

Characterizing rational groups whose irreducible characters vanish only on involutions

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Abstract: A rational group is a finite group whose irreducible complex characters are rational valued. The aim of this paper is to classify rational groups G for which every nonlinear irreducible character vanishes only on involutions.

Key words: Rational group, irreducible character, zero of character

1. Introduction

Let G be a finite group and χ be a nonlinear irreducible ordinary character of G . A well-known theorem of Burnside states that there exists $g \in G$ such that $\chi(g) = 0$; such an element g is called a zero of χ , and we say χ vanishes on g . Zeroes of characters are important in finding the structure of Sylow subgroups of a finite group. Besides well-known theorems related to zeros of characters that usually appear in reference books, e.g.[5], this subject has been well studied by many mathematicians such as Chillag [1]. An important result obtained by Moretó and Navarro [9] applies when zeroes of characters occur on prime order elements. Dolfi et al. in [3] also proved that if p is a prime number and all of the p -elements of G are nonvanishing, then G has a normal Sylow p -subgroup.

Throughout this paper, we use the following notations and terminologies. The order of the group G and the order of the element $g \in G$ are denoted by $|G|$ and $|g|$, respectively. For the prime number p , $O_p(G)$ denotes the unique largest normal p -subgroup of G and $E(p^n)$ denotes the elementary abelian p -group of order p^n . For the elements x and g belonging the group G , by x^g we mean $g^{-1}xg$. The symbol $K : H$ stands for the semidirect product of the groups K and H in which H acts on K . The cyclic group of order n is denoted by \mathbb{Z}_n . We reserve $Z(G)$ for the center of G . If G is a group and N is a normal subgroup of G , then $\text{Irr}(G/N) = \{\chi \in \text{Irr}(G) | N \subseteq \ker \chi\}$. Therefore, in this paper, wherever we choose $\chi \in \text{Irr}(G/N)$, we mean that χ is an irreducible character of G containing N in its kernel.

Here we are interested in classification of rational groups whose nonlinear irreducible characters vanish only on elements of order 2. For this, we need some concepts and theorems of rational groups.

A finite group G is called a rational group or a \mathbb{Q} -group if all irreducible complex characters of G are rational-valued. We recall some relevant theorems from [7], which contains a comprehensive description of \mathbb{Q} -groups.

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Theorem 1.1 *Let G be a nontrivial \mathbb{Q} -group and p be a prime number. If p is a divisor of $|G|$, then $p - 1$ divides $|G|$. In particular, the order of a nontrivial \mathbb{Q} -group is even.*

Theorem 1.2 *Let G be a rational group and N be a normal subgroup of G ; then G/N is also a rational group.*

Theorem 1.3 *Let G be an abelian rational group; then G is an elementary abelian 2-group.*

Here we use [[5], Definition 6.21] to introduce a concept that is needed in the sequel.

Definition 1.1 *Let N be a normal subgroup of G and $\chi \in \text{Irr}(G)$. Then χ is a relative M -character with respect to N if there exists H with $N \subseteq H \subseteq G$ and $\psi \in \text{Irr}(H)$ such that $\psi^G = \chi$ and $\psi_N \in \text{Irr}(N)$. If every $\chi \in \text{Irr}(G)$ is a relative M -character with respect to N , then G is a relative M -group with respect to N .*

Theorem 1.4 *If N is a normal subgroup of G and G/N is nilpotent, then G is a relative M -group with respect to N .*

Proof See [[5], Theorem 6.22 and its subsequent note]. □

For $n \geq 1$, let F_n denote the Frobenius group $E(3^n) : \mathbb{Z}_2$, in which \mathbb{Z}_2 acts on $E(3^n)$ by inverting every nonidentity element. By [2], F_n is a \mathbb{Q} -group. The main result of this paper is as follows.

Main Theorem *If every nonlinear irreducible character of a nonabelian \mathbb{Q} -group G vanishes only on involutions, then $Z(G)$ is an elementary abelian 2-group (possibly trivial) and G is isomorphic to $Z(G) \times F_n$ for some $n \in \mathbb{N}$.*

2. Proof of the main theorem

In order to prove our main theorem, first we prove that in \mathbb{Q} -groups any irreducible character cannot vanish on a p -element unless $p = 2$. We deduce this claim from the following theorem.

Theorem 2.1 *Let G be a finite group and p be a prime divisor of $|G|$. If every nonlinear irreducible character of G vanishes only on p -elements of G , then G has a normal p -complement.*

Proof

We may assume that G is nonabelian. In the case that χ is nonlinear, Burnside's theorem [5] asserts that there exists $g \in G$ such that $\chi(g) = 0$. By assumption, $|g| = p^m$. Since $\chi(g)$ is the sum of $\chi(1)$, p^m th roots of unity, the main theorem of [8] implies that p divides $\chi(1)$. Therefore, p divides the degree of every nonlinear character of G , and hence, by one of Thompson's theorems in [10], the group G has a normal p -complement. □

Corollary 2.1 *With the assumption in Theorem 2.1, if every nonlinear irreducible character of a \mathbb{Q} -group G vanishes only on p -elements, then $p = 2$, and G is solvable.*

Proof The first part follows from Theorems 1.1, 1.2, and 2.1. Again, according to Theorem 2.1, for some integer n , the n th derived subgroup of G is included in a normal 2-complement subgroup that has odd order. Therefore, by the famous theorem of Feit and Thompson, G is solvable. □

Remark 2.1 By Corollary 2.1, G is solvable. On the other hand, it is known that divisors of the order of a solvable \mathbb{Q} -group are included in $\{2, 3, 5\}$ (see [4]). Therefore, divisors of the order of the 2-complement subgroup of G are included in $\{3, 5\}$, but the elements of the 2-complement subgroup of G are not zeros of any irreducible character of G ; hence, by [3], the 2-complement subgroup is nilpotent.

Lemma 2.1 Let G be a finite group such that every nonlinear irreducible character of G vanishes only on involutions. Let P be a Sylow 2-subgroup of G . The following are then true:

(i) If $a \in P \setminus Z(P)$ then $C_G(a) \subseteq P$.

(ii) If G is \mathbb{Q} -group then $Z(G) = O_2(G)$, and if $Z(G) = 1$ then for every nonidentity element $a \in Z(P)$ we have $C_G(a) = P$.

Proof By Theorem 2.1, there exists a normal 2-complement K . If $a \in P \setminus Z(P)$, then there exists $\chi \in Irr(G/K)$ such that $\chi(a) = 0$; otherwise, by Theorem B of [6], we get $a \in Z(P)$, a contradiction. However, if $1 \neq y \in K \cap C_G(a)$, then $\chi(ay) = 0$, and therefore by hypothesis $|ay| = 2$. Thus, $|y| = 2$, which is again a contradiction, and therefore $C_G(a) \subseteq P$.

Now we assume that, G is a \mathbb{Q} -group. By [[7], Corollary 14], $Z(G)$ is an elementary abelian 2-group, and thus $Z(G) \subseteq O_2(G)$. Since both K and $O_2(G)$ are normal in G and their intersection is trivial, $O_2(G)$ centralizes K . Now part (i) implies $O_2(G) \subseteq Z(P)$. Eventually as $O_2(G)$ centralizes both K and P , it centralizes G ; that is, $O_2(G) \subseteq Z(G)$. Therefore, $O_2(G) = Z(G)$.

Suppose that $Z(G) = 1$ and $1 \neq a \in Z(P)$. It is obvious that $P \subseteq C_G(a)$. As G/K is nilpotent, by Theorem 1.4, G is a relative M -group with respect to K . Let $\chi \in Irr(G)$ be nonlinear. Then χ_K cannot be irreducible since otherwise χ would vanish on some element of K , which is not the case. Now by Definition 1.1, χ is induced from some proper subgroup containing K , and thus χ is induced from a subgroup N of index 2. Since N is normal, χ vanishes on $G \setminus N$, and thus every element of $G \setminus N$ is an involution. Let $t \in G \setminus N$ and $n \in N$. Then t and tn are involutions, so $n^t n = tntn = 1$, and $n^t = n^{-1}$. Since conjugation by t defines an automorphism of N , inverting every element, it follows that N is abelian. Now if $x \in Z(P) \cap N$ both P and N are contained in $C_G(x)$. Therefore, $G = PN$ is contained in $C_G(x)$, that is, $x \in Z(G) = 1$. Therefore $Z(P) \cap N = 1$. Now suppose that $y \in K \subseteq N$. Since $a \in G \setminus N$, we have $ay \in G \setminus N$. Thus, ay is an involution and therefore $y^a = y^{-1}$. Now if $y \in C_G(a)$, then $y^a = y$; that is, $|y| = 2$, which is a contradiction as $|K|$ is odd. Therefore, $C_G(a) \subseteq P$ and eventually $C_G(a) = P$. \square

Proof of the main theorem. Now we are ready to prove the main theorem. Suppose P is a Sylow 2-subgroup of G ; then by Theorem 2.1 there is a normal 2-complement K . Now we consider two cases:

Case(1). $Z(G) = 1$. We have $G = KP$ and $K \cap P = 1$. Now by Lemma 2.1 $C_G(a) \subseteq P$ for all $a \in P$, and thus $G = K : P$ is a Frobenius group with a Frobenius complement P . Now as the irreducible characters of G that are induced from K vanish on P , we conclude that P is elementary abelian and therefore, by [2], $G \cong F_n$ for some $n \in \mathbb{N}$.

Case(2). $Z(G) \neq 1$. By Lemma 2.1 $Z(G) = O_2(G)$, but then as $G/Z(G)$ is a \mathbb{Q} -group, again using Lemma 2.1, $Z(G/Z(G)) = Z(G/O_2(G)) = O_2(G/O_2(G)) = 1$. Since every $\chi \in Irr(G/Z(G))$ is an irreducible character of G that has K in its kernel, χ vanishes only on involutions in $G/Z(G)$. Now an argument similar to Case(1) implies $G/Z(G) \cong F_n$. Therefore, $G \cong Z(G) \times F_n$ for some $n \in \mathbb{N}$ and this completes the proof. \square

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