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Finite groups with three conjugacy class sizes of primary and biprimary elements

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Abstract: We determine the structure of finite $\pi(m)$-separable groups if the set of conjugacy class sizes of primary and biprimary elements is $\{1, m, mn\}$, where m and n are two coprime integers.

Key words: Finite groups, conjugacy class sizes, primary and biprimary elements

1. Introduction

We will assume in this paper that any group is finite and G is always a group. For $x \in G$, the conjugacy class containing x will be denoted by x^G and $|x^G|$ will be called the conjugacy class size of x. We say that x is a primary element if its order is a prime power, and x is a biprimary element if its order has exactly two distinct prime divisors. All unexplained notation and terminology are standard; see [6].

There are many considerable works illustrating the relationship between the structure of a group and its conjugacy class sizes. For instance, a classical result due to Itô [7] is that a group G is nilpotent if its set of conjugacy class sizes is $\{1, m\}$ for some fixed integer m. He also proved [8] that G is solvable if the set of its conjugacy class sizes is $\{1, m, n\}$ for positive integers m and n. In [5], Camina proved the following:

Theorem 1.1 *([5, Theorem 3]) Let G be a group. If the set of conjugacy class sizes of G is $\{1, p^a, p^a q^b\}$, p and q being primes, then $G \cong G_0 \times H$, where H is abelian and G_0 contains a normal subgroup of index p, $Q_0 \times P_0$, where Q_0 is an abelian q-subgroup and P_0 is an abelian p-subgroup, neither being central in G, and $Q_0 \times P_0$ is the set of all elements of G_0 of index p^a or 1. Finally, $p^a = p$ and if P is a Sylow p-subgroup of G, then P/P_0 acts fixed-point-freely on Q_0 and $\Phi(P) \leq Z(P)$.*

Recently, Beltrán and Felipe [4] generalized the result above and obtained the following:

Theorem 1.2 *([4, Theorem A]) Let G be a group with no abelian direct factors and suppose that its conjugacy class sizes are $\{1, m, mn\}$, where $m, n > 1$ are coprime. Then G is an $\boldsymbol{F}$-group, $m = p$ for some prime p and G contains an abelian normal subgroup $M = H \times P_0$ of index p, where P_0 is a Sylow p-subgroup of M, and neither H nor P_0 is central in G. Furthermore, M is the set of all elements of G of index 1 or p, and if P is a Sylow p-subgroup of G, then P/P_0 acts fixed-point-freely on $H/\boldsymbol{Z}(G)_{p'}$ and $n = |H/\boldsymbol{Z}(G)_{p'}|$. Also, $|P'| = p$ and $|P/\boldsymbol{Z}(G)_p| = p^2$.*

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On the other hand, some authors replace conditions for all conjugacy classes by conditions referring only to some conjugacy classes to investigate the structure of a group. For instance, Baer [2] proved that a group is solvable if its primary elements have prime power conjugacy class sizes. Furthermore, the authors of this paper studied the influence of primary and biprimary elements on the structure of groups and proved [13] that a solvable group is nilpotent if the set of conjugacy class sizes of primary and biprimary elements is $\{1, m, n, mn\}$ with coprime integers m and n.

In this paper, we go on investigating the structure of groups by conjugacy class sizes of primary and biprimary elements. Our main result is:

Main Theorem Let m and n be two coprime integers. Furthermore, let G be a $\pi(m)$-separable group. Assume that the set of conjugacy class sizes of all primary and biprimary elements of G is exactly $\{1, m, mn\}$. Then $m = p$ for some prime p and G contains an abelian normal subgroup $C = H \times P_0$ of index p, where P_0 is a Sylow p-subgroup of C, and neither H nor P_0 is central in G. Furthermore, C is the set of all primary and biprimary elements of G of conjugacy class sizes 1 or p, and if P is a Sylow p-subgroup of G, then $H/\mathbf{Z}(G)_{p'} \rtimes P/P_0$ is a Frobenius group with $|H/\mathbf{Z}(G)_{p'}| = n$.

Remark 1.1 *The conclusion also holds for solvable groups.*

2. Preliminary results

In this section, we present some results that will be used in the sequel.

Lemma 2.1 *([3, Theorem C]) Let G be a π-separable group. If $x \in G$ with $|x^G|$ a π-number, then $[x^G, x^G] \subseteq \boldsymbol{O}_\pi(G)$. Consequently, $x \in \boldsymbol{O}_{\pi,\pi'}(G)$.*

Lemma 2.2 *([6, Theorem 5.3.4]) Let $P \times Q$ be the direct product of a p group P and a p'-group Q. Suppose that $P \times Q$ acts on a p-group G such that $\mathbf{C}_G(P) \leq \boldsymbol{C}_G(Q)$. Then Q acts trivially on G.*

Lemma 2.3 *([13, Lemma 2.1]) Let G be a π-separable group with π a subset of $\pi(G)$. Then:*

(a) $|x^G|$ is a π-number for every primary π'-element x if and only if G has an abelian Hall π'-subgroup.

(b) $|x^G|$ is a π'-number for every primary π'-element x if and only if $G = \boldsymbol{O}_\pi(G) \times \boldsymbol{O}_{\pi'}(G)$.

Lemma 2.4 *Let G be a p-separable group for a prime p. Assume that there is a p-element of conjugacy class size p^α, which is the highest power of p dividing the conjugacy class size of any $\{p, q\}$-element of G for each prime $q \in \pi(G)$ distinct from p. Then G has a normal p-complement.*

Proof Let x be a p-element of conjugacy class size p^α. Then, for every primary p'-element $v \in \mathbf{C}_G(x)$, we have $|v^{\mathbf{C}_G(x)}| = |\mathbf{C}_G(x) : \mathbf{C}_G(x) \cap \mathbf{C}_G(v)|$, which is a p'-number since p^α is the highest power of p dividing the conjugacy class sizes of all $\{p, q\}$-elements. Then Lemma 2.3 (b) implies that $\mathbf{C}_G(x) = \mathbf{C}_G(x)_p \times \mathbf{C}_G(x)_{p'}$, where $\mathbf{C}_G(x)_{p'}$ is a Hall p'-subgroup of $\mathbf{C}_G(x)$. Clearly, $\mathbf{C}_G(x)_{p'}$ is also a Hall p'-subgroup of G.

Write $H := \mathbf{O}_p(G)$. Then $x \in H$ by Lemma 2.1, which leads to $\mathbf{C}_G(H) \trianglelefteq \mathbf{C}_G(x)$. Note that each primary p'-element $y \in \mathbf{C}_G(x)_{p'}$ satisfies $\mathbf{C}_G(x)_p \leq \mathbf{C}_G(y)$. This forces $\mathbf{C}_H(x) \leq \mathbf{C}_H(y)$. By Lemma 2.2, we obtain that $y \in \mathbf{C}_G(H)$, leading to $|G : \mathbf{C}_G(H)|$ being a p-power. Recall that $\mathbf{C}_G(x)$ is p-nilpotent. Then $\mathbf{C}_G(H)$ has a normal p-complement K. Therefore, K is the normal p-complement of G, as required. □

Remark 2.1 *It is an extension of [5, Theorem 1].*

Lemma 2.5 *([9, Lemma 2.7]) If 1 and m are the only conjugacy class sizes of primary and biprimary elements of a group G, then $G = P \times A$, where $P \in Syl_p(G)$ and A is abelian. In particular, m is a power of p.*

Recall that a nonabelian group G is an **F**-group if, for every $x, y \in G \backslash \mathbf{Z}(G)$, we have that $\mathbf{C}_G(x) \leq \mathbf{C}_G(y)$ implies that $\mathbf{C}_G(x) = \mathbf{C}_G(y)$. In particular:

Lemma 2.6 *([11, Theorem]) G is an $\boldsymbol{F}$-group if and only if one of the following holds:*

(i) G is nonabelian and has an abelian normal subgroup of prime index;

(ii) $G/\boldsymbol{Z}(G)$ is a Frobenius group with Frobenius kernel $K/\boldsymbol{Z}(G)$ and Frobenius complement $L/\boldsymbol{Z}(G)$, where K and L are abelian;

(iii) $G/\boldsymbol{Z}(G)$ is a Frobenius group with Frobenius kernel $K/\boldsymbol{Z}(G)$ and Frobenius complement $L/\boldsymbol{Z}(G)$, where L is abelian, $\boldsymbol{Z}(K) = \boldsymbol{Z}(G)$, $K/\boldsymbol{Z}(G)$ has prime power order, and K is an $\boldsymbol{F}$-group;

(iv) $G/\boldsymbol{Z}(G) \cong S_4$, and V is nonabelian if $V/\boldsymbol{Z}(G)$ is the Klein four-group in $G/\boldsymbol{Z}(G)$;

(v) $G \cong A \times P$, where A is abelian and P is a nonabelian $\boldsymbol{F}$-group of prime power order;

(vi) $G/\boldsymbol{Z}(G) \cong PSL(2, p^n)$ or $PGL(2, p^n)$, $G' \cong SL(2, p^n)$ with $p^n \geq 3$;

(vii) $G/\boldsymbol{Z}(G) \cong PSL(2, 9)$ or $PGL(2, 9)$, $G' \cong PSL(2, 9)$.

3. Special cases of the Main Theorem

First we give a remark that if group G satisfies the assumption of the Main Theorem, then G has no Hall subgroup of order m since, otherwise, each primary $\pi(m)$-element has conjugacy class size 1, implying that G has a central Hall $\pi(m)$-subgroup, which is a contradiction.

Now we consider the case where G has a Hall subgroup of order n. This is necessary because one of the skills we employ in this paper is to analyze the properties of the centralizer of a certain primary element. For instance, suppose that x is a primary $\pi(m)$-element of conjugacy class size mn. The maximality of mn indicates that $\mathbf{C}_G(x) = \mathbf{C}_G(x)_\pi \times \mathbf{C}_G(x)_{\pi'}$, where $\mathbf{C}_G(x)_{\pi'}$ is a Hall π'-subgroup of $\mathbf{C}_G(x)$. If G has a Hall π'-subgroup of order n, then $\mathbf{C}_G(x)_{\pi'}$ is trivial, which makes it impossible to take an element from $\mathbf{C}_G(x)_{\pi'}$.

Here is an example: Let $G = (C_{21} \times C_3) \rtimes C_3$ be a nonabelian group of order 189. The conjugacy class sizes of primary and biprimary elements of G are $\{1, 3, 21\}$ and G has a Sylow 7-subgroup of order 7.

As a result, we first deal with this special case:

Lemma 3.1 *Let $m, n > 1$ be two coprime integers. Furthermore, let G be a $\pi(m)$-separable group. Assume that G has a Hall subgroup of order n and the set of conjugacy class sizes of primary and biprimary elements of G is $\{1, m, mn\}$. Then $m = p$ for some prime p and G contains an abelian normal subgroup $C = H \times P_0$ of index p, where P_0 is a Sylow p-subgroup of C, and neither H nor P_0 is central in G. Furthermore, C is the set of all primary and biprimary elements of G of conjugacy class sizes 1 or p, and if P is a Sylow p-subgroup of G, then $H \rtimes P/P_0$ is a Frobenius group.*

Proof If there exists some prime $r \in \pi(G) \setminus (\pi(m) \cup \pi(n))$, then r does not divide any conjugacy class size of each primary r'-element $y \in G$ as $|y^G| \in \{1, m, mn\}$. By [10, Theorem C], we have $G = R \times N$, where R is a Sylow r-subgroup of G and N is a Hall r'-subgroup of G. As a result, every element $v \in R$ has an

r-number conjugacy class size, yielding to $R \leq \mathbf{Z}(G)$. Hence, $G = A \times B$, where $A \leq \mathbf{Z}(G)$ and B is a Hall $\pi(m) \cup \pi(n)$-subgroup of G. Since central factors are irrelevant in this context, without loss of generality we assume that G is a $(\pi(m) \cup \pi(n))$-group without central factors.

Write $\pi := \pi(m)$. Suppose that H is a Hall subgroup of G of order n. Then every primary π'-element v has conjugacy class size 1 or m. By applying 2.3 (a), it follows that H is abelian.

Let x be a primary π-element of conjugacy class size m. Then for every primary π'-element $v \in \mathbf{C}_G(x)$, we see that $|\mathbf{C}_G(x) : \mathbf{C}_G(x) \cap \mathbf{C}_G(v)| = 1$ or n, by Lemma 2.3 (b), and it follows that $\mathbf{C}_G(x) = \mathbf{C}_G(x)_\pi \times H^g$ for some $g \in G$ since G is π-separable. Moreover, $\mathbf{C}_G(x)_\pi \nleq \mathbf{Z}(G)$ as $x \in C_G(x)_\pi$.

On the other hand, suppose that x is a primary π'-element of conjugacy class size m. We assert that $\mathbf{C}_G(x)$ is abelian. In fact, each primary π-element of $\mathbf{C}_G(x)$ has conjugacy class size 1 or n in $\mathbf{C}_G(x)$, which is a π'-number. According to Lemma 2.3 (a), one has that $\mathbf{C}_G(x)_\pi$ is abelian. Moreover, $\mathbf{C}_G(x)_\pi \nleq \mathbf{Z}(G)$, since otherwise, for each noncentral primary π-element w, we obtain $\mathbf{C}_G(w) \geq \langle w, \mathbf{Z}(G)_\pi \rangle > \mathbf{C}_G(x)_\pi$, which indicates that $|w^G|_\pi < m$ and thus $w \in \mathbf{Z}(G)$, a contradiction to the choice of w.

Take a primary element $z \in \mathbf{C}_G(x)_\pi \setminus \mathbf{Z}(G)$, which exists since G has no Hall π-subgroup of order m. Note that $|H| = n$. Then $x \in \mathbf{C}_G(x)$ forces $|z^G| = m$. As a result, $\mathbf{C}_G(z) = \mathbf{C}_G(z)_\pi \times H^{g_2}$ for some $g_2 \in G$ by the argument in the previous paragraph, where $\mathbf{C}_G(z)_\pi$ is a Hall π-subgroup of $\mathbf{C}_G(z)$. Recall that $x \in \mathbf{C}_G(z)$ and $|H| = n$. Then $\mathbf{C}_G(z) \leq \mathbf{C}_G(x)$ and thus $\mathbf{C}_G(z) = \mathbf{C}_G(x)$ because both x and z have conjugacy class size m. Consequently, $\mathbf{C}_G(x)$ is abelian.

Finally, we show that each element of G has conjugacy class size 1, m or mn. Let $v \in G$ be an arbitrary noncentral element. If v is a primary or a biprimary element, there is nothing to prove. Assume then that v has at least three primary components.

If v has a component of conjugacy class size mn, say v_1, then for another component v_i distinct from v_1, we have that $mn = |v_1^G|$ divides $|(v_1v_i)^G| \in \{1, m, mn\}$, implying $\mathbf{C}_G(v_1v_i) = \mathbf{C}_G(v_1)$. Hence, $|v^G| = mn$.

Now consider the case that each component v_i of v is of conjugacy class size m. Suppose first that v has no π'-component. Let v_i and v_j be two distinct components of v. Then $\mathbf{C}_G(v_i) = \mathbf{C}_G(v_i)_\pi \times H_i$ and $\mathbf{C}_G(v_j) = \mathbf{C}_G(v_j)_\pi \times H_j$, where H_i and H_j are two Hall π'-subgroups of G. As $v_j \in \mathbf{C}_G(v_i)$, we see that $H_i \leq \mathbf{C}_G(v_i) \cap \mathbf{C}_G(v_j)$, yielding $\mathbf{C}_G(v_iv_j) = \mathbf{C}_G(v_i)$. Furthermore, $|v^G| = m$, as wanted; if v has a π'-component, say v_s, then $\mathbf{C}_G(v_s)$ is abelian by the argument above. Since $v_i \in \mathbf{C}_G(v_s)$ for each component v_i of v such that $i \neq s$, we have $\mathbf{C}_G(v_s) = \mathbf{C}_G(v_i)$, which leads to $|v^G| = m$, as required.

Therefore, the set of conjugacy class sizes of G is $\{1, m, mn\}$, and then the theorem holds according to Theorem 1.2. □

Now we work on another special case in which m is a prime power. That is:

Lemma 3.2 *Let G be a p-separable group. Assume that $n > 1$ is a positive integer coprime to p and that α is a positive integer. If the set of conjugacy class sizes of all primary and biprimary elements of G is exactly $\{1, p^\alpha, p^\alpha n\}$, then G contains an abelian normal subgroup $C = H \times P_0$ of index $p^\alpha = p$, where P_0 is a Sylow p-subgroup of C, and neither H nor P_0 is central in G. Furthermore, C is the set of all primary and biprimary elements of G of conjugacy class sizes 1 or p, and if P is a Sylow p-subgroup of G, then $H/\mathbf{Z}(G)_{p'} \rtimes P/P_0$ is a Frobenius group with $|H/\mathbf{Z}(G)_{p'}| = n$.*

Proof According to Lemma 3.1, we may assume that G has no Hall subgroup of order n. Moreover, G

may be assumed as a $(\{p\} \cup \pi(n))$-group without central factors. If $|\pi(n)| = 1$, then the conclusion holds by Theorem 1.1. In the following we assume that $|\pi(n)| \geq 2$ and the proof will be completed in several steps.

Step 1. Let v be a primary element with conjugacy class size p^α. If v is a p-element, then $\mathbf{C}_G(v) = \mathbf{C}_G(v)_p \times \mathbf{C}_G(v)_{p'}$, where $\mathbf{C}_G(v)_{p'}$ is a nontrivial Hall p'-subgroup of G; if v is a p'-element, then $\mathbf{C}_G(v)_p$ is abelian with $\mathbf{C}_G(v)_p \nleq \mathbf{Z}(G)$.

Assume first that v is a p-element. For each primary p'-element $w \in \mathbf{C}_G(v)$, we have $|\mathbf{C}_G(v) : \mathbf{C}_G(v) \cap \mathbf{C}_G(w)| = 1$ or n. By Lemma 2.3 (b), we have $\mathbf{C}_G(v) = \mathbf{C}_G(v)_p \times \mathbf{C}_G(v)_{p'}$, where $\mathbf{C}_G(v)_{p'}$ is a Hall p'-subgroup of $\mathbf{C}_G(v)$, and so is of G. Moreover, $\mathbf{C}_G(v)_{p'}$ is nontrivial because G has no Hall subgroup of order n.

Suppose now that v is a primary p'-element. For any p-element $u \in \mathbf{C}_G(v)$, we have $|\mathbf{C}_G(v) : \mathbf{C}_G(v) \cap \mathbf{C}_G(u)| = 1$ or n. Hence, $\mathbf{C}_G(v)_p$ is abelian by Lemma 2.3 (a). We show that $\mathbf{C}_G(v)_p \nleq \mathbf{Z}(G)$. If not, for every noncentral p-element z, we have $\mathbf{C}_G(v)_p < \langle \mathbf{C}_G(v)_p, z\rangle \leq \mathbf{C}_G(z)$, which implies that $|z^G|_p < p^\alpha$ and thus $|z^G| = 1$, a contradiction to the choice of z.

Step 2. Let v be a q-element of conjugacy class size $p^\alpha n$ for some prime $q \in \pi(G)$. Then $\mathbf{C}_G(v) = \mathbf{C}_G(v)_q \times \mathbf{C}_G(v)_{q'}$, where $\mathbf{C}_G(v)_{q'}$ is an abelian Hall q'-subgroup of $\mathbf{C}_G(v)$. In particular, if $q \in \pi(n)$, then $\mathbf{C}_G(v)_p$ is abelian with $\mathbf{C}_G(v)_p \nleq \mathbf{Z}(G)$.

Let $w \in \mathbf{C}_G(v)$ be an arbitrary primary q'-element. Then $p^\alpha n = |v^G|$ divides $|(vw)^G| \in \{1, p^\alpha, p^\alpha n\}$, which implies that $w \in Z(\mathbf{C}_G(v))$. As a result, $\mathbf{C}_G(v) = \mathbf{C}_G(v)_q \times \mathbf{C}_G(v)_{q'}$ with an abelian Hall q'-subgroup $\mathbf{C}_G(v)_{q'}$. If $q \in \pi(n)$, then $\mathbf{C}_G(v)_p$ is abelian since $\mathbf{C}_G(v)_p \leq Z(\mathbf{C}_G(v))$. Moreover, $\mathbf{C}_G(v)_p \nleq \mathbf{Z}(G)$ by the same argument in Step 1.

We divide the proof into two cases depending on whether G has a p-element of conjugacy class size p^α or not.

Case 1. G has a p-element of conjugacy class size p^α, say x.

Step 3. $\mathbf{C}_G(x) = \mathbf{O}_p(G) \times H$, where $\mathbf{O}_p(G)$ is the Sylow p-subgroup of $\mathbf{C}_G(x)$ and H is a Hall p'-subgroup of G. In particular, $\mathbf{O}_p(G)$ is abelian.

Since x is a p-element of conjugacy class size p^α, by Step 1, we see that $\mathbf{C}_G(x) = \mathbf{C}_G(x)_p \times H$, where H is a Hall p'-subgroup of G. Moreover, Lemma 2.4 indicates that H is the normal p-complement of G. As a result, $\mathbf{O}_p(G) \leq \mathbf{C}_G(H)$. Write $C := \mathbf{C}_G(H)$.

Let $P_0 \in \mathrm{Syl}_p(C)$. Then $C = \mathbf{Z}(H) \times P_0$, which leads to $P_0 \trianglelefteq G$. Moreover, $P_0 = \mathbf{O}_p(G)$ and thus $\mathbf{C}_G(x)_p \leq \mathbf{O}_p(G)$. If $\mathbf{C}_G(x)_p < \mathbf{O}_p(G)$, then for every primary element $v \in H$, it follows that $\mathbf{C}_G(x)_p < \mathbf{O}_p(G) \leq \mathbf{C}_G(v)$, leading to $|v^G|_p < p^a$. As a consequence, $H \leq \mathbf{Z}(G)$. This contradiction shows that $\mathbf{C}_G(x) = \mathbf{O}_p(G) \times H$. Furthermore, $\mathbf{O}_p(G)$ is abelian since $\mathbf{O}_p(G)$ centralizes H.

Step 4. Conclusion in Case 1.

Assume that H is nonabelian. For an arbitrary primary element $y \in H$, we have $|y^G| \in \{1, p^\alpha, p^\alpha n\}$. Notice that $|G : H||y^H| = |y^G||\mathbf{C}_G(y) : \mathbf{C}_H(y)|$ and H is a normal Hall p'-subgroup of G. Then $|y^H| = 1$ or n. By applying Lemma 2.5, we see that $n = q^\beta$ for some prime $q \neq p$, contradicting with $|\pi(n)| \geq 2$. As a consequence, H is abelian.

Write $H = [H, P] \times \mathbf{C}_H(P)$, where P is a Sylow p-subgroup of G. Since $\mathbf{C}_H(P) \leq \mathbf{Z}(G)$, then $\mathbf{C}_H(P) = 1$ as G has no central factors. Now we claim that P/P_0 acts fixed-point-freely on H. Otherwise, there exists a primary element $a \in P \setminus P_0$ and an element $1 \neq b \in H$ such that $a^b = a$. Then we see that $\mathbf{C}_G(b) \geq \langle P_0, a \rangle > \mathbf{O}_p(G) = \mathbf{C}_G(x)_p$, leading to $b \in \mathbf{C}_H(P) = 1$. This contradiction shows that $P/P_0 \ltimes H$ is a Frobenius group and thus P/P_0 is either cyclic or a generalized quaternion group.

Let $z \in P$ be an arbitrary element. Then $|z^P| = |z^G|_p$ as $G = P \ltimes H$. That is, the conjugacy class sizes of P are $\{1, p^\alpha\}$. By [4, Corollary 6], $P/Z(P)$ has exponent p. Since $Z(P) \leq P_0$, we have that P/P_0 also has exponent p. Hence, $|P/P_0| = p$ and $\alpha = 1$, and the theorem is proved.

Case 2. G has no p-element of conjugacy class size p^α.

Step 5. $Z(P) = \mathbf{Z}(G)_p = \mathbf{O}_p(G)$. Moreover, G has a nonabelian Hall π-subgroup H.

Let P be a Sylow p-subgroup of G. Because every element of $\mathbf{O}_p(G)$ has conjugacy class size 1 or $p^\alpha n$, we have $Z(P) = \mathbf{Z}(G)_p \leq \mathbf{O}_p(G)$. Now we show that $\mathbf{O}_p(G) = \mathbf{Z}(G)_p$.

Suppose that $v \in H$ is an arbitrary noncentral primary element. If $|v^G| = p^\alpha$, then $\mathbf{C}_G(v)_p$ is abelian with $\mathbf{C}_G(v)_p \nleq \mathbf{Z}(G)$ by Step 1. Take an element $t \in \mathbf{C}_G(v)_p \backslash \mathbf{Z}(G)$, which exists as G has no Sylow p-subgroup of order p^α. Hence, $|t^G| = p^\alpha n$ by our assumption. By Step 2, we see that $\mathbf{C}_G(t) = \mathbf{C}_G(t)_p \times \mathbf{C}_G(t)_{p'}$ with $\mathbf{C}_G(t)_{p'}$ abelian. Note that $v \in \mathbf{C}_G(t)_{p'}$. Then $\mathbf{C}_G(t) \leq \mathbf{C}_G(v)$ and thus $\mathbf{C}_{\mathbf{O}_p(G)}(t) \leq \mathbf{C}_{\mathbf{O}_p(G)}(v)$. As a result, $v \in \mathbf{C}_G(\mathbf{O}_p(G))$ by Lemma 2.2. On the other hand, if $|v^G| = p^\alpha n$, then $\mathbf{C}_G(v) = \mathbf{C}_G(v)_p \times \mathbf{C}_G(v)_{p'}$ with $\mathbf{C}_G(v)_p$ abelian satisfying $\mathbf{C}_G(v)_p \nleq \mathbf{Z}(G)$ by applying Step 2 again. Select an element $t_1 \in \mathbf{C}_G(v)_p \backslash \mathbf{Z}(G)$. Then we easily see that $\mathbf{C}_G(v) = \mathbf{C}_G(t_1)$. By the same reasoning as above we also have $v \in \mathbf{C}_G(\mathbf{O}_p(G))$. Hence, $H \leq \mathbf{C}_G(\mathbf{O}_p(G))$, implying $\mathbf{O}_p(G) = \mathbf{Z}(G)_p$, as wanted.

If H is abelian, then $|h^G| = 1$ or p^α for every primary element $h \in H$. By Lemma 2.1, $H \leq \mathbf{O}_{p,p'}(G)$. As $\mathbf{O}_p(G) = \mathbf{Z}(G)_p$, it follows that $\mathbf{O}_{p,p'}(G) = H \times \mathbf{Z}(G)_p$ and thus $H \trianglelefteq G$. In this case, we see that the conjugacy class sizes of P are $\{1, p^\alpha\}$. Arguing as in Step 4, we are done. Hence, we consider that H is nonabelian below.

Step 6. Let e be a p-element of conjugacy class size $p^\alpha n$ in G. Then $\mathbf{C}_G(e)$ is abelian.

By Step 2, we have $\mathbf{C}_G(e) = \mathbf{C}_G(e)_p \times \mathbf{C}_G(e)_{p'}$ with $\mathbf{C}_G(e)_{p'}$ a nontrivial abelian Hall p'-subgroup of $\mathbf{C}_G(e)$. If $\mathbf{C}_G(e)_{p'} \leq \mathbf{Z}(G)$, then every primary p'-element has conjugacy class size 1 or p^a in G, which implies that H is abelian by Lemma 2.3 (a), in contradiction to Step 5. Hence there exists a primary element $v_1 \in \mathbf{C}_G(e)_{p'} \backslash \mathbf{Z}(G)$, indicating $\mathbf{C}_G(e) \leq \mathbf{C}_G(v_1)$. If $|v_1^G| = p^\alpha n$, we have that $\mathbf{C}_G(e) = \mathbf{C}_G(v_1)$ is abelian by Step 2; if $|v_1^G| = p^\alpha$, then $\mathbf{C}_G(v_1)_p$ is abelian by Step 1. Note that $|\mathbf{C}_G(v_1) : \mathbf{C}_G(e)| = n$, and then $\mathbf{C}_G(e)_p$ is also abelian. As a consequence, $\mathbf{C}_G(e)$ is abelian.

Step 7. The final contradiction in Case 2.

Let y be a q-element of conjugacy class size p^α. Then $q \in \pi(n)$, which implies that $\mathbf{Z}(H) \neq \mathbf{Z}(G)_{p'}$. We claim that $\mathbf{Z}(H)/\mathbf{Z}(G)_{p'}$ is a q-group. Otherwise, there exists an r-element $d \in \mathbf{Z}(H) \backslash \mathbf{Z}(G)$ such that $q \neq r \in \pi(n)$. Then $H \leq \mathbf{C}_G(d) \cap \mathbf{C}_G(y) = \mathbf{C}_G(dy)$ and hence $\mathbf{C}_G(d) = \mathbf{C}_G(y)$. For every primary $(\pi(n) - \{q\})$-element $w \in \mathbf{C}_G(y)$, we have $|\mathbf{C}_G(y) : \mathbf{C}_G(y) \cap \mathbf{C}_G(w)| = 1$ or n. On the other hand, for every q-element $z \in \mathbf{C}_G(y)$, we obtain $|\mathbf{C}_G(y) : \mathbf{C}_G(y) \cap \mathbf{C}_G(z)| = |\mathbf{C}_G(d) : \mathbf{C}_G(d) \cap \mathbf{C}_G(z)| = 1$ or n. We conclude that every primary p'-element is of conjugacy class size 1 or n in $\mathbf{C}_G(y)$, which follows that $\mathbf{C}_G(y) = \mathbf{C}_G(y)_p \times H$ by Lemma 2.3 (b). Recalling that every noncentral p-element has conjugacy class size $p^\alpha n$, we have that $\mathbf{C}_G(y)_p \leq \mathbf{Z}(P) = \mathbf{Z}(G)_p$, indicating $P \leq \mathbf{Z}(G)$. This contradiction shows that $\mathbf{Z}(H)/\mathbf{Z}(G)_{p'}$ is a q-group. Moreover, every primary and biprimary q'-element has conjugacy class size 1 or $p^\alpha n$ in G. By [1, Theorem A], it follows that n is a prime, contrary to $|\pi(n)| \geq 2$. This completes the proof. □

4. Proof of the Main Theorem

Proof According to Lemma 3.1, we may assume that G has no Hall subgroup of order n. Moreover, if m is a prime power, then we are done according to Lemma 3.2. Hence, in the following we may assume that $|\pi(m)| \geq 2$. Moreover, we may assume that G is a $(\pi(m) \cup \pi(n))$-group without central factors. Write $\pi := \pi(m)$. We divide the proof into several steps.

Step 1. Let v be a primary element with conjugacy class size m. If v is a π-element, then $\mathbf{C}_G(v) = \mathbf{C}_G(v)_\pi \times \mathbf{C}_G(v)_{\pi'}$, where $\mathbf{C}_G(v)_{\pi'}$ is a nontrivial Hall π'-subgroup of G. If v is a π'-element, then $\mathbf{C}_G(v)_\pi$ is abelian with $\mathbf{C}_G(v)_\pi \nleq \mathbf{Z}(G)$.

The proof is similar to the proof of Step 1 of Lemma 3.2.

Step 2. Assume that x is a primary element with conjugacy class size mn. Then $\mathbf{C}_G(x) = \mathbf{C}_G(x)_\pi \times \mathbf{C}_G(x)_{\pi'}$, where $\mathbf{C}_G(x)_\pi$ is the abelian Hall π-subgroup of $\mathbf{C}_G(x)$ such that $\mathbf{C}_G(x)_\pi \nleq \mathbf{Z}(G)$. In particular, if x is a π-element, then $\mathbf{C}_G(x)$ is abelian.

Let x be a q-element for some prime $q \in \pi(G)$. Then for every primary q'-element $v \in \mathbf{C}_G(x)$, we have that $mn = |x^G|$ divides $|(xv)^G| \in \{1, m, mn\}$, which implies that $\mathbf{C}_G(x) = \mathbf{C}_G(x)_q \times \mathbf{C}_G(x)_{q'}$, where $\mathbf{C}_G(x)_{q'}$ is an abelian Hall q'-subgroup of $\mathbf{C}_G(x)$.

If $q \in \pi'$, then we easily see that $\mathbf{C}_G(x)_\pi$ is an abelian Hall π-subgroup of $\mathbf{C}_G(x)$. We show that $\mathbf{C}_G(x)_\pi \nleq \mathbf{Z}(G)$. Otherwise, every primary π-element y satisfying $\mathbf{C}_G(y) \geq \langle y, \mathbf{Z}(G)_\pi \rangle > \mathbf{C}_G(x)_\pi$, implying $y \in \mathbf{Z}(G)$. As a result, G has a central Hall π-subgroup, a contradiction to our assumption.

Suppose then that $q \in \pi$. Then $\mathbf{C}_G(x) = \mathbf{C}_G(x)_\pi \times \mathbf{C}_G(x)_{\pi'}$ with $\mathbf{C}_G(x)_{\pi'}$ an abelian Hall π'-subgroup of $\mathbf{C}_G(x)$. It is obvious that $\mathbf{C}_G(x)_\pi \nleq \mathbf{Z}(G)$ since $x \in \mathbf{C}_G(x)_\pi$. Now we show that $\mathbf{C}_G(x)$ is abelian.

We first prove that $\mathbf{C}_G(x)_\pi/\mathbf{Z}(G)_\pi$ is not a q-group. Otherwise, there exists a noncentral r-element d such that $q \neq r \in \pi$. Clearly, $\mathbf{C}_G(d) \geq \langle \mathbf{Z}(G), d \rangle > \mathbf{C}_G(x)_r$ since $\mathbf{C}_G(x)_r = \mathbf{Z}(G)_r$. This implies that $|d^G|_r < m_r$ and thus $|d^G| = 1$, a contradiction. Hence, there exists an s-element $e \in \mathbf{C}_G(x)_\pi \backslash \mathbf{Z}(G)$ for some prime $s \in \pi$ distinct from q. Moreover, $\mathbf{C}_G(x) \leq \mathbf{C}_G(e)$. If $|e^G| = m$, then $x \in \mathbf{C}_G(e) = \mathbf{C}_G(e)_\pi \times \mathbf{C}_G(e)_{\pi'}$, where $\mathbf{C}_G(e)_{\pi'}$ is a Hall π'-subgroup of G by Step 1. This contradiction shows that $|e^G| = mn$ and $\mathbf{C}_G(x) = \mathbf{C}_G(e)$. Moreover, $\mathbf{C}_G(e)_\pi = \mathbf{C}_G(x)_\pi$ is abelian by Step 2, and so is $\mathbf{C}_G(x)$.

We will divide the proof into two cases depending on whether there exists a primary π-element of conjugacy class size m or not.

Case 1. There is a primary π-element of conjugacy class size m, say x.

Let x be a p-element of conjugacy class size m for some $p \in \pi$ and H be a Hall π'-subgroup of G such that $H \leq \mathbf{C}_G(x)$. Then:

Step 3. $\mathbf{C}_G(x) = \mathbf{C}_G(x)_\pi \times H$, where $\mathbf{C}_G(x)_\pi$ is an abelian Hall π-subgroup of G. In particular, $\pi(\mathbf{C}_G(x)_\pi) = \pi = \pi(\mathbf{C}_G(x)_\pi/\mathbf{Z}(G)_\pi)$.

By Step 1, we obtain $\mathbf{C}_G(x) = \mathbf{C}_G(x)_\pi \times H$, where $\mathbf{C}_G(x)_\pi$ is a nontrivial Hall π-subgroup of $\mathbf{C}_G(x)$ such that $\mathbf{C}_G(x)_\pi \nleq \mathbf{Z}(G)$. Let $w \in \mathbf{C}_G(x)_\pi \backslash \mathbf{Z}(G)_\pi$ be an s-element for some prime $s \in \pi$ distinct from p, which exists by the argument of Step 2. Then $H \leq \mathbf{C}_G(w)$ and hence $|w^G| = m$. Moreover, $H \leq \mathbf{C}_G(x) \cap \mathbf{C}_G(w) = \mathbf{C}_G(wx)$ implies that $|(wx)^G| = m$ and thus $\mathbf{C}_G(w) = \mathbf{C}_G(x)$. This indicates that every primary π-element of $\mathbf{C}_G(x)$ has conjugacy class size 1 or n in $\mathbf{C}_G(x)$. Therefore, by Lemma 2.3 (a), $\mathbf{C}_G(x)_\pi$ is abelian, as wanted.

We now prove that $\pi(\mathbf{C}_G(x)_\pi) = \pi$. If not, there exists a prime $r \in \pi \setminus \pi(\mathbf{C}_G(x)_\pi)$. Suppose that $R \in \mathrm{Syl}_r(G)$ and $y \in R \backslash \mathbf{Z}(G)$. Then $|y^G|_r < |R| = m_r$, which follows that $|y^G| = 1$, a contradiction. If $\pi(\mathbf{C}_G(x)_\pi/\mathbf{Z}(G)_\pi) \neq \pi$, there exists a prime $r \in \pi = \pi(\mathbf{C}_G(x)_\pi)$ such that $\mathbf{C}_G(x)_r = \mathbf{Z}(G)_r$. Then for every r-element $v \in G\backslash\mathbf{Z}(G)$, it follows $\mathbf{C}_G(v) \geq \langle v, \mathbf{Z}(G)_r\rangle > \mathbf{C}_G(x)_r$ that $|v^G|_r < m_r$, leading to $v \in \mathbf{Z}(G)$. This contradiction shows that $\pi(\mathbf{C}_G(x)_\pi/\mathbf{Z}(G)_\pi) = \pi = \pi(\mathbf{C}_G(x)_\pi)$.

Step 4. Every primary element $w \in \mathbf{O}_\pi(G)\backslash\mathbf{Z}(G)$ has conjugacy class size m in G. In particular, $\mathbf{C}_G(w)_\pi < \mathbf{O}_\pi(G)$.

Suppose that there exists a primary element $w \in \mathbf{O}_\pi(G)$ such that $|w^G| = mn$. Then Step 2 indicates that $\mathbf{C}_G(w) = \mathbf{C}_G(w)_\pi \times \mathbf{C}_G(w)_{\pi'}$, where $\mathbf{C}_G(w)_{\pi'}$ is an abelian Hall π'-subgroup of $\mathbf{C}_G(w)$. Moreover, there exists some element $g_0 \in G$ such that $\mathbf{C}_G(w)_{\pi'} \leq H^{g_0}$ because G is π-separable. As a result, for every primary element $t \in \mathbf{C}_G(w)_{\pi'}$, it follows that $\mathbf{C}_G(t) \geq \langle \mathbf{C}_G(w)_\pi, \mathbf{C}_G(x)_\pi^{g_0}\rangle > \mathbf{C}_G(x)_\pi^{g_0}$, so $|t^G|_\pi < m$ since $w \in \mathbf{O}_\pi(G)\backslash\bigcup_{g\in G}\mathbf{C}_G(x)_\pi^g$, leading to $t \in \mathbf{Z}(G)$. This forces that $\mathbf{C}_G(w)_{\pi'} \leq \mathbf{Z}(G)$. Consequently, for every noncentral primary π'-element z, we have $\mathbf{C}_G(w)_{\pi'} < \langle \mathbf{C}_G(w)_{\pi'}, z\rangle \leq \mathbf{C}_G(z)$, yielding $|z^G|_{\pi'} < m_{\pi'}$ and thus $|z^G| = m$. By Lemma 2.3 (a), H is abelian.

Let y be a q-element. Assume first that $q \in \pi$. If $|y^G| = m$, then by Step 3 we see that $\mathbf{C}_G(y)_\pi$ is abelian, and so is $\mathbf{C}_G(y)$ by applying Step 1. If $|y^G| = mn$, then Step 2 indicates that $\mathbf{C}_G(y)$ is also abelian. Hence, we may assume that $q \in \pi'$, leading to $|y^G| = m$. Since G is π-separable, there exists some $g_2 \in G$ such that $y \in H^{g_2}$. Note that H is abelian. Then $\mathbf{C}_G(x)^{g_2} = \mathbf{C}_G(y)$, leading to $\mathbf{C}_G(y) = \mathbf{C}_G(x)^{g_2} = \mathbf{C}_G(x)_\pi^{g_2} \times H^{g_2}$ being abelian. Therefore, we conclude that the centralizer of each noncentral element of G is abelian. This forces that G is an $\mathbf{F}$-group. In particular, G is one of the types in Lemma 2.6. As G is π-separable, we rule out case (vi) and case (vii). Furthermore:

If G is of type (i), then G has a primary element of conjugacy class size prime, in contradiction to $|\pi| \geq 2$;

If G is of type (ii), then $G/\mathbf{Z}(G) = K/\mathbf{Z}(G) \rtimes L/\mathbf{Z}(G)$ is a Frobenius group with Frobenius kernel $K/\mathbf{Z}(G)$ and Frobenius complement $L/\mathbf{Z}(G)$ and K, L are abelian. Let $a \in K$ and $b \in L$ be two noncentral primary elements. Then $|a^G|$ divides $|G : L| = |K/\mathbf{Z}(G)|$ and $|b^G|$ divides $|G : K| = |L/\mathbf{Z}(G)|$. Hence, $|a^G|$ and $|b^G|$ are coprime, also a contradiction;

If G is of type (iii), then $G/\mathbf{Z}(G) = K/\mathbf{Z}(G) \rtimes L/\mathbf{Z}(G)$ is a Frobenius group with complement $L/\mathbf{Z}(G)$ and L is abelian, and $K/\mathbf{Z}(G)$ has prime power order. Let $c \in L$ be a noncentral primary element. Then $|c^G|$ divides $|G : L| = |K/\mathbf{Z}(G)|$ a prime power, in contradiction to our assumption that $|\pi| \geq 2$;

If G is of type (iv), then every primary or biprimary element has conjugacy class size 1, 3, 6, or 8, contrary to our assumption;

If G is of type (v), then each noncentral primary element of G has prime power conjugacy class size, yielding that m is a prime power. The final contradiction shows that the first statement of Step 4 holds.

Let v be an arbitrary element of conjugacy class size m in G. Write $G = \mathbf{C}_G(v)K_1$, where K_1 is a Hall π-subgroup of G containing v. We see easily that $\langle v^G\rangle = \langle v^{\mathbf{C}_G(v)K_1}\rangle = \langle v^{K_1}\rangle \leq K_1$, which follows that $v \in \mathbf{O}_\pi(G)$. Moreover, the argument above implies that for every primary noncentral element $y \in \mathbf{C}_G(v)_\pi$ satisfying $|y^G| = m$ by Step 3. Hence, $y \in \mathbf{O}_\pi(G)$ and $\mathbf{C}_G(v)_\pi \leq \mathbf{O}_\pi(G)$ holds. If $\mathbf{C}_G(v)_\pi = \mathbf{O}_\pi(G)$, then $H \leq \mathbf{C}_G(\mathbf{O}_\pi(G))$, which follows that $\mathbf{O}_{\pi,\pi'}(G) = H \times \mathbf{O}_\pi(G) \trianglelefteq G$. Hence, $H \trianglelefteq G$. This indicates that every primary and biprimary element of K_1 has conjugacy class size 1 or m. Since K_1 is nonabelian, it follows that $|\pi| = 1$ by Lemma 2.5, in contradiction to our assumption. Therefore, $\mathbf{C}_G(v)_\pi < \mathbf{O}_\pi(G)$ and thus the second statement of Step 4 holds.

Step 5. The contradiction of Case 1.

Step 4 implies that $\mathbf{O}_\pi(G) = \mathbf{C}_G(x_1)_\pi \bigcup \cdots \bigcup \mathbf{C}_G(x_t)_\pi$, where $x_i \in \mathbf{O}_\pi(G)\backslash\mathbf{Z}(G)_\pi$ are primary elements satisfying $\mathbf{C}_G(x_i)_\pi \neq \mathbf{C}_G(x_j)_\pi$ when $i \neq j$. In particular, $t > 1$. We claim that $(\mathbf{C}_G(x_i) \cap \mathbf{C}_G(x_j))_\pi = \mathbf{Z}(G)_\pi$ for distinct integers $i, j \in \{1, \cdots, t\}$. Otherwise, there exists a primary element $v \in (\mathbf{C}_G(x_i) \cap \mathbf{C}_G(x_j))_\pi \backslash \mathbf{Z}(G)_\pi$. Since $\mathbf{C}_G(x_i)_\pi$ and $\mathbf{C}_G(x_j)_\pi$ are abelian by Step 3, we have $\mathbf{C}_G(x_i) \leq \mathbf{C}_G(v)$ and $\mathbf{C}_G(x_j) \leq \mathbf{C}_G(v)$, which follows that $\mathbf{C}_G(x_i) = \mathbf{C}_G(v) = \mathbf{C}_G(x_j)$, a contradiction. As a consequence, $\mathbf{O}_\pi(G) \setminus \mathbf{Z}(G)$ is a disjoint union of $\mathbf{C}_G(x_1)_\pi \setminus \mathbf{Z}(G), \cdots, \mathbf{C}_G(x_t)_\pi \setminus \mathbf{Z}(G)$.

Write $\mathbf{C}_G(x_i) = \mathbf{C}_G(x_i)_\pi \times H_i$, where H_i are Hall π'-subgroups of G. We prove that H_i are different conjugates of H. Since G is π-separable, H_i are conjugates of H. If there exists some $i, j \in \{1, \ldots, t\}$ such that $H_i = H_j$, then for every $h \in H_i$ we have $\mathbf{C}_G(h) \geq \mathbf{C}_G(H_i) \geq \langle \mathbf{C}_G(x_i)_\pi, \mathbf{C}_G(x_j)_\pi \rangle > \mathbf{C}_G(x_j)_\pi$, which implies $|h^G|_\pi < m$. As a result, $|h^G| = 1$ and thus $H_i \leq \mathbf{Z}(G)$, a contradiction. On the other hand, for every $g \in G$, we see that $\mathbf{C}_G(x^g) = \mathbf{C}_G(x)_\pi^g \times H^g$. Because $\mathbf{O}_\pi(G) \setminus \mathbf{Z}(G)$ is a disjoint union of $\mathbf{C}_G(x_1)_\pi \backslash \mathbf{Z}(G), \cdots, \mathbf{C}_G(x_t)_\pi \backslash \mathbf{Z}(G)$ with $t > 1$, there exists some $j \in \{1, \ldots, t\}$ such that $\mathbf{C}_G(x)_\pi^g = \mathbf{C}_G(x_j)_\pi$, leading to $H^g = H_j$. Consequently, we conclude that $H_1, \ldots, H_t$ are all the different conjugates of H. Therefore, $t = \frac{|\mathbf{O}_\pi(G)\backslash\mathbf{Z}(G)_\pi|}{|\mathbf{C}_G(x)_\pi\backslash\mathbf{Z}(G)_\pi|} = |G : \mathbf{N}_G(H)|$ because G is π-separable. Furthermore, $t \mid m$ because $\mathbf{C}_G(x) \leq \mathbf{N}_G(H)$ and $|x^G| = m$.

We claim that $|v^G| = m$ for every noncentral biprimary element v in $\mathbf{O}_\pi(G)\backslash\mathbf{Z}(G)_\pi$. Write $v = v_p v_q$, where v_p, v_q are the p-part and the q-part of v, respectively. If v_p (or v_q) is in $\mathbf{Z}(G)$, the conclusion is trivial. Assume that neither v_p nor v_q is in $\mathbf{Z}(G)$. Since $v_p \in \mathbf{C}_G(v_p) \cap \mathbf{C}_G(v_q)$, by the arguments in the first paragraph of this Step we obtain $\mathbf{C}_G(v_p) = \mathbf{C}_G(v_q) = \mathbf{C}_G(v)$, which forces $|v^G| = |v_p^G| = m$.

Let k be an arbitrary primary or biprimary element of $\mathbf{O}_\pi(G)$. Then $|k^G| = 1$ or m. Moreover, $\mathbf{C}_{\mathbf{O}_\pi(G)}(k) = \mathbf{C}_G(k) \cap \mathbf{O}_\pi(G) \leq \mathbf{C}_G(k)_\pi \leq \mathbf{C}_G(k) \cap \mathbf{O}_\pi(G) = \mathbf{C}_{\mathbf{O}_\pi(G)}(k)$, implying $\mathbf{C}_{\mathbf{O}_\pi(G)}(k) = \mathbf{C}_G(k)_\pi$. Hence, $|k^{\mathbf{O}_\pi(G)}| = |\mathbf{O}_\pi(G) : \mathbf{C}_{\mathbf{O}_\pi(G)}(k)| = |\mathbf{O}_\pi(G) : \mathbf{C}_G(k)_\pi| = |K : \mathbf{C}_G(k)_\pi| \frac{|\mathbf{O}_\pi(G)|}{|K|} = |G : \mathbf{C}_G(k)|_\pi \frac{|\mathbf{O}_\pi(G)|}{|K|} = 1$ or $m\frac{|\mathbf{O}_\pi(G)|}{|K|}$. By Lemma 2.5, we have that $m\frac{|\mathbf{O}_\pi(G)|}{|K|}$ is a prime power. Let $|\mathbf{O}_\pi(G) : \mathbf{C}_G(k)_\pi| = r^s$ for some $r \in \pi$. Since $t|\mathbf{C}_G(x)_\pi\backslash\mathbf{Z}(G)_\pi| = |\mathbf{O}_\pi(G)\backslash\mathbf{Z}(G)_\pi|$, we have that $(t - r^s)|\mathbf{C}_G(x)_\pi/\mathbf{Z}(G)_\pi| = t - 1$. As a result, $|\mathbf{C}_G(x)_\pi/\mathbf{Z}(G)_\pi| \mid (t-1)$, which contradicts the fact that $t \mid m$ and $\pi(\mathbf{C}_G(x)_\pi/\mathbf{Z}(G)_\pi) = \pi$ according to Step 3.

Case 2. There is no primary π-element of conjugacy class size m.

In this case, every primary or biprimary π-element is of conjugacy class size 1 or mn. By [12, Main Theorem], we see that m is a prime power, contrary to our assumption. Therefore, the proof is established. □

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