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Real hypersurfaces in complex two-plane Grassmannians whose shape operator is recurrent for the generalized Tanaka–Webster connection

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Abstract: We prove the non-existence of Hopf real hypersurfaces in complex two-plane Grassmannians whose shape operator A is generalized Tanaka–Webster recurrent if the principal curvature of the structure vector field is not equal to trace(A).

Key words: Real hypersurfaces, complex two-plane Grassmannians, Hopf hypersurface, generalized Tanaka–Webster connection, recurrent shape operator

1. Introduction

The generalized Tanaka–Webster connection (from now on, g-Tanaka–Webster connection) for contact metric manifolds was introduced by Tanno ([13]) as a generalization of the connection defined by Tanaka in [12] and, independently, by Webster in [14]. This connection coincides with the Tanaka–Webster connection if the associated CR-structure is integrable. The Tanaka–Webster connection is defined as a canonical affine connection on a non-degenerate, pseudo-Hermitian CR-manifold. A real hypersurface M in a Kähler manifold has an (integrable) CR-structure associated with the almost contact structure (ϕ, ξ, η, g) induced on M by the Kähler structure, but, in general, this CR-structure is not guaranteed to be pseudo-Hermitian. Cho [4] and Tanno [13] defined the g-Tanaka–Webster connection for a real hypersurface of a Kähler manifold by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{1.1}$$

for any X, Y tangent to M, where ∇ denotes the Levi-Civita connection on M, A is the shape operator on M and k is a non-zero real number. In particular, if the real hypersurface satisfies $A\phi + \phi A = 2k\phi$, then the g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka–Webster connection (see [4]).

Now let us denote by $G_2(\mathbb{C}\mathbb{C}^{m+2})$ the set of all complex 2-dimensional linear subspaces in $\mathbb{C}\mathbb{C}^{m+2}$. This Riemannian symmetric space has a remarkable geometric structure. It is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J (see Berndt and Suh [2]). In other words, $G_2(\mathbb{C}\mathbb{C}^{m+2})$ is the unique compact, irreducible Kähler, quaternionic Kähler manifold, which is not a hyper-Kähler manifold.

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Let M be a real hypersurface in $G_2(\mathbb{C}\mathbb{C}^{m+2})$ and N a local normal unit vector field on M. Let also A be the shape operator of M associated to N. The almost contact structure vector field $\xi = -JN$ is said to be a Reeb vector field. Moreover, if $\{J_1, J_2, J_3\}$ is a local basis of $\mathfrak{J}$, we define $\xi_i = -J_i N$, $i = 1, 2, 3$. We will call $\mathbb{D}^\perp = Span\{\xi_1, \xi_2, \xi_3\}$. Its orthogonal complement in TM will be denoted by $\mathbb{D}$.

Berndt and Suh, [2] proved that for a connected hypersurface M in $G_2(\mathbb{C}\mathbb{C}^{m+2})$, $m \geq 3$, both $Span\{\xi\}$ and $\mathbb{D}^\perp$ are invariant under the shape operator A if and only if either (A) M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}\mathbb{C}^{m+2})$, or (B) m is even, say $m = 2n$ and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(C^{m+2})$. Both types of real hypersurfaces have constant principal curvatures.

The Reeb vector field ξ is said to be Hopf if it is invariant under the shape operator A. The 1-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a Hopf foliation of M. We say that M is a Hopf hypersurface in $G_2(\mathbb{C}\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. This is equivalent to the fact that the Reeb vector field is Hopf.

If the shape operator A of M satisfies $(\nabla_X A)Y = 0$ for any vector fields X, Y tangent to M, the shape operator is said to be parallel with respect to the Levi-Civita connection. Suh [10] proved the non-existence of real hypersurfaces in $G_2(\mathbb{C}\mathbb{C}^{m+2})$ with parallel shape operator with respect to the Levi-Civita connection.

On the other hand, Kobayashi and Nomizu [7] introduced the notion of recurrent tensor field of type (r,s) on a manifold M with a linear connection ∇. A non-zero tensor field K of type (r,s) on M is said to be recurrent if there exists a 1-form ω on M such that $\nabla K = K \otimes \omega$.

Suh [11] proved the non-existence of real hypersurfaces in $G_2(\mathbb{C}\mathbb{C}^{m+2})$ with recurrent shape operator with respect to the Levi-Civita connection if $\mathbb{D}$ (respectively, $\mathbb{D}^\perp$) is invariant by the shape operator. Kim et al. [6] showed that this last condition is superfluous.

Jeong et al. [5] considered real hypersurfaces in $G_2(\mathbb{C}\mathbb{C}^{m+2})$ whose shape operator is parallel with respect to the g-Tanaka–Webster connection, that is, $(\hat{\nabla}^{(k)}_X A)Y = 0$ for any X, Y tangent to M and proved that there do not exist Hopf real hypersurfaces in $G_2(\mathbb{C}\mathbb{C}^{m+2})$, $m \geq 3$, with parallel shape operator with respect to the g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$ if $\alpha = g(A\xi, \xi) \neq 2k$.

This paper is devoted to the study of real hypersurfaces in complex two-plane Grassmannians whose shape operator is recurrent with respect to the g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$. That is, there exists a 1-form ω on M such that $(\hat{\nabla}^{(k)}_X A)Y = \omega(X)AY$ for any X, Y tangent to M. We will call $h = trace(A)$. Notice that if $\omega \equiv 0$, the shape operator should be parallel with respect to the g-Tanaka–Webster connection. Thus we will suppose that the 1-form ω does not vanish. We will prove the following

Main Theorem *There do not exist Hopf real hypersurfaces in* $G_2(\mathbb{C}\mathbb{C}^{m+2})$, $m \geq 3$, *whose shape operator is recurrent with respect to the g-Tanaka–Webster connection if* $\alpha = g(A\xi, \xi) \neq h$, *where* $h = TrA$.

2. Preliminaries

For the study of the Riemannian geometry of $G_2(\mathbb{C}\mathbb{C}^{m+2})$ see [1]. All the notations we will use from now on are those in [2] and [3]. We will suppose that the metric g of $G_2(\mathbb{C}\mathbb{C}^{m+2})$ is normalized for the maximal sectional curvature of the manifold to be eight. Then the Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\bar{R}(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ &\quad + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} \\ &\quad + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},\end{aligned} \tag{2.1}$$

where J_1, J_2, J_3 is any canonical local basis of $\mathfrak{J}$.

Let M be a real hypersurface of $G_2(\mathbb{C}\mathbb{C}^{m+2})$, that is, a submanifold of $G_2(\mathbb{C}\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M,g). Let N be a local unit normal field of M and A the shape operator of M with respect to N. The Kähler structure J of $G_2(\mathbb{C}\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ,ξ,η,g). Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu,\xi_\nu,\eta_\nu,g)$ on M.

Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}\mathbb{C}^{m+2}),g)$, for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ there exist three local 1-forms q_1, q_2, q_3 such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{2.2}$$

for any X tangent to $G_2(\mathbb{C}\mathbb{C}^{m+2})$, where subindices are taken modulo 3.

From the expression of the curvature tensor of $G_2(\mathbb{C}\mathbb{C}^{m+2})$ the Gauss equation is given by

$$\begin{aligned}&R(X,Y)Z \\ &\quad = g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\ &\qquad + \sum_{\nu=1}^{3}\{g(\phi_\nu Y,Z)\phi_\nu X - g(\phi_\nu X,Z)\phi_\nu Y - 2g(\phi_\nu X,Y) - 2g(\phi_\nu X,Y)\phi_\nu Z\} \\ &\qquad + \sum_{\nu=1}^{3}\{g(\phi_\nu \phi Y,Z)\phi_\nu \phi X - g(\phi_\nu \phi X,Z)\phi_\nu \phi Y\} \\ &\qquad - \sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(Z)\phi_\nu \phi X - \eta(X)\eta_\nu(Z)\phi_\nu \phi Y\} \\ &\qquad - \sum_{\nu=1}^{3}\{\eta(X)g(\phi_\nu \phi Y,Z) - \eta(Y)g(\phi_\nu \phi X,Z)\}\xi_\nu \\ &\qquad + g(AY,Z)ZX - g(AX,Z)AY\end{aligned} \tag{2.3}$$

for any X, Y, Z tangent to M. The Codazzi equation is also given by

$$
\begin{aligned}
(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\
+ \sum_{\nu=1}^{3} \{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\} \\
+ \sum_{\nu=1}^{3} \{\eta_\nu(\phi X)\phi_\nu \phi Y - \eta_\nu(\phi Y)\phi_\nu \phi X\} \\
+ \sum_{\nu=1}^{3} \{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu
\end{aligned}
\tag{2.4}
$$

for any X, Y tangent to M. The structures of $G_2(\mathbb{C}\mathbb{C}^{m+2})$ give the following

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.5}$$

$$\nabla_X \xi = \phi AX, \tag{2.6}$$

$$\nabla_X \xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{2.7}$$

$$(\nabla_X \phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX, Y)\xi_\nu. \tag{2.8}$$

A real hypersurface of type (A) has three (if $r = \frac{\pi}{2\sqrt{8}}$) or four (otherwise) distinct principal curvatures $\alpha = \sqrt{8}\cot(\sqrt{8}r)$, $\beta = \sqrt{2}\cot(\sqrt{2}r)$, $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$, $\mu = 0$, for some radius $r \in (0, \frac{\pi}{\sqrt{8}})$, with corresponding multiplicities $m(\alpha) = 1$, $m(\beta) = 2$, $m(\lambda) = m(\mu) = 2m-2$. The corresponding eigenspaces can be seen in [2].

A real hypersurface of type (B) has five distinct principal curvatures $\alpha = -2\tan(2r)$, $\beta = 2\cot(2r)$, $\gamma = 0$, $\lambda = \cot(r)$, $\mu = -\tan(r)$, for some $r \in (0, \frac{\pi}{4})$, with corresponding multiplicities $m(\alpha) = 1$, $m(\beta) = 3 = m(\gamma)$, $m(\lambda) = 4m-4 = m(\mu)$. For the corresponding eigenspaces see [2].

In the following we will need the following Proposition, [2],

Proposition 2.1 *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}\mathbb{C}^{m+2})$, $m \geq 3$, such that $A\xi = \alpha\xi$. Then $Y(\alpha) = \xi(\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y)$ for any Y tangent to M.*

and the following Theorem, [8],

Theorem 2.2 *Let M be a connected orientable Hopf real hypersurface in $G_2(\mathbb{C}\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathbb{D}$ if and only if m is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}\mathbb{C}^{m+2})$, where $m = 2n$.*

3. Proof of main theorem

As we suppose that $(\hat{\nabla}_X^{(k)} A)Y = \omega(X)AY$ for any X, Y tangent to M, from (1.1) we get

$$
\begin{aligned}
(\nabla_X A)Y = -g(\phi AX, AY)\xi + \eta(AY)\phi AX + k\eta(X)\phi AY + g(\phi AX, Y)A\xi \\
- \eta(Y)A\phi AX - k\eta(X)A\phi Y + \omega(X)AY
\end{aligned}
\tag{3.1}
$$

for any X, Y tangent to M. As $A\xi = \alpha\xi$, taking $Y = \xi$ in (3.1) we obtain $\nabla_X \alpha\xi = \alpha\phi AX + \alpha\omega(X)\xi$. That is, $X(\alpha)\xi + \alpha\phi AX = \alpha\omega(X)\xi + \alpha\phi AX$. Thus

$$X(\alpha) = \alpha\omega(X) \tag{3.2}$$

for any X tangent to M.

Proposition 3.1 *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}\mathbb{C}^{m+2})$, $m \geq 3$, whose shape operator is recurrent with respect to the g-Tanaka–Webster connection. If $\alpha \neq h$, either $\xi \in \mathbb{D}$ or $\xi \in \mathbb{D}^\perp$.*

Proof From [9] we know that if $\alpha = 0$, a Hopf real hypersurface in $G_2(\mathbb{C}\mathbb{C}^{m+2})$ satisfies either $\xi \in \mathbb{D}$ or $\xi \in \mathbb{D}^\perp$. Therefore we suppose that $\alpha \neq 0$.

We can write $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for a certain $X_0 \in \mathbb{D}$. If $\eta(X_0) = 0$ (respectively, $\eta(\xi_1) = 0$), $\xi \in \mathbb{D}^\perp$ (respectively, $\xi \in \mathbb{D}$). Therefore we suppose $\eta(X_0)\eta(\xi_1) \neq 0$. From (3.1) and the Codazzi equation we have

$$\begin{aligned}
&\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu(X,Y)\xi_\nu\} \\
&+ \sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\} + \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu \\
&= -2g(A\phi AX, Y)\xi - \eta(AX)\phi AY + \eta(AY)\phi AX - k\eta(Y)\phi AX + k\eta(X)\phi AY \\
&\quad + g((\phi A + A\phi)X, Y)A\xi + \eta(X)A\phi AY - \eta(Y)A\phi AX \\
&\quad + k\eta(Y)A\phi X - k\eta(X)A\phi Y + \omega(X)AY - \omega(Y)AX
\end{aligned} \tag{3.3}$$

for any X, Y tangent to M. Taking $X = \xi$ in (3.3) we get

$$\begin{aligned}
&\phi Y + \eta_1(\xi)\phi_1 Y - \sum_{\nu=1}^{3}\eta_\nu(Y)\phi_\nu\xi - 2\sum_{\nu=1}^{3}g(\phi_\nu\xi, Y)\xi_\nu + \sum_{\nu=1}^{3}\eta_\nu(\phi Y)\xi_\nu \\
&= -\alpha\phi AY + k\phi AY + A\phi AY - kA\phi Y + \omega(\xi)AY - \alpha\omega(Y)\xi
\end{aligned} \tag{3.4}$$

for any Y tangent to M. Taking the scalar product of (3.4) and ξ we obtain

$$-4\eta_1(\xi)g(\phi\xi_1, Y) = \alpha\omega(\xi)\eta(Y) - \alpha\omega(Y) \tag{3.5}$$

for any Y tangent to M. As $\alpha \neq 0$, from (3.2) and (3.5)

$$grad(\alpha) = \alpha\omega(\xi)\xi + 4\eta_1(\xi)\phi_1\xi. \tag{3.6}$$

Let $\{E_i\}_{i=1,...,4m-1}$ be an orthonormal basis of eigenvectors of M, and suppose $AE_i = \lambda_i E_i$, $i = 1,...,4m-1$. From (3.1) we have

$$\begin{aligned}\sum_{i=1}^{4m-1} g((\nabla_X A)E_i, E_i) = &- \sum_{i=1}^{4m-1} g(\phi AX, AE_i)g(\xi, E_i)\\ &+ \sum_{i=1}^{4m-1} \eta(AE_i)g(\phi AX, E_i) + k\eta(X)\sum_{i=1}^{4m-1} g(\phi AE_i, E_i)\\ &+ \sum_{i=1}^{4m-1} g(\phi AX, E_i)g(A\xi, E_i) - \sum_{\imath=1}^{4m-1} g(\xi, E_i)g(A\phi AX, E_i)\\ &- k\eta(X)\sum_{i=1}^{4m-1} g(A\phi E_i, E_i) + \omega(X)\sum_{i=1}^{4m-1} g(AE_i, E_i)\\ = &\, h\omega(X),\end{aligned} \tag{3.7}$$

because the other terms are clearly null. This yields $\sum_{i=1}^{4m-1} g(\nabla_X \lambda_i E_i - A\nabla_X E_i, E_i) = \sum_{i=1}^{4m-1} X(\lambda_i) = h\omega(X)$. Thus

$$X(h) = h\omega(X) \tag{3.8}$$

for any X tangent to M. Moreover, $\sum_{i=1}^{4m-1} g((\nabla_{E_i} A)Y, E_i) = g(A\phi AY, \xi) + \omega(AY) = \omega(AY)$. Thus from the Codazzi equation

$$\begin{aligned}\omega(AY) = &\sum_{i=1}^{4m-1} g((\nabla_Y A)E_i + \eta(E_i)\phi Y - \eta(Y)\phi E_i - 2g(\phi E_i, Y)\xi\\ &+ \sum_{\nu=1}^{3}\{\eta_\nu(E_i)\phi_\nu Y - \eta_\nu(Y)\phi_\nu E_i - 2g(\phi_\nu E_i, Y)\xi_\nu\}\\ &+ \sum_{\nu=1}^{3}\{\eta_\nu(\phi E_i)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi E_i\}\\ &+ \sum_{\nu=1}^{3}\{\eta(E_i)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi E_i)\}\xi_\nu, E_i)\\ = &\sum_{i=1}^{4m-1} g(\nabla_Y \lambda_i E_i, E_i) - \sum_{\nu=1}^{3} g(\phi_\nu\xi, \phi_\nu\phi Y)\\ &- \sum_{\nu=1}^{3}\eta_\nu(\phi Y)\sum_{i=1}^{4m-1} g(\phi_\nu\phi E_i, E_i) + \eta_1(\phi Y)\eta(\xi_1)\\ = &\sum_{i=1}^{4m-1} Y(\lambda_i) + 2\eta_1(\xi)\eta_1(\phi Y) - \sum_{\nu=1}^{3}\eta_\nu(\phi Y)trace(\phi_\nu\phi),\end{aligned} \tag{3.9}$$

for any Y tangent to M. As for any $\nu = 1, 2, 3$, $trace(\phi_\nu\phi) = trace(\phi\phi_\nu) = 2\eta_\nu(\xi)$, see for example [9], from (3.9) we obtain

$$\omega(AY) = Y(h) = h\omega(Y) \tag{3.10}$$

for any Y tangent to M. Taking $Y=\xi$ in (3.10) we get $\omega(A\xi)=h\omega(\xi)=\alpha\omega(\xi)$. Thus $(h-\alpha)\omega(\xi)=0$. As we suppose $h\neq\alpha$ it follows

$$\omega(\xi)=0. \tag{3.11}$$

(3.6) and (3.11) yield

$$grad(\alpha)=4\eta(\xi_1)\phi_1\xi. \tag{3.12}$$

We know that for any X,Y tangent to M $g(\nabla_X grad(\alpha),Y)=g(\nabla_Y grad(\alpha),X)$. This yields

$$\begin{aligned}&(g(\phi AX,\xi_1)+g(\xi,\nabla_X\xi_1))g(\phi_1\xi,Y)+\eta_1(\xi)g(\nabla_X\phi_1\xi,Y)\\&=(g(\phi AY,\xi_1)+g(\xi,\nabla_Y\xi_1))g(\phi_1\xi,X)+\eta_1(\xi)g(\nabla_Y\phi_1\xi,X).\end{aligned} \tag{3.13}$$

Taking $Y=\xi$ in (3.13) we obtain $\eta_1(\xi)g(\nabla_X\phi_1\xi,\xi)=g(\xi,\nabla_\xi\xi_1)g(\phi_1\xi,X)+\eta_1(\xi)g((\nabla_\xi\phi_1)\xi,X)$ for any X tangent to M. This gives $-\eta_1(\xi)g(\phi\xi_1,\phi AX)=g(\xi,q_3(\xi)\xi_2-q_2(\xi)\xi_3+\phi_1A\xi)g(\phi_1\xi,X)+\eta_1(\xi)g(-q_2(\xi)\phi_3\xi+q_3(\xi)\phi_2\xi+\eta_1(\xi)A\xi,X)$ for any X tangent to M. As we suppose $\eta_1(\xi)=\eta(\xi_1)\neq0$ we get $-g(A\xi_1,X)+\alpha\eta(\xi_1)\eta(X)=-q_2(\xi)g(\phi_3\xi,X)+q_3(\xi)g(\phi_2\xi,X)+\alpha\eta(\xi_1)\eta(X)$ for any X tangent to M. Therefore

$$A\xi_1=q_2(\xi)\phi_3\xi-q_3(\xi)\phi_2\xi. \tag{3.14}$$

The scalar product of (3.14) and ξ yields $\alpha\eta(\xi_1)=0$. As $\alpha\neq0$, $\eta(\xi_1)=0$ and we arrive at a contradiction.

This finishes the proof of our Proposition. □

Now, if $\xi\in\mathbb{D}$, from Theorem 2.2, M is locally congruent to a real hypersurface of type (B). Therefore consider the case $\xi\in\mathbb{D}^\perp$. We can write $\xi=\xi_1$.

Proposition 3.2 *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}\mathbb{C}^{m+2})$, $m\geq3$, whose shape operator is recurrent with respect to the g-Tanaka–Webster connection. If $\xi\in\mathbb{D}^\perp$ and $\alpha\neq0$, M is locally congruent to a type (A) real hypersurface.*

Proof With our hypothesis and being $\xi=\xi_1$, (3.4) becomes

$$\begin{aligned}&\phi Y+\phi_1Y-\eta_2(Y)\xi_3+\eta_3(Y)\xi_2\\&=-\alpha\phi AY+k\phi AY+A\phi AY-kA\phi Y+\omega(\xi)AY-\alpha\omega(Y)\xi\end{aligned} \tag{3.15}$$

for any Y tangent to M.

The scalar product of (3.15) and ξ, bearing in mind that $\alpha\neq0$, yields $\omega(Y)=\eta(Y)\omega(\xi)$ for any Y tangent to M. From (3.2) we obtain

$$grad(\alpha)=\alpha\omega(\xi)\xi. \tag{3.16}$$

Therefore for any X tangent to M $\nabla_X grad(\alpha)=X(\alpha\omega(\xi))\xi+\alpha\omega(\xi)\phi AX=\omega(\xi)X(\alpha)\xi+\alpha X(\omega(\xi))\xi+\alpha\omega(\xi)\phi AX$. If Y is orthogonal to ξ, $g(\nabla_X grad(\alpha),Y)=\alpha\omega(\xi)g(\phi AX,Y)$. Moreover, if X is also orthogonal to ξ, as $g(\nabla_X grad(\alpha),Y)=g(\nabla_Y grad(\alpha),X)$, we get $\alpha\omega(\xi)g(\phi AX,Y)=\alpha\omega(\xi)g(\phi AY,X)$. Thus either $\omega(\xi)=0$ and ω should vanish, which is impossible, or $g((\phi A+A\phi)X,Y)=0$ for any X,Y orthogonal to ξ. As we also have $(\phi A+A\phi)\xi=0$ we obtain

$$\phi A+A\phi=0. \tag{3.17}$$

From (3.17) it is easy to see that $A^2\phi = \phi A^2$. Now from (3.1) we have $(\nabla_\xi A)\xi_2 = k\phi A\xi_2 - kA\phi\xi_2 + \omega(\xi)A\xi_2 = 2kA\xi_3 + \omega(\xi)A\xi_2$. If $X \in \mathbb{D}$ we get $g((\nabla_\xi A)\xi_2, X) = g(\xi_2, (\nabla_\xi A)X)$ and applying the Codazzi equation this is equal to $g(\xi_2, \alpha\phi AX + \phi X - A\phi AX + \phi_1 X)$. This yields

$$g(A^2\xi_3, X) = (2k-\alpha)g(A\xi_3, X) + \omega(\xi)g(A\xi_2, X) \tag{3.18}$$

for any $X \in \mathbb{D}$. Developing $(\nabla_\xi A)\xi_3$ we have

$$-g(A^2\xi_2, X) = (\alpha - 2k)g(A\xi_2, X) + \omega(\xi)g(A\xi_3, X) \tag{3.19}$$

for any $X \in \mathbb{D}$. If in (3.18) we take ϕX instead of X we get

$$-g(A^2\xi_2, X) = (2k-\alpha)g(A\xi_2, X) - \omega(\xi)g(A\xi_3, X). \tag{3.20}$$

From (3.19) and (3.20) $g(A^2\xi_2, X) = 0$. Similarly $g(A^2\xi_3, X) = 0$ for any $X \in \mathbb{D}$. Thus (3.18) and (3.19) become

$$\begin{aligned} &\omega(\xi)g(A\xi_2, X) + (2k-\alpha)g(A\xi_3, X) = 0 \\ &-(2k-\alpha)g(A\xi_2, X) + \omega(\xi)g(A\xi_3, X) = 0 \end{aligned} \tag{3.21}$$

The matrix of coefficients of this homogeneous linear system has as determinant $(\omega(\xi))^2 + (2k-\alpha)^2$, and as $\omega(\xi) \neq 0$, this determinant does not vanish. This yields $g(A\xi_2, X) = g(A\xi_3, X) = 0$ for any $X \in \mathbb{D}$. Thus $\mathbb{D}$ is A-invariant and M must be locally of type (A). □

From Propositions 3.1 and 3.2, M is locally congruent to a real hypersurface either of type (A) or of type (B).

Let M be a type (A) real hypersurface. Clearly $(\nabla_\xi A)\xi = 0$. For (3.1) to be satisfied we should have $\sqrt{8}cot(\sqrt{8}r)\omega(\xi)\xi = 0$. Therefore $\omega(\xi) = 0$.

As $(\nabla_{\xi_i} A)\xi = \sqrt{8}cot(\sqrt{8}r)\phi A\xi_i - A\phi A\xi_i$, $i = 2,3$, for (3.1) to be satisfied this must be equal to $\sqrt{8}cot(\sqrt{8}r)\phi A\xi_i - A\phi A\xi_i + \omega(\xi_i)A\xi$. Thus $\omega(\xi_i) = 0$, $i = 1,2$.

The same occurs if we take any $X \in T_\lambda$ or $Y \in T_\mu$. This means that for a real hypersurface of type (A) to satisfy our condition we should have $\omega \equiv 0$, which is impossible.

A similar reasoning applied to a real hypersurface of type (B) shows that these real hypersurfaces do not satisfy our condition and our Theorem is proved.

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