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On certain minimal non-$\mathfrak{Y}$-groups for some classes $\mathfrak{Y}$

Ahmet ARIKAN*, **Selami ERCAN**
Department of SSME, Division of Mathematics Education, Gazi Faculty of Education, Gazi University, Ankara, Turkey

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Abstract: Let $\{\theta_n\}_{n=1}^{\infty}$ be a sequence of words. If there exists a positive integer n such that $\theta_m(G) = 1$ for every $m \geq n$, then we say that G satisfies (*) and denote the class of all groups satisfying (*) by $\mathfrak{X}_{\{\theta_n\}_{n=1}^{\infty}}$. If for every proper subgroup K of G, $K \in \mathfrak{X}_{\{\theta_n\}_{n=1}^{\infty}}$ but $G \notin \mathfrak{X}_{\{\theta_n\}_{n=1}^{\infty}}$, then we call G a minimal non-$\mathfrak{X}_{\{\theta_n\}_{n=1}^{\infty}}$-group. Assume that G is an infinite locally finite group with trivial center and $\theta_i(G) = G$ for all $i \geq 1$. In this case we mainly prove that there exists a positive integer t such that for every proper normal subgroup N of G, either $\theta_t(N) = 1$ or $\theta_t(C_G(N)) = 1$. We also give certain useful applications of the main result.

Key words: Locally finite groups, soluble groups, nilpotent groups, sequence of words, outer commutator words

1. Introduction

Let F be a free group generated by an infinite countable set X and consider the words $v(x_1, \ldots, x_n)$ for $n \geq 1$ and $x_1, \ldots, x_n \in X$. We denote by $w(v)$ the number of variables x_i in v, i.e. $w(v) = n$. We mainly refer the reader to [2] to see some properties of words.

Let $\{\theta_n\}_{n=1}^{\infty}$ be a sequence of words and G be a group. If there exists a positive integer n such that for every $m \geq n$ we have $\theta_m(G) = 1$ then we say that G satisfies (*) and denote the class of all groups satisfying (*) by $\mathfrak{X}_{\{\theta_n\}_{n=1}^{\infty}}$. See Section 3 for some examples. For a group $G \in \mathfrak{X}_{\{\theta_n\}_{n=1}^{\infty}}$, we shall use $c(G)$ to denote the least positive integer such that for every $m \geq c(G)$ we have $\theta_m(G) = 1$.

Let $\mathfrak{Y}$ be a class of groups. A group G is called a minimal non-$\mathfrak{Y}$-group, if every proper subgroup of G is in $\mathfrak{Y}$, but G itself is not. In the present paper we consider in this case $\mathfrak{Y} = \mathfrak{X}_{\{\theta_n\}_{n=1}^{\infty}}$ and minimal non-$\mathfrak{X}_{\{\theta_n\}_{n=1}^{\infty}}$-groups. Clearly, for various choices of the sequences $\{\theta_n\}_{n=1}^{\infty}$ we obtain a minimal non-$\mathfrak{X}_{\{\theta_n\}_{n=1}^{\infty}}$-group like minimal non-$\mathfrak{S}$-groups, minimal non-$\mathfrak{H}$-groups, where $\mathfrak{S}$ and $\mathfrak{H}$ are the class of all soluble and hypercentral groups respectively. Therefore, the results we proved here are general in some sense.

In [1] the authors proved useful results for certain minimal non-$\mathfrak{S}$-groups. In the present paper our main aim is to extend the results in [1] to more general contexts:

Theorem 1.1 *Let $\{\theta_n\}_{n=1}^{\infty}$ be a sequence of words and G be an infinite locally finite minimal non-$\mathfrak{X}_{\{\theta_n\}_{n=1}^{\infty}}$-group with trivial center. Assume that $\theta_i(G) = G$ for all $i \geq 1$. Then there exists a positive integer t such that*

*Correspondence: arikan@gazi.edu.tr

for every proper normal subgroup N of G, either $\theta_t(N) = 1$ or $\theta_t(C_G(N)) = 1$, i.e.

$$c(N) \leq t \text{ or } c(C_G(N)) \leq t.$$

In particular, if $c(N) > t$ for a proper normal subgroup N of G, then $c(Z(N)) \leq t$.

2. Proof of Theorem 1.1

Before we embark on the proof of Theorem 1.1, we prove the following general result.

Theorem 2.1 *Let $\{\theta_n\}_{n=1}^{\infty}$ be a sequence of words and let G be an infinite locally finite minimal non-$\mathfrak{X}_{\{\theta_n\}_{n=1}^{\infty}}$-group with trivial center. Assume that $\theta_i(G) = G$ for all $i \geq 1$ and G is not generated by finitely many proper subgroups. Then there exists a finite subgroup U of G, a positive integer t such that for every proper subgroup R of G, either $\theta_t(R) = 1$ or $\theta_t(C_G(R^U)) = 1$, i.e.*

$$c(R) \leq t \text{ or } c(C_G(R^U)) \leq t.$$

Proof Put

$$Y_i := \left\{\theta_i(y_1, \ldots, y_{w(\theta_i)}) : y_1, \ldots, y_{w(\theta_i)} \in G\right\}$$

then by hypothesis $\langle Y_i \rangle = \theta_i(G) = G$ for every $i \geq 1$.

We first show that there exists a nontrivial finite subgroup U of G, a positive integer n such that

$$\bigcap_{\theta_n(x_1,\ldots,x_{w(\theta_n)}) \in Y_n \setminus \langle 1 \rangle} \langle U, x_1, \ldots, x_{w(\theta_n)} \rangle \neq U.$$

Now assume that the assertion is false. Clearly G has elements $y_1^{(1)}, \ldots, y_1^{(w(\theta_1))} \in Y_1$ such that $\theta_1(y_1^{(1)}, \ldots, y_1^{(w(\theta_1))}) \neq 1$. Put $U_1 := \langle y_1^{(1)}, \ldots, y_1^{(w(\theta_1))} \rangle$, then

$$\bigcap_{\theta_2(x_2^{(1)},\ldots,x_2^{(w(\theta_2))}) \in Y_2 \setminus \langle 1 \rangle} \langle U_1, x_2^{(1)}, \ldots, x_2^{(w(\theta_2))} \rangle = U_1$$

by assumption. Let $a \in G \setminus U_1$, then there exist elements

$$y_2^{(1)}, \ldots, y_2^{(w(\theta_2))} \in G$$

such that

$$a \notin \langle U_1, y_2^{(1)}, \ldots, y_2^{(w(\theta_2))} \rangle \text{ and } \theta_2(y_2^{(1)}, \ldots, y_2^{(w(\theta_2))}) \neq 1.$$

Put $U_2 := \langle U_1, y_2^{(1)}, \ldots, y_2^{(w(\theta_2))} \rangle$. Now suppose that we have found elements

$$y_m^{(1)}, \ldots, y_m^{(w(\theta_m))} \in G$$

such that

$$a \notin U_m := \langle U_{m-1}, y_m^{(1)}, \ldots, y_m^{(w(\theta_m))} \rangle \text{ and } \theta_m(y_m^{(1)}, \ldots, y_m^{(w(\theta_m))}) \neq 1$$

for $m > 1$. Then again by assumption

$$\bigcap_{\theta_{m+1}(x_{m+1}^{(1)},\ldots,x_{m+1}^{(w(\theta_{m+1}))}) \in Y_{m+1} \setminus \langle 1 \rangle} \langle U_m, x_m^{(1)}, \ldots, x_{m+1}^{(w(\theta_{m+1}))} \rangle = U_m.$$

Therefore, there exist elements $y_{m+1}^{(1)},\dots,y_{m+1}^{(w(\theta_{m+1}))}\in G$ such that

$$a\notin U_{m+1}:=\langle U_m,y_{m+1}^{(1)},\dots,y_{m+1}^{(w(\theta_{m+1}))}\rangle \text{ and } \theta_{m+1}(y_{m+1}^{(1)},\dots,y_{m+1}^{(w(\theta_{m+1}))})\neq 1.$$

Now put $X:=\bigcup_{i\geq 1}U_i$; then clearly $X\neq G$, since $a\notin X$ and $\theta_n(y_n^{(1)},\dots,y_n^{(w(\theta_n))})\neq 1$ for all $n\geq 1$, i.e. $X\notin\mathfrak{X}_{\{\theta_i\}_{i=1}^{\infty}}$. This contradiction completes the proof of our first assertion. Hence there exists a nontrivial finite subgroup U of G and a positive integer n such that

$$W:=\bigcap_{\theta_n(y_1,\dots,y_{w(\theta_n)})\in Y_n\setminus\langle 1\rangle}\langle U,y_1,\dots,y_{w(\theta_n)}\rangle\neq U.$$

Let $a\in W\setminus U$, then $a\in\langle U,y_1,\dots,y_{w(\theta_n)}\rangle = U\langle U,y_1,\dots,y_{w(\theta_n)}\rangle^U$ for every $y_1,\dots,y_{w(\theta_n)}\in G$ with $\theta_n(y_1,\dots,y_{w(\theta_n)})\neq 1$.

Now put $U=\{u_1,u_2,\dots,u_r\}$ then $a=u_ic$ for some $1\leq i\leq r$ and $c\in\langle U,y_1,\dots,y_{w(\theta_n)}\rangle^U$. Also if $a=u_ic=u_id$ for $d\in\langle U,y_1',\dots,y_{w(\theta_n)}'\rangle^U$ with $\theta_n(y_1',\dots,y_{w(\theta_n)}')\neq 1$, then $c=d$. Now define

$$S_i=\left\{\theta_n(y_1,\dots,y_{w(\theta_n)}):a=u_ib,\ b\in\langle y_1,\dots,y_{w(\theta_n)}\rangle^U\right\},$$

then $Y_n\setminus\langle 1\rangle=\bigcup_{i=1}^r Si$.

If we put

$$K_i=\bigcap_{\theta_n(y_1,\dots,y_{w(\theta_n)})\in S_i}\langle y_1,\dots,y_{w(\theta_n)}\rangle^U,$$

then $K_i\neq\langle 1\rangle$ for every $i\in\{1,\dots,r\}$, since $b\in K_i$.

Let R be a proper subgroup of G with $c(R)>m:=n+1$. Since $\theta_m(R)=\theta_{n+1}(R)\neq 1$ and R satisfies (*) by hypothesis, we have that $\theta_n(R)$ is not contained in $\langle 1\rangle$. Hence there exists a nonnegative integer j such that $\theta_n(R)\cap S_j\neq\emptyset$. Let $\theta_n(y_1,\dots,y_{w(\theta_n)})\in(\theta_n(R)\cap S_j)$ such that $y_1,\dots,y_{w(\theta_n)}\in R$. It follows that

$$K_j\leq\langle y_1,\dots,y_{w(\theta_n)}\rangle^U\leq R^U$$

and hence $C_G(R^U)\leq C_G(K_j)$. Put

$$Z:=\langle C_G(R^U):R\lneqq G,\ c(R)>m\rangle,$$

then we have that

$$Z\leq\langle C_G(K_1),\dots,C_G(K_r)\rangle\neq G$$

by hypothesis. Therefore, $Z\neq G$. Now put $t:=\max\{c(C_G(K_1)),\dots,c(C_G(K_r)),m\}$. If R is a proper subgroup of G such that $c(R)>m$, then $C_G(R^U)\leq C_G(K_i)$ for some i. Hence, $c(C_G(R^U))\leq t$. If $c(R)\leq m$, then we already have that $c(R)\leq t$ and the proof is complete. □

Proof of Theorem 1.1. We argue similarly as in the proof of [1, Theorem 1.1]. First assume that $G=MN$ for some proper normal subgroups M, N of G. Then there is a positive integer s such that $\theta_s(G)\leq M$.

However, this contradicts the hypothesis and so G is not the product of finitely many proper normal subgroups. Put

$$E_i := \bigcap_{\theta_n(y_1,\ldots,y_{w(\theta_n)})\in S_i} \langle y_1,\ldots,y_{w(\theta_n)}\rangle^G$$

for every $i \in \{1,\ldots,r\}$. Since $\langle 1\rangle \neq K_i \leq E_i$ we have $E_i \neq \langle 1\rangle$ for every $1 \leq i \leq r$.

Let N be a proper normal subgroup of G with $c(N) > m := n+1$. Since $\theta_m(N) = \theta_{n+1}(N) \neq \langle 1\rangle$, $\theta_n(N)$ is not contained in $\langle 1\rangle$. Hence there exists a nonnegative integer j such that $\theta_n(N) \cap S_j \neq \emptyset$. Now let $\theta_n(y_1,\ldots,y_{w(\theta_n)}) \in (\theta_n(N) \cap S_j)$ such that $y_1,\ldots,y_{w(\theta_n)} \in N$. It follows that

$$E_j \leq \langle y_1,\ldots,y_{w(\theta_n)}\rangle^G \leq N$$

and hence $C_G(N) \leq C_G(E_j)$. Put

$$V := \langle C_G(N) : N \lhd G,\ c(N) > m\rangle,$$

then we have that

$$V \leq C_G(E_1)\ldots C_G(E_r) \neq G.$$

Therefore, $V \neq G$. Now put $t = \max\{c(C_G(E_1)),\ldots,c(C_G(E_r)),m\}$. If N is a proper normal subgroup of G such that $c(N) > m$, then $C_G(N) \leq C_G(E_i)$ for some i. Hence $c(C_G(N)) \leq t$. If $c(N) \leq m$, then we have $c(N) \leq t$ and the result follows. □

Now we can give some further useful results.

Corollary 2.2 *Let $\{\theta_n\}_{n=1}^{\infty}$ be a sequence of words and let G be an infinite locally finite minimal non-$\mathfrak{X}_{\{\theta_n\}_{n=1}^{\infty}}$-group with trivial center. Assume that $\theta_i(G) = G$ for all $i \geq 1$. If N and M are proper normal subgroups of G such that $c(N) > t$ and $c(M) > t$, then $[N,M] \neq \langle 1\rangle$ and thus $N \cap M \neq \langle 1\rangle$*

Proof If $[N,M] = \langle 1\rangle$, then $N \leq C_G(M)$. However, this is a contradiction by Theorem 2.1. In particular, $N \cap M \neq \langle 1\rangle$. □

Corollary 2.3 *Let $\{\theta_n\}_{n=1}^{\infty}$ be the sequence of words and let G be an infinite locally finite minimal non-$\mathfrak{X}_{\{\theta_n\}_{n=1}^{\infty}}$-group with trivial center. Assume that $\theta_i(G) = G$ for all $i \geq 1$. If*

$$\langle C_G(N) : N \lhd G\rangle = G,$$

then there is a positive integer t such that

$$\langle C_G(N) : N \lhd G,\ c(N) \leq t\rangle = G.$$

Proof We have

$$G = \langle C_G(N) : N \lhd G,\ c(N) \leq t\rangle\langle C_G(N) : N \lhd G,\ c(N) > t\rangle$$

By Theorem 2.1 we follow the result. □

Corollary 2.4 *Let $\{\theta_n\}_{n=1}^{\infty}$ be a sequence of words and let G be an infinite locally finite minimal non-$\mathfrak{X}_{\{\theta_n\}_{n=1}^{\infty}}$-group with trivial center. Assume that $\theta_i(G) = G$ for all $i \geq 1$. If for every proper normal subgroup N of G, $C_G(N) \neq \langle 1 \rangle$, then there is a positive integer t such that*

$$\langle C_G(N) : N \triangleleft G,\ c(N) \leq t \rangle = G.$$

Proof We have $N \leq \langle C_G(\langle x^G \rangle)$ for some $x \in C_G(N)$. This implies

$$\langle C_G(N) : N \triangleleft G \rangle = G.$$

We follow the result by Corollary 2.3. □

3. Certain applications of Theorem 1.1

If $u = u(x_1, \ldots, x_s)$ and $v = v(x_1, \ldots, x_t)$ are two words in F, then the *composite* of u and v, $u \circ v$, is defined as follows (see [3]):

$$u \circ v = u(v(x_1, \ldots, x_t), \ldots, v(x_{(s-1)t+1}, \ldots, x_{st})).$$

Let $\{\omega_n\}_{n=1}^{\infty}$ be a sequence of words. Define $\theta_1 = \omega_1$ and $\theta_i = \omega_i \circ \theta_{i-1}$ for $i \geq 2$, and let G be a group such that $\theta_r(G) = \langle 1 \rangle$ for some positive integer r. Then $\theta_s(G) = \langle 1 \rangle$ for every positive integer $s \geq r$ and thus G satisfies (*).

Clearly if $\delta_n = \underbrace{\gamma_2 \circ \cdots \circ \gamma_2}_{n \text{ times}}$ for $n \geq 0$, where γ_2 is the nilpotent word of two variables (i.e. $\gamma_2(x, y) = [x, y]$) and $\delta_0(x) = x$, then a group G is soluble of derived length at most $k \geq 1$ if and only if $\delta_k(G) = 1$. Hence $\mathfrak{S} \leq \mathfrak{X}_{\{\delta_n\}_{n=1}^{\infty}}$, where $\mathfrak{S}$ is the class of all soluble groups. We also have that the composite of some nilpotent words is called a polynilpotent word, i.e.

$$\gamma_{c_t+1,\ldots,c_1+1} = \gamma_{c_t+1} \circ \cdots \circ \gamma_{c_1+1},$$

where γ_{c_i+1} $(1 \leq i \leq t)$ is a nilpotent word in distinct variables. Then $\mathfrak{P} \leq \mathfrak{X}_{\{\gamma_{c_i+1}\}_{i=1}^{\infty}}$, where $\mathfrak{P}$ denotes the class of all polynilpotent groups. Therefore, our results shall cover a large number of classes of groups.

Corollary 3.1 *Let G be a locally finite group of infinite exponent with trivial center. Let us define $\omega_i(x) = x^{k_i}$ for some $k_i \geq 2$ and for all $i \geq 1$ and assume that G is a minimal non-$\mathfrak{X}_{\{\theta_i\}_{i=1}^{\infty}}$-group, where $\theta_1 = \omega_1$ and $\theta_i = \omega_i \circ \theta_{i-1}$ for $i > 1$. Then there exists a positive integer r such that either $N^r = 1$ or $C_G(N)^r = 1$, i.e. $exp(N) \leq r$ or $exp(C_G(N)) \leq r$ for every proper normal subgroup N of G.*

Proof Assume that $G^n \neq G$ for some positive integer $n \geq 2$. Hence $\theta_m(G^n) = 1$ for some positive integer m. Then G has a finite exponent, a contradiction, and so $\theta_i(G) = G$ for all $i \geq 1$. We also have that there exists a positive integer t such that $\theta_t(N) = 1$ or $\theta_t(C_G(N)) = 1$. Put $r := k_1 \ldots k_t$, then $N^r = 1$ or $C_G(N)^r = 1$, i.e. $\exp(N) \leq r$ or $\exp(C_G(N)) \leq r$ by Theorem 1.1, as desired. □

Corollary 3.2 *Let $\{\gamma_{c_i+1}\}_{i=1}^{\infty}$ be a sequence of nilpotent words and let*

$$\theta_i = \gamma_{c_i+1} \circ \cdots \circ \gamma_{c_1+1}$$

be a sequence of polynilpotent words. If G is a perfect infinite locally finite minimal non-$\{\theta_i\}_{i=1}^{\infty}$-group with trivial center, then there exists a positive integer t such that either

$$\theta_t(N) = (\gamma_{c_t+1} \circ \cdots \circ \gamma_{c_1+1})(N) = 1 \ \textit{or} \ \theta_i(C_G(N)) = (\gamma_{c_t+1} \circ \cdots \circ \gamma_{c_1+1})(C_G(N)) = 1$$

for every proper normal subgroup N of G.

Proof Since G is perfect, we have that $\theta_i(G) = G$ for all $i \geq 1$ and the result follows by Theorem 1.1. □

Let us define the k-Engel word $\epsilon_k(x, y) = [x,_k y]$ for every $k \geq 1$ and $\varepsilon_r = \epsilon_{k_r} \circ \cdots \circ \epsilon_{k_1}$ for every $r \geq 1$ and for some $k_1, \ldots, k_r \geq 1$.

Corollary 3.3 *Let G be an infinite locally finite minimal non-$\{\varepsilon_i\}_{i=1}^{\infty}$-group with trivial center. If $\varepsilon_i(G) = G$ for all $i \geq 1$, then there exists a positive integer t such that either*

$$\varepsilon_t(N) = 1 \ \textit{or} \ \varepsilon_t(C_G(N)) = 1$$

for every proper normal subgroup N of G.

Proof The result follows by Theorem 1.1. □

References

[1] Arıkan A, Sezer S, Smith H. On locally finite minimal non-solvable groups. Cent Eur J Math 2010; 8: 266–273.

[2] Neumann H. Varieties of Groups. New York, NY, USA: Springer-Verlag, 1967.

[3] Moghaddam MRR. On the Schur-Baer Property. J Austral Math Soc (Series A) 1981; 31: 343–361.