i > \lambda_{i+1}$, and here we assume ∂_i acts on the first alphabet $\mathbf{x}$ unless otherwise stated. Indeed, $Y_k(\mathbf{x};\mathbf{y})\partial_k^y = -Y_{k-1}(\mathbf{x};\mathbf{y})$ and $Y_k(\mathbf{x};\mathbf{y})\partial_i^y = 0$ for $i \neq k$. Sometimes we will write $\lambda\partial_i = [\lambda_1, ..., \lambda_{i-1}, \lambda_{i+1}, \lambda_i - 1, ..., \lambda_n]$. Similarly, for a dominant partition $\lambda \in \mathbb{N}^n$, $G_\lambda(\mathbf{x};\mathbf{y}) = \prod_{i=1..n,j=1..\lambda_i}(1 - y_j x_i^{-1})$ and $K_\lambda(\mathbf{x}) = \mathbf{x}^\lambda$ are the *dominant Grothendieck* and *key polynomials* of index λ, respectively. We define Grothendieck (resp., key) polynomials to be all the images of the dominant Grothendieck (resp., key) polynomials under products of π_i's (resp. $\hat{\pi}_i$); for details, see [9]. Here we note that one can easily describe Schubert polynomials using permutations in general for any permutation $\mu \in S_n$, denoted by $X_\mu(\mathbf{x};\mathbf{y})$. Additionally, we have $X_\mu(\mathbf{x};\mathbf{y}) = (-1)^{l(\mu)} X_{\mu^{-1}}(\mathbf{y};\mathbf{x})$, where $l(\mu)$ is the length of μ. For a dominant partition $\lambda \in \mathbb{N}^n$, we note that the integers n, λ_1 denote the lengths of alphabets $\mathbf{x}$, $\mathbf{y}$ in $Y_\lambda(\mathbf{x};\mathbf{y})$, respectively. If $\lambda \in \mathbb{N}^n$ such that $\lambda_i \leq \lambda_{i+1} \leq \cdot\cdot\cdot \leq \lambda_j$, then $Y_\lambda(\mathbf{x};\mathbf{y})$ is symmetric in $x_i, ..., x_j$. Schubert polynomials are nonsymmetric generalizations of the fundamental basis of symmetric functions that are Schur functions. In fact, many properties of the Schur basis can be extended to properties of the $Y_\lambda(\mathbf{x};\mathbf{y})$ bases. Fomin et al. [4] studied quantum Schubert polynomials and they gave a quantum analogue of the results of Bernstein et al. [1] on the cohomology of the flag manifold; they gave the quantum Monk formula.

Unless otherwise stated we will make the following main assumptions and notations: here we use the notation $\mathbf{x}_n := \{x_1, ..., x_n\}$, $\mathbf{y} := \{y_1, y_2, ...\}$.

- $\sigma(u)$: permutation with code u, and $l(u)$: the length of u.
- $\mathbf{x}^w$: the reverse ordering of alphabet $\mathbf{x}$.
- If $u = [u_1, \ldots, u_n] \in \mathbb{N}^n$, then $u^w = [u_n, \ldots, u_1]$.
- If $u \in \mathbb{N}^n$, then $\bar{u} := [u_1 - u_n, \ldots, u_1 - u_2, u_1 - u_1] = u_1^n - u^w$.
- If $v = [v_1, \ldots, v_k, \bar{v}_{k+1}, ..., \bar{v}_n]$, then here $\bar{v}_i := -v_i$.
- For $u \in \mathbb{N}$, $u^n = \overbrace{[u, u, ..., u]}^{n \text{ times}}$ and $\mathbf{1} = \{1, 1, ...\}$.
- $R(A|B) = \prod_{a\in A, b\in B}(a - b)$, where A and B are the two alphabet sets.

3. Rational Schubert polynomials and Cauchy formula

From now on, when we say Schubert polynomial, we mean double Schubert polynomial (dominant or not).

Definition 1 *Let $u \in \mathbb{N}^n$ be a strict dominant partition. The strict rational Schubert polynomial indexed by u*

is defined by

$$Y_u^{rat}(\mathbf{x};\mathbf{y}) = \prod \frac{1}{R(y_{n+1},\ldots,y_{n+v_i}|x_i)} \prod R(x_j|y_n,\ldots,y_{n+v_j+1})$$

with $v = u-(n-1)^n$, *the separation in two blocks corresponding to the values* $v_i \geq 0$ *or* $v_j < 0$. *For a general* Y_u^{rat}, *one needs the strict dominant, which gives it by applying divided differences* $\partial_1,\ldots,\partial_{n-1}$, *which we call* ancestor.

Example 2 *For* $u=[6,4,1,0]$, *we have* $v=[3,1,\bar{2},\bar{3}]$

$$Y_{6410}^{rat}(\mathbf{x};\mathbf{y}) = \frac{(x_3-y_4)(x_3-y_3)(x_4-y_4)(x_4-y_3)(x_4-y_2)}{(y_5-x_1)(y_6-x_1)(y_7-x_1)(y_5-x_2)}.$$

For another example, we see that $Y_{410}^{rat}(\mathbf{x};\mathbf{y})$ *is ancestor of* $Y_{102}^{rat}(\mathbf{x};\mathbf{y})$: $Y_{102}^{rat}(\mathbf{x};\mathbf{y}) = Y_{410}^{rat}(\mathbf{x};\mathbf{y})\partial_1\partial_2$. *Here, the following table illustrates* $Y_{410}^{rat}(\mathbf{x};\mathbf{y})$.

	x_1	x_2	x_3
y_1			
y_2			x_3-y_2
y_3		x_2-y_3	x_3-y_3
y_4	$-x_1+y_4$		
y_5	$-x_1+y_5$		

Proposition 3 *For a strict dominant partition* $u \in \mathbb{N}^n$, *we have*

$$Y_u^{rat}(\mathbf{x};\mathbf{y}) = (-1)^{h_u} Y_{\bar{u}}(x_n,\ldots,x_1;y_N,\ldots y_1)/R(x_1,\ldots,x_n|y_N\ldots y_{n+1})$$

where $\bar{u}=[u_1-u_n,\ldots,u_1-u_2,u_1-u_1]$, $N=u_1+1$, *and* $v=u-(n-1)^n$, $h_u=\sum_{i=1}^{k} v_i, v_i \geq 0$ *for* $i \leq k \leq n$.

Other $Y_r^{rat}(\mathbf{x};\mathbf{y})$ *polynomials are obtained by reordering and decreasing the indices as for Schubert, the factor in the denominator commuting with the divided differences.*

Proof The proof consists in the dominant case just of identifying the factors obtained after multiplication by the resultant $R=\prod_{i=1}^{n}\prod_{j=n+1}^{N}(x_i-y_j)$. □

Notice that, because of reversing the alphabet $\mathbf{x}_n$, a divided difference ∂_i acting on Y_v^{rat} corresponds to $-\partial_{n-i}$ on the Schubert polynomial $Y_{\bar{u}}$ in $\mathbf{x}_n$.

Proposition 4 *Let* $u \in \mathbb{N}^n$ *be a strict dominant partition,* $\bar{u}=u_1^n-u^w$, $N=u_1+1$, $v=u-(n-1)^n$, *and* $\sigma=u+\mathbf{1}$. *Then we have the following properties:*

(a) $Y_\sigma(\mathbf{x};\mathbf{y}).Y_{\bar{u}}(x_n,\ldots,x_1;y_N,\ldots,y_1) = Y_{N^n}(\mathbf{x};\mathbf{y})$,

(b) $Y_u^{rat}(\mathbf{x};\mathbf{y}) = \frac{(-1)^{h_u}Y_{\bar{u}}(x_n,\ldots,x_1;y_N,\ldots,y_1)}{Y_{(N-n)^n}(x_1,\ldots,x_n;y_N,\ldots,y_1)}$, *where* $h_u=\sum_{i=1}^{k} v_i$, *for* $v_i \geq 0$,

(c) $Y_\sigma(\mathbf{x};\mathbf{y}).Y_u^{rat}(\mathbf{x};\mathbf{y})R(\mathbf{x}|y_N,\ldots,y_{n+1})(-1)^{h_u} = Y_{N^n}(\mathbf{x};\mathbf{y})$,

(d) $Y_u^{rat}(\mathbf{x};\mathbf{y}) = (-1)^t \frac{Y_{c\bar{u}}(y_N,\ldots,y_1;x_n,\ldots,x_1)}{Y_{n^{N-n}}(y_{n+1},\ldots,y_N;x_1,\ldots,x_n)}$,

(e) $Y_u^{rat}(\mathbf{x};\mathbf{y}) = (-1)^h \frac{Y_{[-v_n,\dots,-v_{k+1}]}(\mathbf{x}^{\mathbf{w}};\mathbf{y}^w)}{Y_{[v_1,\dots,v_k]}(\mathbf{x};y_{n+1},\dots,y_N)}$,

where $c\bar{u}$ denotes the conjugate of $\bar{u}$, $t = n(N-n)+h_u+\sum_{i=1}^{n}\bar{u}_i$, and Y_σ, Y_{N^n} are Schubert polynomials indexed by σ and N^n, respectively.

Proof (a)–(e) are obvious by the definition. For the others, it is easy to see that $R(\mathbf{x}_n|y_N,\dots,y_{n+1}) = Y_{(N-n)^n}(x_1,\dots,x_n;y_N,\dots y_{1+n})$ and we use the relation $X_\mu(\mathbf{x},\mathbf{y}) = (-)^{l(\mu)}X_{\mu^{-1}}(\mathbf{y},\mathbf{x})$. Since $v = u-(n-1)^n$, there exists a natural number k such that $v_i \geq 0$ for $i \leq k$ and $v_i \leq 0$ for $k \leq n$. By the definition, if $h_u = \sum_{i=1}^{k} v_i$, then we have

$$\begin{aligned} Y_u^{rat}(\mathbf{x};\mathbf{y}) &= (-1)^{h_u}\frac{Y_{\bar{u}}(x_n,\dots,x_1;y_N,\dots y_1)}{Y_{(N-n)^n}(x_1,\dots,x_n;y_N,\dots y_1)} \\ &= (-1)^{h_u+\sum_{i=1}^{n}\bar{u}_i}\frac{Y_{c\bar{u}}(y_N,\dots,y_1;x_n,\dots,x_1)}{Y_{c(N-n)^n}(y_{n+1},\dots,y_N;x_1,\dots,x_n)} \\ &= (-1)^{n(N-n)+h_u+\sum_{i=1}^{n}\bar{u}_i}\frac{Y_{c\bar{u}}(y_N,\dots,y_1;x_n,\dots,x_1)}{Y_{n^{N-n}}(y_{n+1},\dots,y_N;x_1,\dots,x_n)} \end{aligned}$$

where $c\bar{u}$ denotes the conjugate of $\bar{u}$. □

For example, $\lambda = [4,1,0]$, $Y_{410}^{rat}(\mathbf{x};\mathbf{y}) = \frac{Y_{430}(\mathbf{x}_3^w;\mathbf{y}_5^w)}{Y_2(\mathbf{x};\mathbf{y})}$, and we see that there is a relation explained as a diagram between indices.

$Y_{430}(\mathbf{x}_3^w;\mathbf{y}_5^w)$

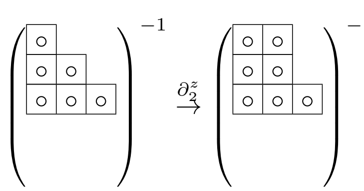
∘	∘	∘	∘
x_2-y_2	∘	∘	∘
x_1-y_2	x_1-y_3	x_1-y_4	x_1-y_5

$Y_{410}(\mathbf{x};y_2,\dots,y_5)$

Now we consider the Cauchy kernel $K_n(\mathbf{z};\mathbf{x}) := \frac{1}{\prod_{i+j\leq n+1}(z_j-x_i)}$. Actually, $K_n(\mathbf{z};\mathbf{x})$ is the inverse of the maximal Schubert polynomial in an alphabets of n letters, i.e. $K_n(\mathbf{z};\mathbf{x}) = Y_{n,n-1,\dots 1}(\mathbf{z};\mathbf{x})^{-1}$. It can be easily calculated that

$$K_n(\mathbf{z};\mathbf{x})\partial_i^z = K_n(\mathbf{z};\mathbf{x}).(z_{i+1}-x_{n+1-i})^{-1}, \text{ for } i \leq n-1.$$

When we apply ∂_i^z to the $K_n(\mathbf{z};\mathbf{x})$, actually we add a box corresponding to the $i+1$-th row and $(n+1-i)$-th column of the diagram of $Y_{n,n-1,\dots 1}(\mathbf{z};\mathbf{x})^{-1}$. For example, for $n=3$, we have $K_3(\mathbf{z};\mathbf{x}) = \{(z_1-x_1)(z_1-x_2)(z_1-x_3)(z_2-x_1)(z_2-x_2)(z_3-x_1)\}^{-1}$, and $K_3(\mathbf{z};\mathbf{x})\partial_2^z = K_3(\mathbf{z};\mathbf{x})(z_3-x_2)^{-1}$.

$K_3(\mathbf{z};\mathbf{x}) = Y_{321}(\mathbf{z};\mathbf{x})^{-1} =$ 

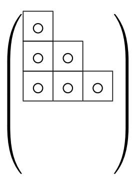
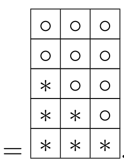
Let v be a dominant code of length n and $\sigma(v)$ be permutation of v. Then there exists a positive integer k such that $\sigma(v)$ is an element of the symmetric group S_k. Let m be the smallest such k. Actually, m corresponds to N in Proposition 3.

In this work, from now on m denotes the integer that was described in the last paragraph unless otherwise stated.

We define $R(\mathbf{z}|\mathbf{x}) = \prod_{j=1,\dots,m,i=1,\dots,n}(z_j - x_i)$ and $\Upsilon := (-1)^{\frac{nm-n(n+1)}{2}}$. Therefore, we have:

$$\begin{aligned} K_n(\mathbf{z};\mathbf{x}).R(\mathbf{z}|\mathbf{x}) &= Y_{n,n-1,\dots 1}(\mathbf{z};\mathbf{x})^{-1} Y_{n^m}(\mathbf{z};\mathbf{x}) \\ &= Y_{m-1,m-2,..,m-n}(x_n,\dots,x_1;z_m,z_{m-1}\dots,z_1) \\ &= \Upsilon . Y_{n^{m-n},n-1,\dots 2,1}(z_m,z_{m-1}\dots,z_1;x_n,\dots,x_1). \end{aligned}$$

For example, for $n=3$, $m=5$, we illustrate the following diagram:

$K_3(\mathbf{z};\mathbf{x}) =$  $^{-1}$ and $K_3(\mathbf{z};\mathbf{x}).R(\mathbf{z}|\mathbf{x}) =$ .

The Newton nterpolation formula (NIF) gives us the chance of expanding the polynomial $K_n(\mathbf{z};\mathbf{x}).R(\mathbf{z}|\mathbf{x})$. We choose the alphabet $\mathbf{z}^w$, $\mathbf{y}^w$ and we express the NIF in the basis $Y_\alpha(\mathbf{z}^w;\mathbf{y}^w)$. Let c_α be the coefficient of the polynomial Y_α. By the formula, c_α can be calculated as

$$c_\alpha = (K_n(\mathbf{z};\mathbf{x}).R(\mathbf{z}|\mathbf{x}))\partial^z_{\sigma(\alpha)^{-1}}|_{\mathbf{z}^w=\mathbf{y}^w},$$

and then we obtain a Schubert polynomial indexed by u, and we have

$$\begin{aligned} K_n(\mathbf{z};\mathbf{x}).R(\mathbf{z}|\mathbf{x}) &= \sum_{\alpha\in\mathbb{N}^n}\{(K_n(\mathbf{z};\mathbf{x}).R(\mathbf{z}|\mathbf{x}))\partial^z_{\sigma(\alpha)^{-1}}\}|_{\mathbf{z}^w=\mathbf{y}^w} Y_\alpha(\mathbf{z}^w;\mathbf{y}^w) \\ &= \sum_{\alpha\in\mathbb{N}^n} Y_u(\mathbf{y}^w;\mathbf{x}^w)\ Y_\alpha(\mathbf{z}^w;\mathbf{y}^w) \end{aligned}$$

where $u := [m-1, m-2, .., m-n] - \alpha$. Because of the symmetry, we have $R(y_1,\dots,y_n|x_{n+1},\dots,x_m) = (-1)^{n(m-n)}R(x_{n+1},\dots,x_m|y_1,\dots,y_n)$. Now we remember Proposition 3 (b); if $c_\alpha = Y_u(\mathbf{y^w};\mathbf{x^w})$ and u is dominant, then

$$\frac{c_\alpha}{R(y_1,\dots,y_n|x_{n+1},\dots,x_m)} = (-1)^{h_r}\frac{Y_u(\mathbf{y^w};\mathbf{x^w})}{R(y_1,\dots,y_n|x_{n+1},\dots,x_m)} = Y_r^{rat}(\mathbf{y};\mathbf{x}),$$

where $r = u_1^n - u^w = \rho^w + \alpha^w$.

$$\begin{aligned} \frac{c_\alpha}{R(z_1,\dots,z_m|x_1,\dots,x_n)} &= (-1)^{h_r}\frac{Y_r^{rat}(\mathbf{y};\mathbf{x}).R(y_1,\dots,y_n|x_{n+1},\dots,x_m)}{R(z_1,\dots,z_m|x_1,\dots,x_n)} \\ &= (-1)^{h_r}\frac{Y_r^{rat}(\mathbf{y};\mathbf{x})Y_{(m-n)^n}(y_1,\dots,y_n;x_{n+1},\dots,x_m)}{Y_{n^m}(z_1,\dots,z_m;x_1,\dots,x_n)}. \end{aligned}$$

If u is not dominant, we need to find its ancestor. Let us say $u = \beta\partial_{i_1}\dots\partial_{i_s}$ and $R = R(y_1,\dots,y_n|x_{n+1},\dots,x_m)$. If $\frac{Y_\beta}{R} = Y_s^{rat}$, then $\frac{Y_\beta}{R}\partial_{i_1}\dots\partial_{i_t} = Y_s^{rat}\partial_{n-i_1}\dots\partial_{n-i_t}$ and we obtain $r = s\partial_{n-i_1}\dots\partial_{n-i_t}$.

Finally, collecting what we have obtained above, we have the following theorem.

Theorem 5 *Let K_n denote the Cauchy kernel, and let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be alphabets. Then we have*

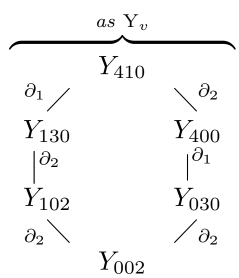
$$K_n(\mathbf{z};\mathbf{x}) = \sum_{\alpha\in\mathbb{N}^n} (-1)^{h_r} \frac{Y_{(m-n)^n}(y_1,...,y_n;x_{n+1},...,x_m)}{Y_{n^m}(z_1,...,z_m;x_1,...,x_n)} Y_r^{rat}(\mathbf{y};\mathbf{x}) Y_\alpha(\mathbf{z^w};\mathbf{y^w}),$$

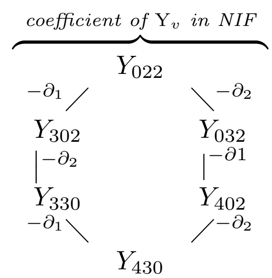
where α is a code of length n and m is the smallest positive integer such that $\sigma(\alpha)\in S_m$ is the permutation of α, and $\Phi = [m-n,...,m-2,\ m-1]$, $h_r = \sum_{i=1}^{k} v_i$, for $v_i \geq 0$, $v = r-(n-1)^n$, $\Phi - \alpha = \beta\partial_{i_1}...\partial_{i_t}$, and Y_β is the numerator of Y_s^{rat}, $r = \begin{cases} \rho^w + \alpha^w, & \Phi\text{-}\alpha \text{ is dominant} \\ s\partial_{n-i_1}...\partial_{n-i_t}, & \text{otherwise.} \end{cases}$

Example 6 *For $v=[4,1,0]$, we have $K_3(\mathbf{z};\mathbf{x}).R(\mathbf{z};\mathbf{x}) = Y_{432}(\mathbf{x}^w;\mathbf{z}^w)$. In fact,*

$$\begin{aligned} Y_{432}(\mathbf{x}^w;\mathbf{z}^w) &= Y_{210}(\mathbf{x}_3^w;\mathbf{z}_3^w).Y_{2^3}(x_1,x_2,x_3;z_4,z_5) \\ &= Y_{210}(\mathbf{x}_3^w;\mathbf{z}_3^w).Y_{3^2}(z_5,z_4;x_1,x_2,x_3) = Y_{3321}(\mathbf{z}^w;\mathbf{x}^w). \end{aligned}$$

The following diagrams show the action of braid relations on v.

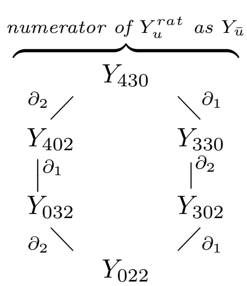


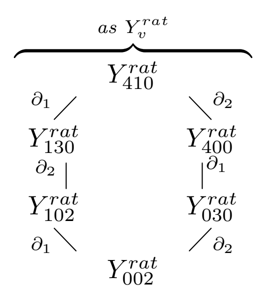


For Y_{410}^{rat}, the numerator polynomial is Y_{430} in the reversed alphabet. Now let Y_{022} be the numerator of a rational Schubert polynomial and R be the resultant; see Proposition 3. First we need to find its ancestor: $Y_{022} = Y_{430}\partial_1\partial_2\partial_1$, and then

$$Y_{410}^{rat} = \frac{Y_{430}}{R} \Rightarrow \frac{Y_{430}\partial_1\partial_2\partial_1}{R} = \frac{Y_{022}}{R} = Y_r^{rat} \Rightarrow \frac{Y_{022}}{R} = Y_{410}^{rat}\partial_2\partial_1\partial_2 = Y_{002}^{rat}.$$

Similarly, one can see that $Y_{030}^{rat} = \frac{Y_{302}}{R}$, $Y_{102}^{rat} = \frac{Y_{032}}{R}$, $Y_{400}^{rat} = \frac{Y_{330}}{R}$, $Y_{130}^{rat} = \frac{Y_{402}}{R}$.





Dividing $Y_{432}(\mathbf{x}^w;\mathbf{z}^w)$ by $Y_2(x_1,x_2,x_3;z_4,z_5)$, we obtain

$$(-1)^2\frac{Y_{210}(\mathbf{x}_3^w;\mathbf{z}_3^w)}{Y_2(\mathbf{x_3};z_4,z_5)} = Y_{410}^{rat}(\mathbf{x},\mathbf{z}) = \frac{Y_{430}(\mathbf{x}^w;\mathbf{z}^w)}{Y_{2^3}(\mathbf{x_3};z_4,z_5)}.$$

Then $Y_{432}(\mathbf{x}^w;\mathbf{z}^w) = Y^{rat}_{410}(\mathbf{x};\mathbf{z}).Y_2(\mathbf{x};z_4,z_5).Y_{2^3}(\mathbf{x};z_4,z_5)$ *and we get*

$$Y^{rat}_{410}(\mathbf{x};\mathbf{z}) = \sum_{v\in\mathbb{N}^n} \frac{(-1)^{h_r} Y^{rat}_r(\mathbf{y};\mathbf{x}) Y_\alpha(\mathbf{z^w};\mathbf{y^w}) Y_{2^3}(\mathbf{y};x_4,x_5)}{Y_{2^3}(\mathbf{x};z_4,z_5) Y_2(\mathbf{x};z_4,z_5)}$$

where r *is the same as in Theorem 5. In general, we state the following proposition.*

Proposition 7

$$Y^{rat}_{m-1,n-2,n-3,\ldots,1,0}(\mathbf{x};\mathbf{z}) = \sum_{v\in\mathbb{N}^n} \frac{(-1)^{m-n} Y^{rat}_r(\mathbf{y};\mathbf{x}) Y_v(\mathbf{z}^w;\mathbf{y}^w)}{Y_{(m-n)^n}(\mathbf{x};z_{n+1},\ldots,z_m).Y_{(m-n)}(\mathbf{x};z_{n+1},\ldots,z_m)},$$

where r *is the same as in Theorem 5.*

Proof Properties of Schubert polynomials give us the following relations:

$$\begin{aligned} Y_{m-1,\ldots,m-n}(\mathbf{x}^w_n;\mathbf{z}^w_m) &= Y_{n-1,\ldots,1}(\mathbf{x}^w;z_n,\ldots,z_1).Y_{(m-n)^n}(\mathbf{x};z_{n+1},\ldots,z_m) \\ &= (-1)^{(m-n)n} Y_{n-1,\ldots,1}(\mathbf{x}^w;z_n,\ldots,z_1).Y_{n^{m-n}}(z_{n+1},\ldots,z_m;\mathbf{x}) \\ &= (-1)^{(m-n)n} Y_{n^{m-n},n-1,\ldots 2,1}(z_m,z_{m-1}\ldots,z_1;\mathbf{x}^w) \end{aligned}$$

and

$$\begin{aligned} (-1)^{m-n} Y^{rat}_{m-1,n-2,n-3,\ldots,1,0}(\mathbf{x};\mathbf{z}) &= \frac{Y_{n-1,\ldots,1,0}(\mathbf{x}^w_n;z_n,\ldots,z_1)}{Y_{m-n}(\mathbf{x}_n;z_{n+1},\ldots,z_m)} \\ &= \frac{Y_{m-1,m-2\ldots,m-n+1,0}(\mathbf{x}^w_n;\mathbf{z}^w)}{Y_{(m-n)^n}(\mathbf{x}_n;z_{n+1},\ldots,z_m)}. \end{aligned}$$

Hence, as a result, we obtain

$$Y^{rat}_{m-1,n-2,n-3,\ldots,1,0}(\mathbf{x};\mathbf{z}) = \frac{(-1)^{m-n} Y_{m-1,\ldots,m-n}(\mathbf{x}^w_n;\mathbf{z}^w_m)}{Y_{(m-n)}(\mathbf{x};z_{n+1},\ldots,z_m).Y_{(m-n)^n}(\mathbf{x};z_{n+1},\ldots,z_m)},$$

and Theorem 5 gives Proposition 7. □

Corollary 8 *The following statements hold for the Cauchy kernel:*

(a) $K_n(\mathbf{z};\mathbf{x}) = \frac{(-1)^{(m-n)(n+1)} Y^{rat}_{m-1,n-2,n-3,\ldots,1,0}(\mathbf{x};\mathbf{z}) Y_{(m-n)}(\mathbf{x};z_{n+1},\ldots,z_m)}{Y_{n^n}(\mathbf{z};\mathbf{x})}$.

(b) $K_n(\mathbf{z};\mathbf{x}) = (-1)^{(m-n)} \frac{Y^{rat}_\rho(\mathbf{z};\mathbf{x})}{Y_{n^n}(\mathbf{z};\mathbf{x})}$, *where* $\rho = [n-1,\ldots,1,0]$.

Proof (a) By the proof of Theorem 5 and Proposition 7 we have

$$\begin{aligned} K_n(\mathbf{z};\mathbf{x}) Y_{n^m}(\mathbf{z};\mathbf{x}) &= Y_{m-1,\ldots,m-n}(\mathbf{x}^w_n;\mathbf{z}^w_m) \\ &= (-1)^{(m-n)+(m-n)n} Y^{rat}_{m-1,n-2,n-3,\ldots,1,0}(\mathbf{x};\mathbf{z}).S \end{aligned}$$

where $S = Y_{(m-n)}(\mathbf{x};z_{n+1,\ldots,m}) Y_{(m-n)^n}(z_{n+1,\ldots,m};\mathbf{x})$ and

$$K_n(\mathbf{z};\mathbf{x}) = \frac{(-1)^{(m-n)(n+1)} Y^{rat}_{m-1,n-2,n-3,\ldots,1,0}(\mathbf{x};\mathbf{z}).Y_{(m-n)}(\mathbf{x};z_{n+1},\ldots,z_m)}{Y_{n^n}(\mathbf{z};\mathbf{x})}.$$

(b) Since $\frac{K_n(\mathbf{z};\mathbf{x}).Y_{n^m}(\mathbf{z};\mathbf{x})}{Y_{(m-n)^n}(\mathbf{x};z_{n+1},\ldots,z_m)} = \frac{Y_{m-1,\ldots,m-n}(\mathbf{x}_n^w;\mathbf{z}_m^w)}{Y_{(m-n)^n}(\mathbf{z};\mathbf{x})}$, we obtain

$(-1)^{m-n}K_n(\mathbf{z};\mathbf{x}).Y_{n^n}(\mathbf{z};\mathbf{x}) = Y_\rho^{rat}(\mathbf{z};\mathbf{x})$, where $\rho = [n-1, n-2, ..., 1, 0]$. □

Proposition 9 $\prod_{\substack{i=n+1,\ldots,m,\\ j=1,\ldots,n}}(z_i - x_j) = \sum_{v\in\mathbb{N}^n} Y_u(\mathbf{y^w};\mathbf{x^w})Y_{v-\rho^w}(\mathbf{z}^w;\mathbf{y}^w)$, *where* $\rho = [n-1,...,1,0]$ *and* $u = (m-1)^n - \rho^w - v$.

Proof We have seen that $K_n(\mathbf{z};\mathbf{x})R(\mathbf{z}|\mathbf{x}) = \sum_{v\in\mathbb{N}^n} Y_u(\mathbf{y^w};\mathbf{x^w})Y_v(\mathbf{z}^w;\mathbf{y}^w)$; see Theorem 5. Using ∂_ω^z, we obtain

$$K_n(\mathbf{z};\mathbf{x})R(\mathbf{z}|\mathbf{x})\partial_\omega^z = K_n(\mathbf{z};\mathbf{x})\partial_\omega^z R(\mathbf{z}|\mathbf{x}) = \sum_{v\in\mathbb{N}^n} Y_u(\mathbf{y^w};\mathbf{x^w})(Y_v(\mathbf{z}^w;\mathbf{y}^w)\partial_\omega^z)$$

where ω is the maximum permutation. In this case, the following results hold.

1) $K_n(\mathbf{z};\mathbf{x})\partial_\omega^z = \prod_{\substack{i=1,\ldots,n,\\ j=1,\ldots,n}}(z_i - x_j)^{-1}$;

2) $K_n(\mathbf{z};\mathbf{x})\partial_\omega^z R(\mathbf{z},\mathbf{x}) = \prod_{\substack{i=n+1,\ldots,m,\\ j=1,\ldots,n}}(z_i - x_j)$;

3) $Y_v(\mathbf{z}^w;\mathbf{y}^w)\partial_\omega^z = Y_s(\mathbf{z}^w;\mathbf{y}^w)$, where $s = [v_n, v_{n-1}-1, ..., v_1-(n-1)] = v - \rho^w$.

Finally, (1)-(2)-(3) give us the proposition. □

4. Dual Schur and rational Schubert polynomials

Molev constructed dual Schur functions starting from the $(1-xy)$ description; see [12]. By changing variables by $x_i \to 1/x_i$, one can start equally from the $(x-y)$ description. Here, we outline both approaches. Dual Schur functions are defined as follows:

$(1-yx)$ **picture**: As in [12], p. 15, for given partition λ, define

$$A_{ij} = \begin{cases} x_i^{\lambda_{j+n-j}}\frac{1}{(1-a_0x_i)(1-a_{-1}x_i)\cdots(1-a_{-(\lambda_j-j)}x_i)}, & j \leqslant \lambda_j \\ x_i^{\lambda_{j+n-j}}(1-a_1x_i)\cdots(1-a_{j-\lambda_j-1}x_i]), & \lambda_j < j. \end{cases}$$

Let d be the number of elements on the diagonal. Note that d is determined by the inequality $\lambda_{d+1} \le d \le \lambda_d$. Consequently, if $j \leqslant d$, then $j \leqslant \lambda_j$; otherwise, if $j > d$, then $\lambda_j < j$.

$(x-y)$ **picture**: Plug $1/x_i$ for x_i in A_{ij}:

$$(A_{ij})_{x_i\leftarrow\frac{1}{x_i}} = \begin{cases} \frac{1}{x_i^{n-1}}\frac{1}{(x_i-a_0)(x-a_{-1})\cdots(x_i-a_{-(\lambda_j-j)})}, & j \leqslant \lambda_j \\ \frac{1}{x_i^{n-1}}(x_i-a_1)\cdots(x_i-a_{j-\lambda_j-1}), & \lambda_j < j. \end{cases}$$

For simplicity, define $\widetilde{A}_{ij} := x_i^{n-1}A_{ij}$. The dual Schur function corresponding to λ is obtained by applying the operator ∂_ω to the product $\prod A_{ii}$, and we denote it by $DualShur(\lambda)$. Equivalently, the dual Schur function for λ can also be obtained by applying the operator ∂_ω to the product $\prod \widetilde{A}_{ii}$. Starting from $(x-y)^\pm$, we introduce a graphical display first. Given a partition λ, we put an empty box in (i,k) if the term $x_i - a_k$ appears in $\widetilde{A}_{ii}$. Columns of the diagram represent factors of the form $x_i - a_k$. In the case of $\lambda = [4,3,1,0]$, the display becomes the following.

$\widetilde{A}_{11}$	$\widetilde{A}_{22}$	$\widetilde{A}_{33}$	$\widetilde{A}_{44}$
x_1	x_2	x_3	x_4
		$x_3 - a_1]$	$x_4 - a_3$ $x_4 - a_2$ $x_4 - a_1$
$x_1 - a_0$ $x_1 - a_{-1}$ $x_1 - a_{-2}$ $x_1 - a_{-3}$	$x_2 - a_0$ $x_2 - a_{-1}$		

The dual Schur function for a given partition λ is defined to be $(\prod \widetilde{A}_{ii})\partial_\omega$ as follows: multiplying by the resultant $R = R(x_1, \ldots, x_4 \,|a_0, \ldots,\ a_3)$, one obtains a polynomial product of factors $(x_i - a_j)$, whose image under ∂_ω is by definition $\pm$ a Grassmannian Schubert polynomial in the alphabets. For $\lambda = [4,3,1,0]$, the display would be:

	x_1	x_2	x_3	x_4
a_3				□
a_2				□
a_1			□	□
a_0			□	□
a_{-1}			□	□
a_{-2}		□	□	□
a_{-3}		□	□	□

and $DualSchur([4,3,1,0]) = \frac{Y_{0134}}{R(x_4,\ldots,x_1 \,|\, a_{-3},\ldots,a_0)}$. Here,

$$Y_{7520}(x_4, ..., x_1; a_{-3}, ..., a_3)\partial_\omega = Y_{7520}\partial_3\partial_2\partial_1\partial_3\partial_2\partial_3 = Y_{0134}.$$

It is convenient to use two sets of parameters y_i and z_j instead of $a_{-i+1},\ i \geq 0$ and $a_j,\ \ j = 1, ..., n$, respectively.

Proposition 10 *There exists a correspondence between dominant rational Schubert polynomials and dual Schur polynomials.*

Proof For a given dominant λ with length n, let $v = \lambda - (n-1)^n$ and let N_v denote the product of the boxes of the graphical display. If $N_v := (-1)^{h_\lambda} Y_\lambda^{rat}(\mathbf{x};\mathbf{y})|_{\substack{y_i \to z_{n+1-i} \\ y_{n+i} \to y_i}}$ we have

$$DualSchur(\beta) = ((-1)^{h_\lambda} Y_\lambda^{rat}(\mathbf{x};\mathbf{y})|_{\substack{y_i \to z_{n+1-i} \\ y_{n+i} \to y_i}} R)\partial_\omega^x$$

where $\beta = \lambda - (n-1)^n + [0, 1, ..., n-1]$ and $h_\lambda = \sum_{i=1}^{k} v_i,\ v_i \geq 0$ for $i \leq k$. For any given dominant dual Schur polynomial indexed by β, since $Y_{\beta+\rho}^{rat} = \frac{(-1)^{h_{\beta+\rho}} Y_{\overline{\beta+\rho}}}{R}$, we have

$$Y_{\beta+\rho}^{rat}\partial_\omega^x = \frac{(-1)^{h_{\beta+\rho}} Y_{[0,\beta_1-\beta_2,\ldots,\beta_1-\beta_n]}|_{\substack{y_i \to z_{n+1-i} \\ y_{n+i} \to y_i}}}{R(\mathbf{x}|y_{n+1}, ..., y_N)|_{y_{n+i} \to y_i}} = DualSchur(\beta)$$

where $\rho = [n-1, ..., 1, 0]$. □

Now we want to give a formula that is related to those for key polynomials and the kernel $\frac{1}{\prod_{i+j\leq n+1}(1-x_iy_j)}$. For a given dominant λ with length n, we take $v=\lambda-[0,1,...,n-1]=[v_1,v_2,...,v_k,\bar{v}_{k+1},...,\bar{v}_n]$. Then we have

$$\begin{aligned}\frac{Y_{[v_n,v_{n-1},\dots,v_{k+1}]}(\mathbf{x}^w;\mathbf{z}).R}{Y_{[v_1,v_2,\dots,v_k]}(\mathbf{x};\mathbf{y})} &= \frac{Y_{[v_n,v_{n-1},\dots,v_{k+1}]}(\mathbf{x}^w;\mathbf{z})Y_{v_1^n}(\mathbf{x};\mathbf{y})}{Y_{[v_1,v_2,\dots,v_k]}(\mathbf{x};\mathbf{y})}\\ &= Y_\xi(\mathbf{x}^w;\mathbf{y}^{\mathbf{w}},\mathbf{z})\end{aligned}$$

where $\xi=[v_n+v_1,v_{n-1}+v_1,...,v_{k+1}+v_1,v_1-v_k,...,v_1-v_1]$ and $R=\prod_{\substack{i=1,\dots,n,\\ j=1,\dots,v_1}}(x_i-y_j)$. Hence $DualSchur(\lambda)=Y_\xi(\mathbf{x}^w;\mathbf{y}^{\mathbf{w}},\mathbf{z})\partial^x_\omega$.

By changing variables by $x_i\to 1/x_i$, we have $x^{\lambda+\rho}\frac{Y_{[v_n,v_{n-1},\dots,v_{k+1}]}(\mathbf{1};\mathbf{x}^w\mathbf{z})}{Y_{[v_1,v_2,\dots,v_k]}(\mathbf{1};\mathbf{xy})}$, where $\mathbf{xy}=\{x_iy_j : i,j\geqslant 1\}$. For example, $Y_{21}(\mathbf{1};\mathbf{xy})=(1-x_\mathbf{1}y_\mathbf{1})(1-x_\mathbf{1}y_\mathbf{2})(1-x_\mathbf{2}y_\mathbf{1})$. The transformation $x_i\to 1/x_i$ preserves ∂_ω up to global symmetric factor $\mathbf{x}^{n-1}$.

$$x^{\lambda+\rho}\frac{Y_{[v_n,v_{n-1},\dots,v_{k+1}]}(\mathbf{1};\ \mathbf{x}^w\mathbf{z})}{Y_{[v_1,v_2,\dots,v_k]}(\mathbf{1};\ \mathbf{xy})}\xrightarrow{\partial^x_\omega} DualSchur(\lambda).$$

On the other hand, we choose the starting point that is part of the denominator $\mathbf{r}=[r,r-1,...,1,\underbrace{0,..,0}_{k-r\ times}]$.

Now we define $\partial_\tau=\prod_{i=0}^{r-1}\partial_{\tau_i}$, and $\partial_{\tau_i}=\prod_{j=0}^{v_{(i+1)}-r+i-1}\partial_{\tau_{(r-i+j)}}$. Let us take $\mathbf{b}=[\mathbf{r},v_{k+1},...,v_n]$. If $\gamma=\mathbf{b}+[0,1,...,n-1]=[r^r,r+1,...,k-1,v_{k+1}+k,...,v_n+(n-1)]$, then we have

$$\begin{aligned}\mathbf{x}^{\gamma+\rho}\frac{Y_{[v_n,v_{n-1},\dots,v_{k+1}]}(\mathbf{1};\mathbf{x}^w\mathbf{z})}{Y_{[r,r-1,\dots,1]}(\mathbf{1};\mathbf{xy})} &\xrightarrow{\partial^y_\tau}\mathbf{x}^{\lambda+\rho}\frac{Y_{[v_n,v_{n-1},\dots,v_{k+1}]}(\mathbf{1};\ \mathbf{x}^w\mathbf{z})}{Y_{[v_1,v_2,\dots,v_k]}(\mathbf{1};\mathbf{xy})}\\ &\xrightarrow{\partial^x_\omega} DualSchur(\lambda).\end{aligned}$$

Now we remember that $Y_{[r,r-1,\dots,1]}(\mathbf{1};\mathbf{xy})=\frac{1}{\prod_{i+j\leq r+1}(1-x_iy_j)}$, and we know how to expand this with respect to key polynomials; see [6].

$$\begin{aligned}DualSchur(\lambda) &= x^{\gamma+\rho}\frac{Y_{[v_n,v_{n-1},\dots,v_{k+1}]}(\mathbf{1};\ \mathbf{x}^w\mathbf{z})}{Y_{[r,r-1,\dots,1]}(\mathbf{1};\mathbf{xy})}\partial^y_\tau\partial^x_\omega\\ &= \left(x^{\gamma+\rho}Y_{[v_n,v_{n-1},\dots,v_{k+1}]}(\mathbf{1};\mathbf{x}^w\mathbf{z})\left(\frac{1}{Y_{[r,r-1,\dots,1]}(\mathbf{1};\mathbf{xy})}\right)\partial^y_\tau\right)\partial^x_\omega\\ &= \sum_{\alpha\in\mathbb{N}^r}(K_\alpha(\mathbf{y})\partial^y_\tau)\left(x^{\gamma+\rho}Y_{[v_n,v_{n-1},\dots,v_{k+1}]}(\mathbf{1};\mathbf{x}^w\mathbf{z})\hat{K}_{\alpha w}(\mathbf{x})\right)\partial^x_\omega,\end{aligned}$$

where $\hat{K}$ denotes the adjoint key polynomial. Take $Y_{[v_n,v_{n-1},\dots,v_{k+1}]}:=Y_{[v_n,v_{n-1},\dots,v_{k+1}]}(\mathbf{1};\mathbf{x}^w\mathbf{z})$. Using the formula for the action of ∂^x_ω, we have

$$\begin{aligned}DualSchur(\lambda) &= \sum_{\alpha\in\mathbb{N}^r}(K_\alpha(\mathbf{y})\partial^y_\tau)\sum_{\sigma\in S_r}\frac{(-1)^{l(\sigma)}}{\Delta}(x^{\gamma+\rho}Y_{[v_n,v_{n-1},\dots,v_{k+1}]}\hat{K}_{\alpha w}(\mathbf{x}))\sigma\\ &= \sum_{\alpha\in\mathbb{N}^r}(K_\alpha(\mathbf{y})\partial^y_\tau)\sum_{\sigma\in S_r}\frac{(-1)^{l(\sigma)}}{\Delta}Y_{[v_n,v_{n-1},\dots,v_{k+1}]}\sigma.(x^{\gamma+\rho}\hat{K}_{\alpha w}(\mathbf{x}))\sigma\end{aligned}$$

where $\Delta=\prod_{1\leq i<j\leq n}(x_i-x_j)$. Hence, we see that DualSchur polynomials can be expressed as a sum of key polynomials in the alphabet $\mathbf{y}$, multiplied by the product of Schubert polynomials in $\mathbf{x}$, $\mathbf{z}$ and key polynomials in $\mathbf{x}$. If we combine what we have achieved, we can express the following theorem.

Theorem 11 *For a given dominant partition λ with length n, using the same notation above, $DualSchur(\lambda)$*

is equal to

$$\sum_{\alpha\in\mathbb{N}^r}\sum_{\sigma\in S_r}(K_\alpha(\mathbf{y})\partial_\tau^y)\frac{(-1)^{l(\sigma)}}{\Delta}Y_{[v_n,v_{n-1},\dots,v_{k+1}]}(\mathbf{1};\mathbf{x}^w\mathbf{z})\sigma(x^{\gamma+\rho}\hat{K}_{\alpha w}(\mathbf{x}))\sigma$$

where $\Delta=\prod_{1\le i<j\le n}(x_i-x_j)$*, and* S_r *is the* r*-th symmetric group.*

5. Rational Grothendieck and rational key polynomials

Now we are going to give the definition of rational Grothendieck and key polynomials.

Definition 12 *Let* $u\in\mathbb{N}^n$ *be a strict dominant partition. The strict rational Grothendieck and key polynomials indexed by* u *are defined by*

$$G_u^{rat}(\mathbf{x};\mathbf{y})=\prod_{v_i\ge 0}\frac{1}{R(\frac{y_{n+1}}{x_i},\dots,\frac{y_{n+v_i}}{x_i}|1)}\prod_{v_j<0}R(\mathbf{1}|\frac{y_n}{x_j},\dots,\frac{y_{n+v_j+1}}{x_j})$$

$$K_u^{rat}(\mathbf{x})=\prod_{v_i\ge 0}\frac{1}{R(0\dots,0|x_i)}\prod_{v_j<0}R(x_j|0,\dots,0)$$

with $v=u-(n-1)^n$*, respectively. For a general* G_u^{rat}*, (resp.,* K_u^{rat}*) one needs the strict dominant, given by applying divided differences* π_i*(resp.,* $\hat{\pi}_i$*),* $i\le n-1$*, which we call* ancestor*.*

Example 13 *For* $u=[6,4,1,0]$*, we have* $K_{6410}^{rat}=\frac{x_4^3x_3^2}{x_1^3x_2}$ *and*

$$G_{6410}^{rat}(\mathbf{x};\mathbf{y})=\frac{\left(1-\frac{y_4}{x_3}\right)\left(1-\frac{y_3}{x_3}\right)\left(1-\frac{y_4}{x_4}\right)\left(1-\frac{y_3}{x_4}\right)\left(1-\frac{y_2}{x_4}\right)}{\left(\frac{y_5}{x_1}-1\right)\left(\frac{y_6}{x_1}-1\right)\left(\frac{y_7}{x_1}-1\right)\left(\frac{y_5}{x_2}-1\right)}.$$

Similarly, in the case of the Schubert polynomial, G_{102}^{rat} *is the image under* $\pi_1\pi_2$ *of* G_{410}^{rat} *under* $\pi_1\pi_2$*, and then* 410 *is the ancestor* 102*.*

The following proposition can be proved similarly to Proposition 3.

Proposition 14 *Let* $u\in\mathbb{N}^n$ *be a strict dominant partition. Then we have*

$$G_u^{rat}(\mathbf{x};\mathbf{y})=(-1)^{h_u}G_{\bar{u}}(x_n,\dots,x_1;y_N,\dots y_1)/\prod_{i=1,\dots,n}R(1|\frac{y_N}{x_i},\dots,\frac{y_{n+1}}{x_i})$$

$$K_u^{rat}(\mathbf{x})=(-1)^{h_u}K_{\bar{u}}(x_n,\dots,x_1)/(x_1,\dots,x_n)^{N-n}$$

with $\bar{u}=[u_1-u_n,\dots,u_1-u_2,u_1-u_1]$*,* $N=u_1+1$*,* $h_u=\sum_{i=1}^{k}v_i$ *for* $v_i\ge 0$*, and* $v=u-(n-1)^n$*.*

For example, we have $G_{532}(\mathbf{x};\mathbf{y})=\mathbf{x}^{-(532)}Y_{532}(\mathbf{x};\mathbf{y})=\frac{Y_{532}(\mathbf{x};\mathbf{y})}{K_{532}(\mathbf{x})}$ and
$G_{6410}^{rat}(\mathbf{x};\mathbf{y})=\frac{x_4^{-3}x_3^{-2}}{x_1^{-3}x_2^{-1}}\frac{(x_4-y_2)(x_4-y_3)(x_4-y_4)(x_3-y_3)(x_3-y_4)}{(x_2-y_5)(x_1-y_5)(x_1-y_6)(x_1-y_6)}=\frac{Y_{6410}^{rat}(\mathbf{x};\mathbf{y})}{K_{6410}^{rat}(\mathbf{x})}$.

We note that, because of reversing the alphabet, a divided difference π_i^x acting on G_v^{rat} corresponds to π_{n+1-i}^x on the Grothendieck polynomial in $\mathbf{x}_n$.

Proposition 15 *Let $u \in \mathbb{N}^n$ be a strict dominant partition. Then one can easily seen that*

1) $Y_u(\mathbf{x};\mathbf{y})|_{\mathbf{y}=0} = K_u(\mathbf{x})$ and $Y_u^{rat}(\mathbf{x};\mathbf{y})|_{\mathbf{y}=0} = K_u^{rat}(\mathbf{x})$,
2) $G_u(\mathbf{x};\mathbf{y}) = \frac{Y_u(\mathbf{x};\mathbf{y})}{K_u(\mathbf{x})}$,
3) $G_u^{rat}(\mathbf{x};\mathbf{y}) = \frac{Y_u^{rat}(\mathbf{x};\mathbf{y})}{K_u^{rat}(\mathbf{x})}$.

Proof It is clear by the definition. □

In the dominant case, rational Grothendieck and key polynomials have similar properties of Schubert polynomials listed in Proposition 4. It is clear that rational Grothendieck (resp., key) polynomials can be written as a proportion of two Grothendieck (resp., key) polynomials, depending on the given strict dominant partition $u \in \mathbb{N}^n$, in the following way:

$$G_u^{rat}(\mathbf{x};\mathbf{y}) = (-1)^{h_u} \frac{G_{[-v_n,\dots,-v_{k+1}]}(\mathbf{x}^w;\mathbf{y}^w)}{G_{[v_1,\dots,v_k]}(\mathbf{x};y_{n+1},\dots,y_N)}$$

where $v = u-(n-1)^n$ and $h_u = \sum_{i=1}^{k} v_i,\ v_i \geq 0$ for $i \leq k$.

Now we consider the Grassmannian $Gr(n, N+n)$ and Schubert class $\tau_{dual(\sigma)}$ as the dual class of τ_σ. Rational polynomials $Y_u^{rat}(\mathbf{x};\mathbf{0})$ and $Y_{dual(u)}^{rat}(\mathbf{x};\mathbf{0})$ correspond to classes $\tau_\sigma, \tau_{dual(\sigma)}$, respectively, and $Y_u^{rat}(\mathbf{x};\mathbf{0})Y_{dual(u)}^{rat}(\mathbf{x};\mathbf{0}) = \mp 1$.

Proposition 16 *Let τ_σ and τ_μ be Schubert classes. Then we have*

$$\tau_\sigma.\tau_\mu = \begin{cases} 1, & K_\sigma^{rat}(\mathbf{x}).K_\mu^{rat}(\mathbf{x}) = \mp 1 \\ 0, & \textit{otherwise.} \end{cases}$$

Proof Let $u \in \mathbb{N}^n$ be a strict dominant partition and $\sigma = u + \mathbf{1}$. If u is not strict dominant, we use its ancestor. Let $\bar{u} = u_1^n - u^w$, $N = u_1 + 1$, $v = u-(n-1)^n$, and $h_u = \sum_{i=1}^{k} v_i,\ v_i \geq 0$, where u^w is the reverse ordering of u. Then we obtain

$$\begin{aligned} Y_{(N-n)^n}(\mathbf{x};y_{n+1},\dots,y_N)Y_u^{rat}(\mathbf{x};\mathbf{y}) &= (-1)^{h_u} Y_{\bar{u}}(x_n,\dots,x_1;y_N,\dots,y_1) \\ Y_{(N-n)^n}(\mathbf{x};y_{n+1},\dots,y_N)Y_{\bar{u}}^{rat}(\mathbf{x};\mathbf{y}) &= (-1)^{h_{\bar{u}}} Y_u(x_n,\dots,x_1;y_N,\dots,y_1). \end{aligned}$$

Here we note that $\bar{u}$ is the dual of σ. We get ordinary Schubert polynomial $Y_u(\mathbf{x};\mathbf{y})$ for $y = \mathbf{0}$, which is equal to the key polynomial $K_u(\mathbf{x})$, and we have $Y_u^{rat}(\mathbf{x};\mathbf{0}) = K_u^{rat}(\mathbf{x})$. Additionally, we have $Y_u^{rat}(\mathbf{x};\mathbf{0}) = (-1)^{h_u} \frac{K_{[-v_n,\dots,-v_{k+1}]}(x_n,\dots,x_1)}{K_{[v_1,\dots,v_k]}(x_1,\dots,x_n)}$ and $Y_{dual(u)}^{rat}(\mathbf{x};\mathbf{0}) = (-1)^{h_{\bar{u}}} \frac{K_{[v_1,\dots,v_k]}(x_n,\dots,x_1)}{K_{[-v_n,\dots,-v_{k+1}]}(x_1,\dots,x_n)}$, by Proposition 4. The Young diagram for $Y_{dual(u)}^{rat}(\mathbf{x};\mathbf{0})$ can be obtained by the Young diagram for $Y_u^{rat}(\mathbf{x};\mathbf{0})$ rotated by $180°$. Therefore, $Y_u^{rat}(\mathbf{x};\mathbf{0})Y_{dual(u)}^{rat}(\mathbf{x};\mathbf{0}) = \mp 1$. If τ_μ is dual class to τ_σ, then μ, σ are dual partitions, and $\tau_\sigma.\tau_\mu$ is 1 and 0 otherwise. Hence, we have proven the Proposition 16. □

Example 17 *Let us take* $u = [4,1,0]$. *Then we have* $\sigma = [5,2,1]$ *and* $dual(\sigma) = \bar{u} = [4,3,0]$. *The rational Schubert polynomials* $Y^{rat}_{410}(\mathbf{x};\mathbf{0}) = \frac{x_3^2 x_2}{x_1^2}$ *and* $Y^{rat}_{430}(\mathbf{x};\mathbf{0}) = \frac{x_3^2}{-x_1^2 x_2}$ *correspond to diagrams*

	*	*		
		*		
			*	*

and

	*	*		
			*	
			*	*

, respectively. Hence, $Y^{rat}_{410}(\mathbf{x};\mathbf{0})Y^{rat}_{430}(\mathbf{x};\mathbf{0}) = -1$.

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