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OD-characterization of some alternating groups

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Abstract: Let G be a finite group. Moghaddamfar et al. defined prime graph $\Gamma(G)$ of group G as follows. The vertices of $\Gamma(G)$ are the primes dividing the order of G and two distinct vertices p, q are joined by an edge, denoted by $p \sim q$, if there is an element in G of order pq. Assume $|G| = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ with $P_1 < \cdots < p_k$ and nature numbers α_i with $i = 1, 2, \cdots, k$. For $p \in \pi(G)$, let the degree of p be $\deg(p) = |\{q \in \pi(G) \mid q \sim p\}|$, and $D(G) = (\deg(p_1), \deg(p_2), \cdots, \deg(p_k))$. Denote by $\pi(G)$ the set of prime divisor of $|G|$. Let $GK(G)$ be the graph with vertex set $\pi(G)$ such that two primes p and q in $\pi(G)$ are joined by an edge if G has an element of order $p \cdot q$. We set $s(G)$ to denote the number of connected components of the prime graph $GK(G)$. Some authors proved some groups are OD-characterizable with $s(G) \geq 2$. Then for $s(G) = 1$, what is the influence of OD on the structure of groups? We knew that the alternating groups A_{p+3}, where $7 \neq p \in \pi(100!)$, A_{130} and A_{140} are OD-characterizable. Therefore, we naturally ask the following question: if $s(G) = 1$, then is there a group OD-characterizable? In this note, we give a characterization of A_{p+3} except A_{10} with $s(A_{p+3}) = 1$, by OD, which gives a positive answer to Moghaddamfar and Rahbariyan's conjecture.

Key words: Order component, element order, alternating group, degree pattern, prime graph, Simple group

1. Introduction

In this short paper, all groups under study are finite, and for a simple group, we mean a non-Abelian simple group. Let G be a group. Then $\omega(G)$ denotes the set of orders of its elements of G and $\pi(G)$ denotes the set of prime divisors of $|G|$. Associated to $\omega(G)$ a graph is called a prime graph of G, which is denoted by $GK(G)$. The vertex set of $GKG)$ is $\pi(G)$, and two distinct vertices p, q are joined by an edge if $p \cdot q \in \omega(G)$, which is denoted by $p \sim q$.

Throughout this paper, we also use the following symbols. For a finite group G, the socle of G is defined as the subgroup generated by the minimal normal subgroup of G, denoted by $\mathrm{Soc}(G)$. $\mathrm{Syl}_p(G)$ denotes the set of all Sylow p-subgroups of G, where $p \in \pi(G)$, G_r denotes the Sylow r-subgroup of G for $r \in \pi(G)$. S_n and A_n denote the symmetric and alternating groups of degree n, respectively. Let $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ denote the automorphism and outer-automorphism groups of G, respectively. The other symbols are standard (see [2], for instance).

Moghaddamfar et al. introduced the following concept, which attracted the attention of some authors (see [1]).

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Definition 1.1 *[12] Let G be a finite group and $|G| = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where p_is are primes and α_is are integers. For $p \in \pi(G)$, let $\deg(p) := |\{q \in \pi(G)|p \sim q\}|$, which we call the degree of p. We also define $D(G) := (\deg(p_1), \deg(p_2), \cdots, \deg(p_k))$, where $p_1 < p_2 < \cdots < p_k$. We call $D(G)$ the degree pattern of G*

Since not all groups are OD-characterizable, Moghaddamfar et al. introduced the following.

Given a finite group M, denote by $h_{OD}(M)$ the number of isomorphism classes of finite groups G such that (1) $|G| = |M|$ and (2) $D(G) = D(M)$.

Definition 1.2 *[12] A finite group M is called k-fold OD-characterizable if $h_{OD}(M) = k$. Moreover, a 1-fold OD-characterizable group is simply called an OD-characterizable group.*

To date, we knew that some groups are k-fold OD-characterizable (see Tables 1 and 2 and corresponding references of [1]).

In particular, related to alternating groups, we have the following results.

Proposition 1.3 *A finite group G is OD-characterizable if G is one of the following groups:*

(1) The alternating groups A_p, A_{p+1}, and A_{p+2}, where p is a prime [11].

(2) The alternating groups A_{p+3}, where p is a prime and $7 \neq p \in \pi(100!)$ [6, 10].

Proposition 1.4 *Alternating group A_{10} is 2-fold OD-characterizable.*

We set $s(G)$ to denote the number of connected components of the prime graph $GK(G)$. Some authors proved that some special groups with $s(G) \geq 2$ are OD-characterizable. However, if $s(G) = 1$, the author has proved that the alternating groups A_{27} are 6-fold OD-characterizable. Therefore there is a question: which of the alternating groups is OD-characterizable? Related to $s(G) = 1$ for the alternating group, Moghaddamfar and Rahbariyan gave the following conjecture about the alternating group A_{p+3}.

Conjecture. [10, pp. 665, Conjecture 1] Let $p \neq 7$ be a prime. Then the alternating group A_{p+3} is OD-characterizable.

Inspired by the works of [6, 10], we generalize some authors' results and show that the alternating groups A_{p+3} with $s(A_{p+3}) = 1$ are OD-characterizable by using the classification of finite simple groups, which gives a positive answer regarding Moghaddamfar and Rahbariyan's conjecture. In fact, we prove the following.

Main Theorem. The alternating groups A_{p+3} except for A_{10} are OD-characterizable.

2. Preliminary results

In this section, we will give some results that will be used.

Lemma 2.1 *[15] Let $S = P_1 \times P_2 \times \cdots \times P_r$, where P_i's are an isomorphic non-abelian simple group. Then* $\mathrm{Aut}(S) = (\mathrm{Aut}(P_1) \times \mathrm{Aut}(P_2) \times \cdots \times \mathrm{Aut}(P_r)) \cdot S_r$.

Lemma 2.2 *[16] The group S_n (or A_n) has an element of order $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, where p_1, p_2, $\cdots$, p_s are distinct primes and α_1, α_2, $\cdots$, α_s are natural numbers, if and only if $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq n$ (or $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq n$ for m odd, and $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq n-2$ for m even).*

As a corollary of Lemma 2.1, we have the following result.

Lemma 2.3 *Let A_n (or S_n) be an alternating (or symmetric group) of degree n. Then the following hold.*

(1) Let $p, q \in \pi(A_n)$ be odd primes. Then $p \sim q$ if and only if $p + q \leq n$.

(2) Let $p \in \pi(A_n)$ be odd prime. Then $2 \sim p$ if and only if $p + 4 \leq n$.

(3) Let $p, q \in \pi(S_n)$. Then $p \sim q$ if and only if $p + q \leq n$.

By [2], we have that $|A_n| = n!/2$ and $|S_n| = n!$.

Let $\exp(n, r) = a$ denote that $r^a \mid n$ but $r^{a+1} \nmid n$.

Lemma 2.4 *Let A_{p+3} be an alternating group of degree $p+3$, where p is a prime. Then the following hold.*

(1) $\exp(|A_{p+3}|, 2) = \sum_{i=1}^{\infty}[\frac{p+3}{2^i}] - 1$. *In particular,* $\exp(|A_{p+3}|, 2) \leq p + 2$.

(2) $\exp(|A_{p+3}|, r) = \sum_{i=1}^{\infty}[\frac{p+3}{r^i}]$ *for each* $r \in \pi(A_{p+3})\backslash\{2\}$. *Furthermore,* $\exp(|A_{p+3}|, r) < \frac{p-1}{2}$, *where* $3 \leq r \in \pi(A_{p+3})$. *In particular, if* $r > [\frac{p+3}{2}]$, *then* $\exp(|A_{p+3}|, r) = 1$.

Proof See [9] □

Lemma 2.5 *Let A_{p+3} be an alternating group of degree $p+3$ with $p+2$ composite and p prime. Suppose that $|\pi(A_{p+3})| = d$. Then the following hold.*

(1) $\deg(2) = d - 2$. *In particular,* $2 \sim r$ *for all* $r \in \pi(A_{p+3}) \setminus \{p\}$.

(2) $\deg(3) = d - 1$.

(3) $\deg(p) = 1$. *In particular,* $p \sim r$ *where* $r \in \pi(A_{p+3})$ *if and only if* $r = 3$.

Proof From Lemmas 2.3 and 2.4, we have the desired result. □

Lemma 2.6 *Let G be a group with $D(G) = D(A_{p+3})$ and $|G| = |A_{p+3}|$, where p is a prime such that $p + 2$ is composite. Suppose that $|\pi(A_{p+3})| = d$. Then*

(1) $\deg(2) = \deg(5) = d - 2$, $\deg(3) = d - 1$ *and* $\deg(p) = 1$. *Hence $GK(G)$ is a connected graph.*

(2) If K is the maximal normal soluble subgroup of G, then K is an ω-group, where $\omega = \pi(3(p-1))$. In particular, G is insoluble.

Proof. See [6].

Lemma 2.7 *Let L be nonabelian simple groups. Then the orders and their outer-automorphism of L are as listed in Tables 1, 2, and 3.*

Proof See [7]. □

Table 1. The simple classical groups.

L	Lie; rank **L**	d	O	$\|L\|$
$L_n(q)$	$A_{n-1}(q)$ $n-1$	$(n, q-1)$	$2df$, if $n \geq 3$; df, if $n=2$	$\frac{1}{d}q^{n(n-1)/2}\prod_{i=2}^{n}(q^i-1)$
$U_n(q)$	${}^2A_{n-1}(q)$ $[n/2]$	$(n, q+1)$	$2df$, if $n \geq 3$ df, if $n=2$	$\frac{1}{d}q^{n(n-1)/2}\prod_{i=2}^{n}(q^i-(-1)^i)$
$PSp_{2m}(q)$	$C_m(q)$ m	$(2, q-1)$	df, $m \geq 3$; $2f$, if $m=2$	$\frac{1}{d}q^{m^2}\prod_{i=1}^{m}(q^{2i}-1)$
$\Omega_{2m+1}(q)$ q odd	$B_m(q)$ m	2	$2f$	$\frac{1}{2}q^{m^2}\prod_{i=1}^{m}(q^{2i}-1)$
$P\Omega^{+}_{2m}(q)$ $m \geq 3$	$D_m(q)$ m	$(4, q^m-1)$	$2df$, if $m \neq 4$ $6df$, if $m=4$	$\frac{1}{d}q^{m(m-1)(q^m-1)\prod_{i=1}^{m-1}(q^{2i}-1)}$
$P\Omega^{-}_{2m}(q)$ $m \geq 2$	${}^2D_m(q)$ $m-1$	$(4, q^m+1)$	$2df$	$\frac{1}{d}q^{m(m-1)(q^m+1)\prod_{i=1}^{m-1}}(q^{2i}-1)$

Table 2. The simple exceptional groups.

L	**L**	d	O	$\|L\|$
$G_2(q)$	2	1	f, if $p \neq 3$ $2f$, if $p=3$	$q^6(q^2-1)(q^6-1)$
$F_4(q)$	4	1	$(2,p)f$	$q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)$
$E_6(q)$	6	$(3, q-1)$	$2df$	$\frac{1}{d}q^{36}\prod_{i\in\{2,5,6,8,9,12\}}(q^i-1)$
$E_7(q)$	7	$(2, q-1)$	df	$\frac{1}{d}q^{63}\prod_{i\in\{2,6,8,10,12,14,18\}}(q^i-1)$
$E_8(q)$	8	1	f	$q^{120}\prod_{i\in\{2,8,12,14,18,20,24,30\}}(q^i-1)$
${}^2B_2(q), q=2^{2m+1}$	1	1	f	$q^2(q^2+1)(q-1)$
${}^2G_2(q), q=3^{2m+1}$	1	1	f	$q^3(q^3+1)(q-1)$
${}^2F_4(q), q=2^{2m+1}$	2	1	f	$q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$
${}^3D_4(q)$	2	1	$3f$	$q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$
${}^2E_6(q)$	4	$(3, q+1)$	$2df$	$\frac{1}{d}q^{36}\prod_{i\in\{2,5,6,8,9,12\}}(q^i-(-1)^i)$

Lemma 2.8 *Let a, b, and n be positive integers such that $(a,b)=1$. Then there exists a prime p with the following properties:*

- *p divides $a^n - b^n$,*
- *p does not divide $a^k - b^k$ for all $k < n$,*

with the following exceptions: $a=2, b=1$; $n=6$ and $a+b=2^k$; $n=2$.

Proof See [17]. □

Lemma 2.9 *Let $q > 1$ be an integer, m be a natural number, and p be an odd prime. If p divides $q-1$, then $(q^m-1)_p = m_p \cdot (q-1)_p$.*

Proof See Lemma 8(1) of [4]. □

Table 3. The simple sporadic groups.

L	d	O	$\lvert L\rvert$
M_{11}	1	1	$2^4 \cdot 3^2 \cdot 5 \cdot 11$
M_{12}	2	2	$2^6 \cdot 3^3 \cdot 5 \cdot 11$
M_{22}	12	2	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
M_{23}	1	1	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
M_{24}	1	1	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
J_1	1	1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$
J_2	2	2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$
J_3	3	2	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$
J_4	1	1	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$
HS	2	2	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$
Suz	6	2	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
McL	3	2	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$
Ru	2	1	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$
$He(F_7)$	1	2	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$
Ly	1	1	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$
ON	3	2	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$
Co_1	2	1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$
Co_2	1	1	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Co_3	1	1	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Fi_{22}	6	2	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
Fi_{23}	1	1	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
Fi'_{24}	3	2	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
$HN(F_5)$	1	2	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$
$Th(F_3)$	1	1	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$
$BM(F_2)$	2	1	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$
$M(F_1)$	1	1	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

Remark 2.10 *If $b = 1$, the prime p is called the Zsigmondy prime. If p is a Zsigmnody of $a^n - 1$, then Fermat's little theorem shows that $n \mid p-1$. Put $Z_n(a) = \{p : p$ is a Zsigmondy prime of $a^n - 1\}$. If $r \in Z_n(a)$ and $r \mid a^m - 1$, then $n \mid m$.*

Lemma 2.11 *If $n \geq 6$ is a natural number, then there are at least $s(n)$ prime numbers p_i such that $\frac{n+1}{2} < p_i < n$. Here*

- *$s(n) = 6$ for $n \geq 48$;*
- *$s(n) = 5$ for $42 \leq n \leq 47$;*
- *$s(n) = 4$ for $38 \leq n \leq 41$;*
- *$s(n) = 3$ for $18 \leq n \leq 37$;*
- *$s(n) = 2$ for $14 \leq n \leq 17$;*
- *$s(n) = 1$ for $6 \leq n \leq 13$.*

In particular, for every natural number $n > 6$, there exists a prime p such that $\frac{n+1}{2} < p < n-1$, and for every natural number $n > 3$, there exists an odd prime number p such that $n - p < p < n$.

Proof See Lemma 1 of [8]. □

Lemma 2.12 *Let X be a finite simple non-abelian group, $x \in \mathrm{Out}(X)$, and $|x|$ a prime greater than 7. Then $\pi(\mathrm{Out}(X))$ contains a number greater than $2|x|$.*

Proof See [14, Lemma 11]. □

Lemma 2.13 *Let G be a finite non-abelian simple group and p is the largest prime divisor of $|G|$ with $p \| |G|$. Then $p \nmid |\mathrm{Out}(G)|$.*

Proof By Lemma 2.7, G is isomorphic to alternating group, simple group of Lie type, or sporadic simple groups.

If G is an alternating group, then $|\mathrm{Out}(A_n)| = 2$ if $n \geq 5$ and $n \neq 6$; $|\mathrm{Out}(A_n)| = 4$ if $n = 6$ (see [7]). Hence we have the desired result.

If G is a sporadic simple group, then by Table 3 we have the result.

Therefore, we only consider that G is a simple group of Lie type. By hypothesis, if $p < 3$, then G is a $\{2, 3\}$-group that is soluble by Burnside's theorem (see [5], for instance). Hence in the following, let $p \geq 5$. Suppose the contrary; then $p \mid |\mathrm{Out}(G)|$. We mainly consider three cases.

Case 1. Let $G \cong A_{n-1}(q)$. Then $p \mid f$ or $p \mid d$. Obviously, $p \nmid q$ (in fact, if $p \mid q$, then $q = p^f$ and hence, by Lemma 2.8 there is a prime r such that $r \mid p^{tf} - 1$ and $r > p$).

(1) If $p \mid f$, then let $f = p \cdot m$ for some integer m and hence, $p \mid (r^{pm})^i - 1$ for some integer i and prime r. It follows that $(pmi) \mid p - 1$, which contradicts Remark 2.10.

(2) If $p \mid d$ and $d = (n, q - 1)$, then we can assume that $q = p \cdot m + 1$ for some integer m. It follows that $p \mid (p \cdot m + 1)^i - 1$ for some integer i and hence by Lemma 2.9, $(m, p) = 1$, and $i \mid p - 1$. Thus we can assume that $n = p$.

If $i = 1$, then $n = 2$ and hence $p = 3, q = 4$, in this case, $G \cong L_2(4) \cong A_5$ by [2]. It follows that $3 < 5 \mid |G|$, which contradicts the hypothesis.

If $i \geq 2$ and $m = 1$, then by Lemma 2.11, there is a prime r with $p \leq (p + 1)^{i-1} - 1 < r < (p + 1)^i - 1$, which contradicts the maximality of p.

If $i \geq 2$ and $m \geq 2$, we also can rule out this case as "$i \geq 2$ and $m = 1$".

Case 2. Let $G \cong^2 A_{n-1}(q)$. Then $p \mid f$ or $p \mid d$.

(1) If $p \mid f$, then we write $f = pt$ for some integer t and hence $p \mid (r^{pt})^i - (-1)^i$ for some integer i and prime r.

If i is even, then $p \mid (r^{pt})^i - 1$ and so $pti \mid p - 1$, a contradiction.

If i is odd, then $p \mid (r^{pt})^i + 1$ and hence $p \mid (r^{2pt})^i - 1$. We also have $2pti \mid p - 1$ or $pt \mid p - 1$ by Remark 2.10, a contradiction.

(2) If $p \mid d$ and $d = (n, q + 1)$, then let $q = pt - 1$ for some integer t, $p \mid (pt - 1)^i - (-1)^i$ for some integer i.

If i is even, then $p \mid (pt-1)^i - 1$ and so by Remark 2.10, $i \leq p-1$. Thus we assume that $n = p$. Hence we rule out this case as the case "Case 1(2)"

If i is odd, then $p \mid (pt-1)^i + 1$ and so by Remark 2.10, $2i \leq p-1$ or $i \mid p-1$. Thus we assume that $n = p$. If $i = 1$, then $n = 2 = p$ and so $q = 2$. Hence $G \cong^2 A_1(2) \cong 3^2.Q_8$ by [2, pp. xv] and G is soluble, which contradicts the hypothesis. If $i \geq 3$, then by Lemma 2.11, there is a prime r with $p \leq (pt-1)^{i-1} + 1 < r < (pt-1)^i + 1$, which contradicts the hypothesis.

Case 3. G is isomorphic to one of the other simple groups of Lie type.
We can rule out as "Case 1 or Case 2".
The proof is completed. □

Remark 2.14 *In the proof of Lemma 2.13, if $p \mid |\mathrm{Out}(G)|$, then by Lemmas 2.12 and 2.11, there is a prime r such that $p < r < 2p$, which contradicts the hypothesis of Lemma 2.13.*

3. Main theorem and its proof

Since A_p, A_{p+1}, A_{p+2} are OD-characterizable, we only consider when $p+2$ is composite, namely, we have the following result.

Theorem 3.1 *If G is a finite group such that $D(G) = D(A_{p+3})$ and $|G| = |A_{p+3}|$, where $7 \neq p$ is a prime, and $p+2$ is not prime, then G is isomorphic to A_{p+3}.*

Proof Since $p \neq 7$ and $p+2, p+4$ are primes, we can assume that $p \geq 13$. We will prove the theorem by a series of lemmas. □

Lemma 3.2 *Let K be the maximal normal soluble subgroup of G. Then K is a π-group, where $\pi = \pi(3(p-1))$. In particular, G is insoluble.*

Proof By Lemma 2.6, G is insoluble and if K is the maximal normal soluble subgroup of G, then K is a π-group, where $\pi = \pi(3(p-1))$. □

Lemma 3.3 *The quotient group G/K is an almost simple group. In fact, $S \lesssim G/K \lesssim \mathrm{Aut}(S)$.*

Proof Let $\overline{G} = G/K$ and $S = \mathrm{Soc}(G)$. Then $S = B_1 \times B_2 \times \cdots \times B_m$, where $B_i (1 \leq i \leq m)$ are non-abelian simple groups and $S \lesssim \overline{G} \lesssim \mathrm{Aut}(S)$. In the following, we will prove that $m = 1$.

Let $m \geq 2$. Then we have that $p \nmid |S|$. For otherwise, $2 \sim p$ and hence $\deg(p) \geq 2$ contradicting Lemma 2.5. Thus for every i, $B_i \in \mathfrak{F}_p$, where $\mathfrak{F}_p$ is the set of non-abelian finite simple groups S such that $p \in \pi(G) \subsetneqq \{2, 3, 5, \cdots, p\}$ and p is a prime. By Lemma 3.2, $p \nmid |K|$ and so $p \in \pi(\overline{G}) \subsetneqq \pi(\mathrm{Aut}(S))$. Hence $p \mid |\mathrm{Out}(S)|$. We know that

$$\mathrm{Out}(S) = \mathrm{Out}(S_1) \times \mathrm{Out}(S_2) \times \cdots \times \mathrm{Out}(S_r),$$

where the groups $S_j (j = 1, 2, \cdots, r)$ are direct products of all isomorphic B_i's such that

$$S = S_1 \times S_2 \times \cdots \times S_r.$$

Therefore, for certain j, p divides the order of an outer-automorphism of a direct product S_j of t isomorphic simple groups B_i for some $1 \leq j \leq m$. Since $B_i \in \mathfrak{F}_p$, it follows from Lemma 2.13 that $p \nmid |\mathrm{Out}(B_i)|$. However, by Lemma 2.1, $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(S_j)|^t \cdot t!$. Thus $t \geq p$. Since B_j is a non-abelian simple group, $4^p \mid |\mathrm{Aut}(B_i)|^t$, and hence $2^{2p} \mid |G|$, which contradicts Lemma 2.4. Hence $m = 1$ and $S = B_i$. □

Lemma 3.4 *The order $|S|$ of S is divisible by p.*

Proof If $r > \frac{p+3}{2}$, then by Lemma 2.4, $r \| |G|$. Assume that $p \nmid |S|$. Then by Lemma 3.3 $p \mid |K|$ or $p \mid |\mathrm{Out}(S)|$.

If $p \mid |K|$, then by Lemma 3.2, $p \in \pi(p-1)$. Thus $p \leq p-1$, a contradiction. Therefore, $p \mid |\mathrm{Out}(S)|$, which contradicts Lemma 2.13. □

Lemma 3.5 *S is isomorphic to A_n with $n = p, p+1, p+2, p+3$.*

Proof By hypothesis and Lemma 3.4, $|G_p| = |G|_p = |S_p| = p$. According to the classification of simple groups, we see that the possibilities for S are the alternating groups A_n with $n \geq 6$, one of the 26 sporadic simple groups, or simple groups of Lie type.

- **Case 1.** $S \cong A_n$ with $n \geq 6$.

  Then $n = p, p+1, p+2$, $n = p+3$ or $p+k$ with $k \geq 4$. If $n = p+k$ and $k \geq 4$, then order consideration rules out this case. Therefore, $S \cong A_n$ with $n = p, p+1, p+2, p+3$.

- **Case 2.** S is not isomorphic to a sporadic simple group according to [2].

- **Case 3.** S is isomorphic to a simple group of Lie type.

  Let q be a prime power.

  - 1. $S \cong B_n(q)$ with $n \geq 2$.

    In this situation, by hypothesis, $\pi(G) = \{2, 3, 5, 7, \cdots, p\}$ and so

    $$\frac{1}{(2, q-1)} q^{n^2} \prod_{i=1}^{n} (q^{2i} - 1) \mid p!.$$

    It follows that $p \mid q$ or $p \mid \prod_{i=1}^{n}(q^{2i}-1)$. If $p \mid q$, then q is a power of p. Since $|G_p| = p$ by hypothesis, this is impossible as $n \geq 2$. Therefore, $p \mid \prod_{i=1}^{n}(q^{2i}-1)$. It follows that $p \mid q^{2t} - 1$ for some $1 \leq t \leq n$ as p is prime. If $p \mid q^2 - 1$, then $p \mid q^4 - 1$ and hence $p \mid q^{2n} - 1$. Since $|G_p| = p$, then $p \nmid q^{2n-2} - 1$. Then, without loss of generality, we assume that $p = q^n - 1$ or $p = q^n + 1$ and hence $2 \mid q$ by Lemma 2.8. By Fermat's little theorem, $n \leq (p-1)/2$ and so $n^2 \leq n$ by Lemma 2.4, a contradiction.
  - 2. $S \cong D_n(q)$ with $n \geq 4$.

    Therefore, we have

    $$\frac{1}{(4, q^n - 1)} q^{n(n-1)} (q^n - 1) \prod_{i=1}^{n-1} (q^{2i} - 1) \mid p!.$$

Since the Sylow p-subgroup of G is of order p, $p \nmid q$ as otherwise, $q = p$ and thus $n = 1$, a contradiction. It follows that $p \mid q^n - 1$ or $p \mid q^{2t} - 1$ for some integer $1 \leq t \leq n-1$. If $p \mid q^2 - 1$, then $p \mid q^{2t} - 1$ and hence $p \mid q^{2n-2} - 1$ or $p \mid q^n - 1$. If $p \mid q^n - 1$, then since $|G_p| = p$ we can assume that $p = q^n - 1$ and hence by Lemma 2.8, $2 \mid q$. By Remark 2.10, $n \mid p-1$ and so $n+3 \leq p+2$. By Lemma 2.4, $\frac{n(n+1)}{2} \leq n+3$ and hence $n = 3$, a contradiction.

– 3. $S \cong {}^2A_n(q)$ with $n \geq 2$.

In this situation,

$$\frac{1}{(n+1, q+1)} q^{\frac{1}{2}n(n+1)} \prod_{i=1}^{n} (q^{i+1} - (-1)^{i+1}) \mid p!.$$

Since the Sylow p-subgroup of G is of order p and $n \geq 2$, we obtain that $p \mid q^{t+1} - (-1)^{t+1}$ for some integer $1 \leq t \leq n$.

Let n be odd. Then $p \mid q^{n+1} + 1$. If q is odd, then $2 \| q^{n+1} + 1$ and hence we assume that $p = \frac{q^{n+1}+1}{2}$, contradicting Lemma 2.8. Hence q is even. We can assume that $p = q^{n+1} + 1$ is a Mersenne prime. Obviously $p \mid q^{2(n+1)} - 1$ and hence by Remark 2.10, $2(n+1) \mid p-1$. It follows from Lemma 2.4 that $\frac{n(n+1)}{2} \leq 2(n+1) + 3$ and so $n = 5, 3$. Order consideration and Lemma 2.13 imply that it is impossible.

Let n be even. Then $p \mid q^{n+1} - 1$. If q is odd, then by Lemma 2.9, $p \mid q - 1$ and hence we assume that $p = \frac{q^{n+1}-1}{q-1}$. Therefore, $n+1 \leq p-1$. By Lemma 2.4, $\frac{n(n+1)}{2} \leq \frac{n+1}{2}$, a contradiction. Thus q is even. Similarly we have $n+4 \leq p+2$ and $\frac{n(n+1)}{2} \leq n+4$. Therefore, $n = 2, 4, 6$. Order consideration and Lemma 2.13 rule out this case.

– 4. $S \cong E_8(q)$.

Therefore, we have

$$q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1) \mid p!.$$

It follows that

$$p \mid q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1).$$

Hence $p \mid q^t - 1$, where $t \in \{14, 18, 20, 24, 30\}$.

Let $t = 14$. If q is odd, then by Lemma 2.11, there is a prime $r > p$, a contradiction. Hence $p \mid q^{30} - 1$ and by Remark 2.10, $30 + 3 \leq p + 2$. It follows from Lemmas 2.9 and 2.4 that $2^{14} \cdot (q-1)_2^8 \leq 33$, a contradiction. If q is odd, then similarly we have $q^{120} \mid 2^{33}$, a contradiction. Similarly, we can exclude that $H/K \cong E_6(q), E_7(q)$ and $F_4(q)$.

– 5. $S \cong G_2(q)$.

Then we have $q^6(q^6-1)(q^2-1) \mid p!$. It follows that $p \mid q^6 - 1$ or $p \mid q^2 - 1$. If $p \mid q^2 - 1$, then $p \mid q^6 - 1$. Hence we only consider $p \mid q^6 - 1$ and hence $6 \mid p - 1$. If q is odd, then $6 \mid 3$, a contradiction. Hence q is even,

– 6. $S \cong^2 E_6(q)$.

It is easy to see that

$$\frac{1}{(3,q+1)}q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1) \mid p!.$$

It follows that $p \mid q^t-1$ with $t=12,8$, or $p \mid q^k+1$ with $k=9,5$.

Let $t = 12$. If q is odd, then $2 \mid q-1$ and $2 \mid q+1$. It follows from Lemma 2.4 that $|S|_2 = 2^7 \cdot (q-1)_2^4 \cdot (q+1)_2^2$ and $\exp(|S|,2) \geq 15$. On the other hand, $15 \leq p+2$. We have $q = 3$ and so $p = 73$. Order consideration rules this out. If q is even, then by Lemma 2.4, $36 \mid 2^m+1$, a contradiction. Similarly we can rule out "$t=8$".

Let $t=9$. If q is odd, then similarly we have $\exp(|S|,2) \geq 15$. On the other hand, $18 \leq p+2$. Thus we also have $q=3$ and so $p=703$. Order consideration rules this out. If q is even, $36 \mid q^9+3$, a contradiction. Similarly, we can rule out "$t=5$".

– 7. $S \cong^2 B_2(q)$ with $q = 2^{2m+1}$.

It follows that $q^2(q^2+1)(q-1) \mid p!$. Thus $p \mid q^2+1$ or $p \mid q-1$.

Let $p \mid q^2+1$. We can assume that $p = q^2+1$ and hence, $m = 0$. By [2, pp. xv], $S \cong 5:4$ is soluble, a contradiction.

Let $p \mid q-1$, then we can assume that $p = 2^{2m+1}-1$ and hence $2m+1$ is a prime. Thus by Lemma 2.4, $4m+2 \mid 2^{2m+1}+1$, a contradiction.

Similarly $S \ncong^2 F_4(2^{2m+1})$.

– 8. $S \cong^2 G_2(q)$, $q = 3^{2n+1}$ with $n \geq 1$.

We see that $q^3(q^3+1)(q-1) \mid p!$. It follows that $p \mid q^3+1$ or $p \mid q-1$. If $p \mid q^3+1$, then we can assume that $p = \frac{q^3+1}{4}$ and so $6n+3 \mid \frac{q^2+9}{2}$. It follows that $n=1$ and $p=73$. We can rule out this case by order consideration. If $p \mid q-1$ and $r \mid q$, then there exists a Frobenius group of $r \cdot p$ with a Kernel of order r and a complement of order p respectively, and so there is an element of order $r \cdot p$, which contradicts the fact that $\deg(p) = 1$.

– 9. $S \cong^3 D_4(q)$.

We have $q^{12}(q^8+q^4+1)(q^6-1)(q^2-1) \mid p!$. In this case, since G has a Sylow p-subgroup of order p, then $p \mid q^8+q^4+1$, or $q \mid q^6-1$. If $p \mid q^8+q^4+1$, then by Remark 2.10, $12 \mid p-1$. If q is odd, then $12l \mid 6$, a contradiction.

If $p \mid q^6-1$, then $6 \mid p-1$ and similarly we also can rule this out.

Similarly we can rule out this case "$p \mid q^2-1$".

– 10. $S \cong A_n(q)$ with $n \geq 1$.

It is easy to get

$$\frac{1}{(n+1,q-1)}q^{n(n+1)/2}\prod_{i=1}^{n}(q^{i+1}-1) \mid p!.$$

It follows that $p \mid \prod_{i=1}^{n}(q^{i+1}-1)$ and so $p \mid q^{t+1}-1$ for some integer $t = n, n-1$.

Let $t = n-1$. Then $p \mid q^n - 1$ and so $n \le p-1$. If q is odd, then by Lemma 2.9 $|S|_2 = (q-1)_2^n \cdot \prod_{i=1}^{n}(i+1)_2$ and hence $\exp(|S|, 2) \ge \frac{3n}{2}$. By Lemma 2.4, we conclude that $\frac{3n}{2} \le n+3$ and $n \le 6$. Order consideration can rule out this case. If q is even, then similarly $\exp(|S|, 2) \ge \frac{n(n+1)}{2}$ and hence $\frac{n(n+1)}{2} \le n+3$. Thus we get $n \le 3$; order consideration rules this out.

Let $t = n$. Then similarly we can rule out "$t = n-1$".

This completes the proof of the lemma. □

Lemma 3.6 *G is isomorphic to A_{p+3}.*

Proof By Lemma 3.3, $S \le G/K \le \mathrm{Aut}(S)$. By Lemma 3.5, $S \cong A_n$ with $n = p, p+1, p+2, p+3$. We consider the following cases.

Case 1. $S \cong A_p$.

Therefore, $A_p \le G/K \le S_p$.

If $G/K \cong A_p$, then order consideration of G, we have that $|K| = (p+1)(p+2)(p+3)$. Obviously $2 \in \pi(K)$. It follows that there is an element of order $2 \cdot p$, which contradicts the fact that $\deg(p) = 1$.

If $G/K \cong S_p$, we have $|K| = (p+1)(p+2)(p+3)/2$ and also $2 \in \pi(K)$. It means that $2 \sim p$, contradicting $\deg(p) = 1$.

Case 2. $S \cong A_{p+1}$.

In this case, $A_{p+1} \le G/K \le S_{p+1}$.

If $G/K \cong A_{p+1}$, then $|K| = (p+2)(p+3)$. Obviously $2 \in \pi(K)$ and so there exists an element of order $2 \cdot p$. It follows that $\deg(p) \ge 2$, a contradiction.

If $G/K \cong S_{p+1}$, then $|K| = (p+2)(p+3)/2$. If $2 \nmid |K|$, then there is a prime r such that $p > r > 3$ and $r \le \frac{p+3}{2}$. It follows that there exists an element of order $r \cdot p$ and hence $r \sim p$, contradicting Lemma 2.5. If $4 \mid |K|$, then also we can rule out this case.

Case 3. $S \cong A_{p+2}$.

We have $A_{p+2} \le G/K \le S_{p+2}$.

If $G/K \cong A_{p+2}$, then $|K| = p+3$. Obviously $2 \in \pi(K)$, we rule out this case as "Case 1".

If $G/K \cong S_{p+2}$, then $|K| = (p+3)/2$. We rule out this case as "Case 2".

Case 4. $S \cong A_{p+3}$.

It is easy to get $A_{p+3} \le G/K \cong S_{p+3}$.

If $G/K \cong S_{p+3}$, then $(p+3)! \mid \frac{(p+3)!}{2}$, a contradiction.

If $G/K \cong A_{p+3}$, then $K = 1$ and hence $G \cong A_{p+3}$.

This completes the proof of the Lemma and also of the main theorem. □

4. Some applications

We knew that alternating groups A_p, A_{p+1}, and A_{p+2}, where p is a prime, are OD-characterizable (see [11]) and by our main theorem, we have the following.

Theorem 4.1 *The alternating group A_n except A_{10} with $n = p, p+1, p+2, p+3$ are OD-characterization.*

Shi gave the following conjecture.

Conjecture [13] Let G be a group and H a finite simple group. Then $G \cong H$ if and only if (a) $\omega(G) = \omega(H)$ and (b) $|G| = |H|$.

Then we have the following corollary.

Corollary 4.2 *Let G be a group and $p \geq 5$ is a prime. Then $G \cong A_n$ where $n = p, p+1, p+2, p+3$ if and only if $\omega(G) = \omega(A_n)$ and $|G| = |A_n|$.*

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