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The prime tournaments T with $|W_5(T)| = |T| - 2$

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Abstract: We consider a tournament $T = (V, A)$. For $X \subseteq V$, the subtournament of T induced by X is $T[X] = (X, A \cap (X \times X))$. A module of T is a subset X of V such that for $a, b \in X$ and $x \in V \setminus X$, $(a, x) \in A$ if and only if $(b, x) \in A$. The trivial modules of T are $\emptyset$, $\{x\}(x \in V)$, and V. A tournament is prime if all its modules are trivial. For $n \geq 2$, W_{2n+1} denotes the unique prime tournament defined on $\{0, \ldots, 2n\}$ such that $W_{2n+1}[\{0, \ldots, 2n-1\}]$ is the usual total order. Given a prime tournament T, $W_5(T)$ denotes the set of $v \in V$ such that there is $W \subseteq V$ satisfying $v \in W$ and $T[W]$ is isomorphic to W_5. B.J. Latka characterized the prime tournaments T such that $W_5(T) = \emptyset$. The authors proved that if $W_5(T) \neq \emptyset$, then $|W_5(T)| \geq |V| - 2$. In this article, we characterize the prime tournaments T such that $|W_5(T)| = |V| - 2$.

Key words: Tournament, prime, embedding, critical, partially critical

1. Introduction

1.1. Preliminaries

A *tournament* $T = (V(T), A(T))$ (or (V, A)) consists of a finite set V of *vertices* together with a set A of ordered pairs of distinct vertices, called *arcs*, such that for all $x \neq y \in V$, $(x, y) \in A$ if and only if $(y, x) \notin A$. The *cardinality* of T, denoted by $|T|$, is that of $V(T)$. Given a tournament $T = (V, A)$, with each subset X of V is associated the *subtournament* $T[X] = (X, A \cap (X \times X))$ of T induced by X. For $X \subseteq V$ (resp. $x \in V$), the subtournament $T[V \setminus X]$ (resp. $T[V \setminus \{x\}]$) is denoted by $T - X$ (resp. $T - x$). Two tournaments $T = (V, A)$ and $T' = (V', A')$ are *isomorphic*, which is denoted by $T \simeq T'$, if there exists an *isomorphism* from T onto T', i.e. a bijection f from V onto V' such that for all x, $y \in V$, $(x, y) \in A$ if and only if $(f(x), f(y)) \in A'$. We say that a tournament T' *embeds* into T if T' is isomorphic to a subtournament of T. Otherwise, we say that T *omits* T'. The tournament T is said to be *transitive* if it omits the tournament $C_3 = (\{0, 1, 2\}, \{(0, 1), (1, 2), (2, 0)\})$. For a finite subset V of $\mathbb{N}$, we denote by $\overrightarrow{V}$ the usual *total order* defined on V, i.e., the transitive tournament $(V, \{(i, j) : i < j\})$.

Some notations are needed. Let $T = (V, A)$ be a tournament. For two vertices $x \neq y \in V$, the notation $x \longrightarrow y$ signifies that $(x, y) \in A$. Similarly, given $x \in V$ and $Y \subseteq V$, the notation $x \longrightarrow Y$ (resp. $Y \longrightarrow x$) means that $x \longrightarrow y$ (resp. $y \longrightarrow x$) for all $y \in Y$. Given $x \in V$, we set $N_T^+(x) = \{y \in V : x \longrightarrow y\}$. For all

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$n \in \mathbb{N} \setminus \{0\}$, the set $\{0, \ldots, n-1\}$ is denoted by $\mathbb{N}_n$.

Let $T = (V, A)$ be a tournament. A subset I of V is a *module* [11] (or a *clan* [7]) of T provided that for all $x \in V \setminus I$, $x \longrightarrow I$ or $I \longrightarrow x$. For example, $\emptyset$, $\{x\}$, where $x \in V$, and V are modules of T, called *trivial* modules. A tournament is *prime* [4] (or *primitive* [7]) if all its modules are trivial. Notice that a tournament $T = (V, A)$ and its *dual* $T^\star = (V, \{(x, y) : (y, x) \in A\})$ share the same modules. Hence, T is prime if and only if $T^\star$ is.

For $n \geq 2$, we introduce the tournament W_{2n+1} defined on $\mathbb{N}_{2n+1}$ as follows: $W_{2n+1}[\mathbb{N}_{2n}] = \overrightarrow{\mathbb{N}_{2n}}$ and $N^+_{W_{2n+1}}(2n) = \{2i : i \in \mathbb{N}_n\}$ (see Figure 1). In 2003, B.J. Latka [8] characterized the prime tournaments omitting the tournament W_5. In 2012, the authors were interested in the set $W_5(T)$ of the vertices x of a prime tournament $T = (V, A)$ for which there exists a subset X of V such that $x \in X$ and $T[X] \simeq W_5$. They obtained the following.

Theorem 1 ([1]) *Let T be a prime tournament into which W_5 embeds. Then $| W_5(T) | \geq | T | - 2$. If, in addition, $|T|$ is even, then $|W_5(T)| \geq |T| - 1$.*

Our main result in this paper, presented in [3] without detailed proof, gives a characterization of the class $\mathcal{T}$ of the prime tournaments T on at least 3 vertices such that $|W_5(T)| = |T| - 2$. This answers [1, Problem 4.4].

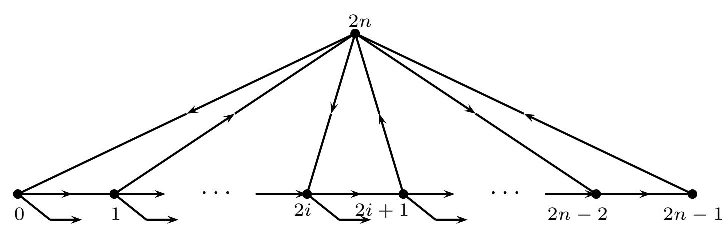

Figure 1. W_{2n+1}

1.2. Partially critical tournaments and the class $\mathcal{T}$

Our characterization of the tournaments of the class $\mathcal{T}$ requires the study of their partial criticality structure, a notion introduced as a weakening of the notion of criticality defined in Section 2. These notions are defined in terms of critical vertices. A vertex x of a prime tournament T is *critical* [10] if $T - x$ is not prime. The set of noncritical vertices of a prime tournament T was introduced in [9]. It is called the *support* of T and is denoted by $\sigma(T)$. Let T be a prime tournament and let X be a subset of $V(T)$ such that $| X | \geq 3$; we say that T is *partially critical according to* $T[X]$ (or $T[X]$*-critical*) [6] if $T[X]$ is prime and if $\sigma(T) \subseteq X$. We will see that: *for* $T \in \mathcal{T}$, $V(T) \setminus W_5(T) = \sigma(T)$. Partially critical tournaments are characterized by M.Y. Sayar in [9]. In order to recall this characterization, we first introduce the tools used to this end. Given a tournament $T = (V, A)$, with each subset X of V, such that $| X | \geq 3$ and $T[X]$ is prime, are associated the following subsets of $V \setminus X$:

- $\langle X \rangle = \{x \in V \setminus X : x \longrightarrow X \text{ or } X \longrightarrow x\}$.
- For all $u \in X$, $X(u) = \{x \in V \setminus X : \{u, x\} \text{ is a module of } T[X \cup \{x\}]\}$.
- $\mathrm{Ext}(X) = \{x \in V \setminus X : T[X \cup \{x\}] \text{ is prime}\}$.

The family $\{X(u) : u \in X\} \cup \{\text{Ext}(X), \langle X\rangle\}$ is denoted by p_X^T.

Lemma 1 ([7]) *Let $T = (V, A)$ be a tournament and let X be a subset of V such that $|X| \geq 3$ and $T[X]$ is prime. The nonempty elements of p_X^T constitute a partition of $V \setminus X$ and satisfy the following assertions:*

- *For $u \in X$, $x \in X(u)$, and $y \in V \setminus (X \cup X(u))$, if $T[X \cup \{x, y\}]$ is not prime, then $\{u, x\}$ is a module of $T[X \cup \{x, y\}]$.*
- *For $x \in \langle X\rangle$ and $y \in V \setminus (X \cup \langle X\rangle)$, if $T[X \cup \{x, y\}]$ is not prime, then $X \cup \{y\}$ is a module of $T[X \cup \{x, y\}]$.*
- *For $x \neq y \in Ext(X)$, if $T[X \cup \{x, y\}]$ is not prime, then $\{x, y\}$ is a module of $T[X \cup \{x, y\}]$.*

Furthermore, $\langle X\rangle$ is divided into $X^- = \{x \in \langle X\rangle : x \longrightarrow X\}$ and $X^+ = \{x \in \langle X\rangle : X \longrightarrow x\}$. Similarly, for all $u \in X$, $X(u)$ is divided into $X^-(u) = \{x \in X(u) : x \longrightarrow u\}$ and $X^+(u) = \{x \in X(u) : u \longrightarrow x\}$. We then introduce the family $q_X^T = \{\text{Ext}(X), X^-, X^+\} \cup \{X^-(u) : u \in X\} \cup \{X^+(u) : u \in X\}$.

A *graph* $G = (V(G), E(G))$ (or (V, E)) consists of a finite set V of vertices together with a set E of unordered pairs of distinct vertices, called *edges*. Given a vertex x of a graph $G = (V, E)$, the set $\{y \in V, \{x, y\} \in E\}$ is denoted by $N_G(x)$. With each subset X of V is associated the *subgraph* $G[X] = (X, E \cap \binom{X}{2})$ of G induced by X. An isomorphism from a graph $G = (V, E)$ onto a graph $G' = (V', E')$ is a bijection f from V onto V' such that for all $x, y \in V$, $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$. We now introduce the graph G_{2n} defined on $\mathbb{N}_{2n}$, where $n \geq 1$, as follows. For all $x, y \in \mathbb{N}_{2n}$, $\{x, y\} \in E(G_{2n})$ if and only if $|y - x| \geq n$ (see Figure 2).

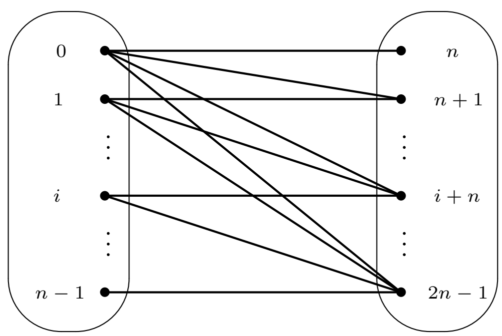

Figure 2. G_{2n}

A graph G is *connected* if for all $x \neq y \in V(G)$, there is a sequence $x_0 = x, \ldots, x_m = y$ of vertices of G such that for all $i \in \mathbb{N}_m$, $\{x_i, x_{i+1}\} \in E(G)$. For example, the graph G_{2n} is connected. A *connected component* of a graph G is a maximal subset X of $V(G)$ (with respect to inclusion) such that $G[X]$ is connected. The set of the connected components of G is a partition of $V(G)$, denoted by $\mathcal{C}(G)$. Let $T = (V, A)$ be a prime tournament. With each subset X of V such that $|X| \geq 3$ and $T[X]$ is prime, is associated its *outside* graph G_X^T defined by $V(G_X^T) = V \setminus X$ and $E(G_X^T) = \{\{x, y\} \in \binom{V \setminus X}{2} : T[X \cup \{x, y\}] \text{ is prime}\}$. We now present the characterization of partially critical tournaments.

Theorem 2 ([9]) *Consider a tournament $T = (V, A)$ with a subset X of V such that $|X| \geq 3$ and $T[X]$ is prime. The tournament T is $T[X]$-critical if and only if the assertions below hold.*

1. *Ext*$(X) = \emptyset$.

2. *For all* $u \in X$, *the tournaments* $T[X(u) \cup \{u\}]$ *and* $T[\langle X \rangle \cup \{u\}]$ *are transitive.*

3. *For each* $Q \in \mathcal{C}(G_X^T)$, *there is an isomorphism* f *from* G_{2n} *onto* $G_X^T[Q]$ *such that* Q_1, $Q_2 \in q_X^T$, *where* $Q_1 = f(\mathbb{N}_n)$ *and* $Q_2 = f(\mathbb{N}_{2n} \setminus \mathbb{N}_n)$. *Moreover, for all* $x \in Q_i$ $(i = 1$ *or* $2)$, $|N_{G_X^T}(x)| = |N^+_{T[Q_i]}(x)| + 1$ (*resp.* $n - |N^+_{T[Q_i]}(x)|$) *if* $Q_i = X^+$ *or* $X^-(u)$ (*resp.* $Q_i = X^-$ *or* $X^+(u)$), *where* $u \in X$.

The next corollary follows from Theorem 2 and Lemma 1.

Corollary 1 *Let* T *be a* $T[X]$*-critical tournament,* T *is entirely determined up to isomorphy by giving* $T[X]$, q_X^T *and* $\mathcal{C}(G_X^T)$. *Moreover, the tournament* T *is exactly determined by giving, in addition, either the graphs* $G_X^T[Q]$ *for any* $Q \in \mathcal{C}(G_X^T)$, *or the transitive tournaments* $T[Y]$ *for any* $Y \in q_X^T$.

We underline the importance of Theorem 2 and Corollary 1 in our description of the tournaments of the class $\mathcal{T}$. Indeed, these tournaments are introduced up to isomorphy as C_3-critical tournaments T defined by giving $\mathcal{C}(G^T_{\mathbb{N}_3})$ in terms of the nonempty elements of $q^T_{\mathbb{N}_3}$. Figure 3 illustrates a tournament obtained from such information. We refer to [10, Discussion] for more details about this purpose.

We now introduce the class $\mathcal{H}$ (resp. $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$, $\mathcal{L}$) of the C_3-critical tournaments H (resp. I, J, K, L) such that:

- $\mathcal{C}(G^H_{\mathbb{N}_3}) = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^-, \mathbb{N}_3^+ \cup \mathbb{N}_3^-(1)\}$ (see Figure 3);
- $\mathcal{C}(G^I_{\mathbb{N}_3}) = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^+(2), \mathbb{N}_3^+(1) \cup \mathbb{N}_3^-(0)\}$;
- $\mathcal{C}(G^J_{\mathbb{N}_3}) = \{\mathbb{N}_3^+(1) \cup \mathbb{N}_3^-, \mathbb{N}_3^-(1) \cup \mathbb{N}_3^-(0)\}$;
- $\mathcal{C}(G^K_{\mathbb{N}_3}) = \{\mathbb{N}_3^+(1) \cup \mathbb{N}_3^-, \mathbb{N}_3^+(0) \cup \mathbb{N}_3^-(2)\}$;
- $\mathcal{C}(G^L_{\mathbb{N}_3}) = \{\mathbb{N}_3^+(1) \cup \mathbb{N}_3^-, \mathbb{N}_3^+(0) \cup \mathbb{N}_3^-(2), \mathbb{N}_3^+ \cup \mathbb{N}_3^-(0)\}$.

Notice that for $\mathcal{X} = \mathcal{H}$, $\mathcal{I}$, $\mathcal{J}$ or $\mathcal{K}$, $\{|V(T)|: T \in \mathcal{X}\} = \{2n+1 : n \geq 3\}$ and $\{|V(T)|: T \in \mathcal{L}\} = \{2n+1 : n \geq 4\}$. We denote by $\mathcal{H}^\star$ (resp. $\mathcal{I}^\star$, $\mathcal{J}^\star$, $\mathcal{K}^\star$, $\mathcal{L}^\star$) the class of the tournaments $T^\star$, where $T \in \mathcal{H}$ (resp. $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$, $\mathcal{L}$).

Remark 1 *We have* $\mathcal{H}^\star = \mathcal{H}$ *and* $\mathcal{I}^\star = \mathcal{I}$.

Proof Let $T \in \mathcal{H}$. The permutation f of $V(T)$ defined by $f(1) = 0$, $f(0) = 1$, and $f(v) = v$ for all $v \in V(T) \setminus \{0, 1\}$ is an isomorphism from $T^\star$ onto a tournament T' of the class $\mathcal{H}$. Let now $T \in \mathcal{I}$ and let x be the unique vertex of $\mathbb{N}_3^+(2)$ such that $|N^+_{T[\mathbb{N}_3^+(2)]}(x)| = 0$. The permutation g of $V(T)$ defined by $g(1) = 0$, $g(0) = 1$, $g(x) = 2$, $g(2) = x$, and $g(v) = v$ for $v \in V(T) \setminus \{0, 1, 2, x\}$ is an isomorphism from $T^\star$ onto a tournament T' of the class $\mathcal{I}$. □

By setting $\mathcal{M} = \mathcal{H} \cup \mathcal{I} \cup \mathcal{J} \cup \mathcal{J}^\star \cup \mathcal{K} \cup \mathcal{K}^\star \cup \mathcal{L} \cup \mathcal{L}^\star$, we state our main result as follows.

Theorem 3 *Up to isomorphy, the tournaments of the class* $\mathcal{T}$ *are those of the class* $\mathcal{M}$. *Moreover, for all* $T \in \mathcal{M}$, *we have* $V(T) \setminus W_5(T) = \sigma(T) = \{0, 1\}$.

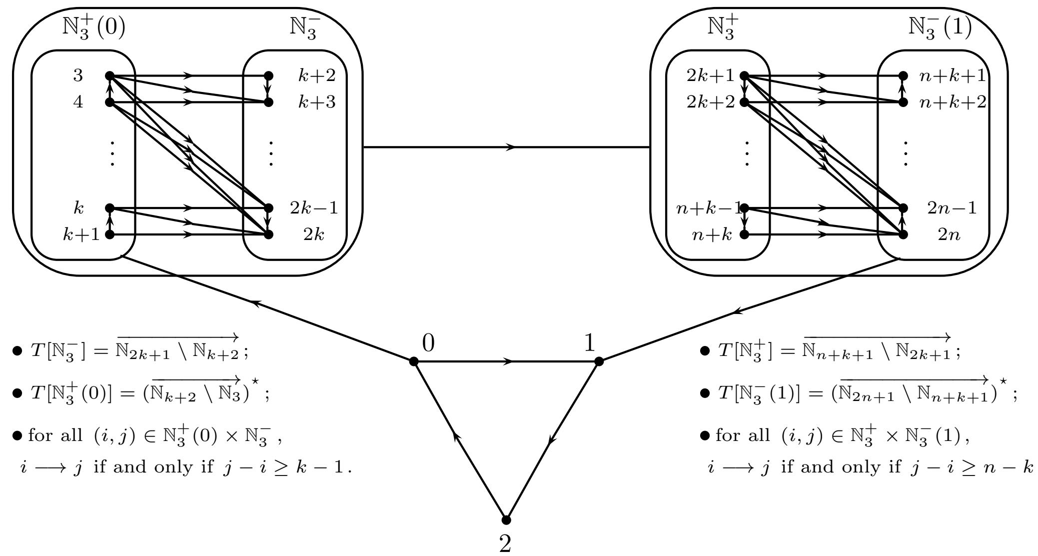

Figure 3. A tournament T of the class $\mathcal{H}$

2. Critical tournaments and tournaments omitting W_5

We begin by recalling the characterization of the critical tournaments and some of their properties. A prime tournament $T = (V, A)$, with $|T| \geq 3$, is *critical* if $\sigma(T) = \emptyset$, i.e. if all its vertices are critical. In order to present the critical tournaments, characterized by J.H. Schmerl and W.T. Trotter in [10], we introduce the tournaments T_{2n+1} and U_{2n+1} defined on $\mathbb{N}_{2n+1}$, where $n \geq 2$, as follows:

- $A(T_{2n+1}) = \{(i,j) : j - i \in \{1, \ldots, n\} \text{ mod. } 2n+1\}$ (see Figure 4).
- $A(T_{2n+1}) \setminus A(U_{2n+1}) = A(T_{2n+1}[\{n+1, \ldots, 2n\}])$ (see Figure 5).

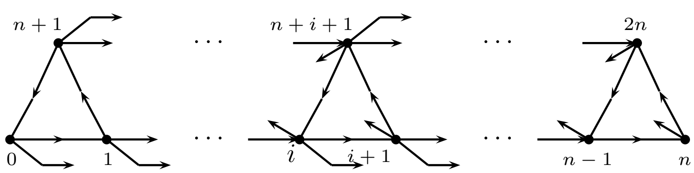

Figure 4. T_{2n+1}

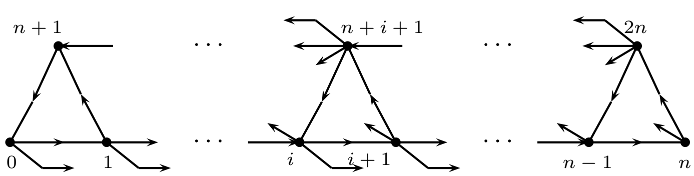

Figure 5. U_{2n+1}

Theorem 4 ([10]) *Up to isomorphy, T_{2n+1}, U_{2n+1}, and W_{2n+1}, where $n \geq 2$, are the only critical tournaments.*

Notice that a critical tournament is isomorphic to its dual. Moreover, as a tournament on 4 vertices is not prime, we have:

Fact 1 *Up to isomorphy, T_5, U_5, and W_5 are the only prime tournaments on 5 vertices.*

As mentioned in [2], the next remark follows from the definition of the critical tournaments.

Remark 2 *Up to isomorphy, the prime subtournaments on at least 5 vertices of T_{2n+1} (resp. U_{2n+1}, W_{2n+1}), where $n \geq 2$, are the tournaments T_{2m+1} (resp. U_{2m+1}, W_{2m+1}), where $2 \leq m \leq n$.*

To recall the characterization of the prime tournaments omitting W_5, we introduce the *Paley* tournament P_7 defined on $\mathbb{N}_7$ by $A(P_7) = \{(i,j) : j-i \in \{1,2,4\} \text{ mod. } 7\}$. Notice that for all $x \neq y \in \mathbb{N}_7$, $P_7 - x \simeq P_7 - y$, and let $B_6 = P_7 - 6$.

Theorem 5 ([8]) *Up to isomorphy, the prime tournaments on at least 5 vertices and omitting W_5 are the tournaments B_6, P_7, T_{2n+1}, and U_{2n+1}, where $n \geq 2$.*

3. Some useful configurations

In this section, we introduce a number of configurations that occur in the proof of Theorem 3. These configurations involve mainly partially critical tournaments. We begin with the two following lemmas obtained in [2].

Lemma 2 ([2]) *If B_6 embeds into a prime tournament T on 7 vertices and if $T \not\simeq P_7$, then $|W_5(T)| = 7$.*

Lemma 3 ([2]) *Let T be a U_5-critical tournament on 7 vertices. If $T \not\simeq U_7$, then $W_5(T) \cap \{3,4\} \neq \emptyset$.*

Lemma 4 specifies the C_3-critical tournaments with a connected outside graph. It follows from the examination of the different possible configurations obtained by using Theorem 2.

Lemma 4 *Given a C_3-critical tournament T on at least 5 vertices, if $G^T_{\mathbb{N}_3}$ is connected, then T is critical. More precisely, the different configurations are as follows where $i \in \mathbb{N}_3$ and $i+1$ is considered modulo 3.*

1. *If $\mathcal{C}(G^T_{\mathbb{N}_3}) = \{\mathbb{N}_3^-(i) \cup \mathbb{N}_3^+(i+1)\}$, then $T \simeq T_{2n+1}$ for some $n \geq 2$.*
2. *If $\mathcal{C}(G^T_{\mathbb{N}_3}) = \{\mathbb{N}_3^- \cup \mathbb{N}_3^+(i)\}$, $\{\mathbb{N}_3^+ \cup \mathbb{N}_3^-(i)\}$, $\{\mathbb{N}_3^+(i) \cup \mathbb{N}_3^+(i+1)\}$, or $\{\mathbb{N}_3^-(i) \cup \mathbb{N}_3^-(i+1)\}$, then $T \simeq U_{2n+1}$ for some $n \geq 2$.*
3. *If $\mathcal{C}(G^T_{\mathbb{N}_3}) = \{\mathbb{N}_3^- \cup \mathbb{N}_3^-(i)\}$, $\{\mathbb{N}_3^+ \cup \mathbb{N}_3^+(i)\}$, or $\{\mathbb{N}_3^+(i) \cup \mathbb{N}_3^-(i+1)\}$, then $T \simeq W_{2n+1}$ for some $n \geq 2$.*

For a transitive tournament T, recall that $\min T$ denotes its smallest element and $\max T$ its largest.

Lemma 5 *Given a C_3-critical tournament T on at least 5 vertices, if $T[\mathbb{N}_3 \cup e] \simeq T_5$ for all $e \in E(G^T_{\mathbb{N}_3})$, then $T \simeq T_{2n+1}$ for some $n \geq 2$.*

Proof Let T be a C_3-critical tournament on at least 5 vertices such that for all $e \in E(G^T_{\mathbb{N}_3})$, $T[\mathbb{N}_3 \cup e] \simeq T_5$. Given $e \in E(G^T_{\mathbb{N}_3})$, by using Lemma 4 and Remark 2, $e = \{v, v'\}$, where $v \in \mathbb{N}_3^-(i)$, $v' \in \mathbb{N}_3^+(i+1)$, $i \in \mathbb{N}_3$ and $i+1$ is considered modulo 3. Then, by Theorem 2, the connected components of T are the nonempty elements of the family $\{\mathbb{N}_3^-(j) \cup \mathbb{N}_3^+(j+1)\}_{j \in \mathbb{N}_3}$, where $j+1$ is considered modulo 3. The tournament T is critical. Indeed, by using Theorem 2, for each $k \in \mathbb{N}_3$, $\{\max T[\mathbb{N}_3^+(k+1) \cup \{k+1\}], \min T[\mathbb{N}_3^-(k+2) \cup \{k+2\}]\}$, where

$k+1$ and $k+2$ are considered modulo 3, is a nontrivial module of $T-k$. It follows that $T \simeq T_{2n+1}$ for some $n \geq 2$ by Remark 2. □

Lemma 6 *Given a U_5-critical tournament, if $T[\mathbb{N}_5 \cup e] \simeq U_7$ for all $e \in E(G^T_{\mathbb{N}_5})$, then $T \simeq U_{2n+1}$ for some $n \geq 2$.*

Proof The subsets X of $\mathbb{N}_7$ such that $U_7[X] \simeq U_5$ are the sets $\mathbb{N}_7 \setminus \{i,j\}$, where $\{i,j\} = \{0,4\}$, $\{4,1\}$, $\{1,5\}$, $\{5,2\}$, $\{2,6\}$, or $\{6,3\}$. By observing $q^{U_7}_X$ for such subsets X and by Theorem 2, we deduce that the elements of $\mathcal{C}(G^T_{\mathbb{N}_5})$ are the nonempty elements among the following six sets: $\mathbb{N}^+_5 \cup \mathbb{N}^-_5(0)$, $\mathbb{N}^+_5(0) \cup \mathbb{N}^+_5(3)$, $\mathbb{N}^-_5(1) \cup \mathbb{N}^-_5(3)$, $\mathbb{N}^+_5(1) \cup \mathbb{N}^+_5(4)$, $\mathbb{N}^-_5(2) \cup \mathbb{N}^-_5(4)$, and $\mathbb{N}^-_5 \cup \mathbb{N}^+_5(2)$. Suppose first that $|\,\mathcal{C}(G^T_{\mathbb{N}_5})\,| = 6$. The tournament T is critical. Indeed, by using Theorem 2, $\{\min T[\mathbb{N}^+_5], \max T[\mathbb{N}^+_5(3)]\}$ (resp. $\{\min T[\mathbb{N}^-_5(3)], \max T[\mathbb{N}^+_5(4)]\}$, $\{\min T[\mathbb{N}^-_5(4)], \max T[\mathbb{N}^-_5]\}$, $\{\min T[\mathbb{N}^-_5(1)], \max T[\mathbb{N}^+_5(0)]\}$, $\{\min T[\mathbb{N}^-_5(2)], \max T[\mathbb{N}^+_5(1)]\}$) is a nontrivial module of $T-0$ (resp. $T-1$, $T-2$, $T-3$, $T-4$). By Remark 2, $T \simeq U_{2n+1}$ for some $n \geq 8$. Suppose now that $|\,\mathcal{C}(G^T_{\mathbb{N}_5})\,| \leq 5$. Then T embeds into a U_5-critical tournament T' with $|\,\mathcal{C}(G^{T'}_{\mathbb{N}_5})\,| = 6$. By the first case, $T' \simeq U_{2n+1}$ for some $n \geq 8$ and thus $T \simeq U_{2n+1}$ for some $n \geq 2$ by Remark 2. □

Lemma 7 *Let $T = (V,A)$ be a $T[X]$-critical tournament with $|\,V \setminus X\,| \geq 2$, let $Q = \mathbb{N}_{2n}$ be a connected component of G^T_X such that $G^T_X[Q] = G_{2n}$, and let $e = \{i, i+n\}$, where $i \in \mathbb{N}_n$. Then the tournament $T-e$ is $T[X]$-critical. Moreover, Q is included in any subset Z of V such that $T[Z] \simeq W_5$ and $Z \cap (V \setminus (Q \cup W_5(T-e)) \neq \emptyset$.*

Proof For $n \geq 2$, the function

$$f_i : \quad Q \setminus e \longrightarrow \mathbb{N}_{2n-2}$$

$$k \quad \longmapsto \begin{cases} k & \text{if } 0 \leq k \leq i-1 \\ k-1 & \text{if } i+1 \leq k \leq n+i-1 \\ k-2 & \text{if } n+i+1 \leq k \leq 2n-1, \end{cases}$$

is an isomorphism from $G_{2n}-e$ onto G_{2n-2}. It follows from Theorem 2 that $T-e$ is $T[X]$-critical. Now suppose that there is $Z \subseteq V$ such that $T[Z] \simeq W_5$ and $Z \cap (V \setminus (Q \cup W_5(T-e)) \neq \emptyset$. Therefore, we have $|\,Z \cap e\,| = 1$ or $e \subset Z$. Suppose for a contradiction that $|\,Z \cap e\,| = 1$, and set $\{z\} = Z \cap e$. As $\text{Ext}(V \setminus e) = \emptyset$, then by Lemma 1, either $z \in \langle V' \rangle$ or $z \in V'(u)$, where $V' = V \setminus e$ and $u \in V'$. If $z \in \langle V' \rangle$, then $Z \setminus \{z\}$ is a nontrivial module of $T[Z]$, a contradiction. If $z \in V'(u)$, then $u \notin Z$, otherwise $\{u, z\}$ is a nontrivial module of $T[Z]$. Thus, $T[Z'] \simeq W_5$, where $Z' = (Z \setminus \{z\}) \cup \{u\} \subset V \setminus e$. A contradiction because $Z' \cap (V \setminus W_5(T-e)) \neq \emptyset$. Finally, for all $e' \in \{\{j, j+n\} : j \in \mathbb{N}_n\}$, the bijection f from $V \setminus e$ onto $V \setminus e'$, defined by $f_{|V \setminus Q} = \text{Id}_{V \setminus Q}$ and $f_{|Q \setminus e}$ $= f_j^{-1} \circ f_i$, is an isomorphism from $T-e$ onto $T-e'$. It follows that $V \setminus (Q \cup W_5(T-e')) = V \setminus (Q \cup W_5(T-e))$. Thus, as proved above, $e' \subset Z$, so that $Q \subset Z$. □

4. Proof of Theorem 3

We begin by establishing the partial criticality structure of the tournaments of the class $\mathcal{T}$. For this purpose, we use the notion of minimal tournaments for two vertices. Given a prime tournament $T=(V,A)$ of cardinality ≥ 3 and two distinct vertices $x \neq y \in V$, T is said to be *minimal* for $\{x,y\}$ (or $\{x,y\}$*-minimal*) when for all proper subset X of V, if $\{x,y\} \subset X$ ($|X| \geq 3$), then $T[X]$ is not prime. These tournaments were introduced and characterized by A. Cournier and P. Ille in [5]. From this characterization, the following fact, observed in [1], is obtained by a simple and quick verification.

Fact 2 ([1, 5]) *Up to isomorphy, the tournaments C_3 and U_5 are the unique minimal tournaments for two vertices T such that $|W_5(T)| \leq |T|-2$. Moreover, $\{3,4\}$ is the unique unordered pair of vertices for which U_5 is minimal.*

Proposition 1 *Let $T=(V,A)$ be a tournament of the class $\mathcal{T}$. Then the vertices of $W_5(T)$ are critical and there exists $z \in W_5(T)$ such that $T[(V \setminus W_5(T)) \cup \{z\}] \simeq C_3$. In particular, T is $T[(V \setminus W_5(T)) \cup \{z\}]$-critical.*

Proof By Theorem 1, $|T|$ is odd and ≥ 7. First, suppose by contradiction that there is $\alpha \in W_5(T)$ such that $T-\alpha$ is prime. Since $|T-\alpha|$ is even and ≥ 6 with $|V(T-\alpha) \setminus W_5(T-\alpha)| \geq 2$, then by Theorems 1 and 5, $T-\alpha \simeq B_6$ and $T \not\simeq P_7$. A contradiction by Lemma 2. Second, let X be a minimal subset of V such that $V \setminus W_5(T) \subset X$ ($|X| \geq 3$) and $T[X]$ is prime, so that $T[X]$ is $(V \setminus W_5(T))$-minimal. By Fact 2, $T[X] \simeq C_3$ or U_5. Suppose, toward a contradiction that $T[X] \simeq U_5$ and take $T[X]=U_5$. By Fact 2, $V \setminus W_5(T)=\{3,4\}$. As T is U_5-critical, then by Lemma 6 and Theorem 5, there exists $e \in E(G_X^T)$ such that $T[X \cup e]$ is prime and not isomorphic to U_7. It follows from Lemma 3, that there exists a subset Z of $X \cup e$ such that $T[Z] \simeq W_5$ and $Z \cap (V \setminus W_5(T)) \neq \emptyset$, a contradiction. □

Now, we prove Theorem 3 for tournaments on 7 vertices.

Proposition 2 *Up to isomorphy, the class $\mathcal{M}$ and the class $\mathcal{T}$ have the same tournaments on 7 vertices. Moreover, for each tournament T on 7 vertices of the class $\mathcal{M}$, we have $V(T) \setminus W_5(T) = \sigma(T) = \{0,1\}$.*

Proof Let $T=(V,A)$ be a tournament on 7 vertices of the class $\mathcal{M}$. $T \in \mathcal{M} \setminus (\mathcal{L} \cup \mathcal{L}^\star)$ because the tournaments of the class $\mathcal{L}$ have at least 9 vertices. Let $e \in E(G_{\mathbb{N}_3}^T)$. By Lemma 4, $T-e \simeq U_5$ or T_5. By Lemma 7, if there exists a subset $Z \subset V$ such that $T[Z] \simeq W_5$, then $e \subset Z$. It follows that $V \setminus \mathbb{N}_3 \subset Z$. Thus $V \setminus W_5(T)=\{0,1\}$ by verifying that $T-\{1,2\} \not\simeq W_5$, $T-\{0,2\} \not\simeq W_5$ and $T-\{0,1\} \simeq W_5$. As T is C_3-critical, $\sigma(T)=\{0,1\}$ from the following. First, $T-2$ is not prime because $\{0\} \cup \mathbb{N}_3^- \cup \mathbb{N}_3^+(0)$ (resp. $\{1\} \cup \mathbb{N}_3^+(0)$, $\{0,1\} \cup \mathbb{N}_3^-(0) \cup \mathbb{N}_3^-(1)$, $\{1\} \cup \mathbb{N}_3^+(0)$) is a nontrivial module of $T-2$ if $T \in \mathcal{H}$ (resp. $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$). Second, by Lemma 1, we have $\text{Ext}(X)=\{0,1\}$, where $X=V \setminus \{0,1\}$, because $\{0,1\} \cap \langle X \rangle = \emptyset$, and for all $u \in X$, $\{0,1\} \cap X(u) = \emptyset$ because $V \setminus W_5(T)=\{0,1\}$.

Conversely, let T be a tournament on 7 vertices of the class $\mathcal{T}$. By Proposition 1, we can assume that T is C_3-critical with $V(T) \setminus W_5(T) \subset \mathbb{N}_3$. By Lemma 4 and Theorem 5, $|\mathcal{C}(G_{\mathbb{N}_3}^T)|=2$. We distinguish the following cases.

- $\mathbb{N}_3^+ \neq \emptyset$ and $\mathbb{N}_3^- \neq \emptyset$. By Theorem 2, $|\mathbb{N}_3^-|=|\mathbb{N}_3^+|=1$. Therefore, we can assume that $\mathbb{N}_3(0) \neq \emptyset$ and $\mathbb{N}_3(2)=\emptyset$. It suffices to verify that $|\mathbb{N}_3(0)|=|\mathbb{N}_3^+(0)|=1$ because, in this case, by using Theorem 2 and Lemma 4, $T \in \mathcal{H}$. By using again Theorem 2 and Lemma 4, we verify the following. First, if $|\mathbb{N}_3(0)|=2$, then $\mathcal{C}(G^T_{\mathbb{N}_3}) = \{\mathbb{N}_3^+ \cup \mathbb{N}_3^-(0), \mathbb{N}_3^- \cup \mathbb{N}_3^+(0)\}$. Therefore, $T - \{0,1\} \simeq T - \{0,2\} \simeq W_5$, a contradiction. Second, if $|\mathbb{N}_3^-(0)| = 1$, then $\mathcal{C}(G^T_{\mathbb{N}_3}) = \{\mathbb{N}_3^+ \cup \mathbb{N}_3^-(0), \mathbb{N}_3^- \cup \mathbb{N}_3^+(1)\}$. Therefore, $T \simeq U_7$, a contradiction by Theorem 5.

- $\langle \mathbb{N}_3 \rangle = \emptyset$. By Theorem 2, we can assume that $|\mathbb{N}_3^-(0)| = |\mathbb{N}_3^+(0)| = 1$. We have $|\mathbb{N}_3(1)| = 1$. Otherwise, by Theorem 2 and Lemma 4, we can suppose that $\mathcal{C}(G^T_{\mathbb{N}_3}) = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^+(1), \mathbb{N}_3^-(0) \cup \mathbb{N}_3^-(1)\}$. Therefore, $T - \{1,2\} \simeq T - \{0,2\} \simeq W_5$, a contradiction. We have also $\mathcal{C}(G^T_{\mathbb{N}_3}) = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3(2), \mathbb{N}_3^-(0) \cup \mathbb{N}_3(1)\}$. Otherwise, again by Theorem 2 and Lemma 4, $\mathcal{C}(G^T_{\mathbb{N}_3}) = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^+(1), \mathbb{N}_3^-(0) \cup \mathbb{N}_3^-(2)\}$, so that $T \simeq U_7$, a contradiction by Theorem 5. Thus, we distinguish four cases. If $|\mathbb{N}_3^-(2)| = |\mathbb{N}_3^+(1)| = 1$, then $T \simeq T_7$, which contradicts Theorem 5. If $|\mathbb{N}_3^+(2)| = |\mathbb{N}_3^-(1)| = 1$, then $T - \{0,2\} \simeq T - \{0,1\} \simeq W_5$, a contradiction. If $|\mathbb{N}_3^+(2)| = |\mathbb{N}_3^+(1)| = 1$, then $T \in \mathcal{I}$. If $|\mathbb{N}_3^-(2)| = |\mathbb{N}_3^-(1)| = 1$, then T is isomorphic to a tournament of the class $\mathcal{I}$ with $V(T) \setminus W_5(T) = \{0,2\}$.

- $\emptyset \neq \langle \mathbb{N}_3 \rangle \in q^T_{\mathbb{N}_3}$. By interchanging T and $T^\star$, we can suppose that $\langle \mathbb{N}_3 \rangle = \mathbb{N}_3^-$. In this case, $|\mathbb{N}_3^-| = 1$ by Theorem 2. First, suppose that $|\mathbb{N}_3(0)| = 2$ and $|\mathbb{N}_3(1)| = 1$. By Theorem 2 and Lemma 4, $\mathcal{C}(G^T_{\mathbb{N}_3}) = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^-, \mathbb{N}_3^-(0) \cup \mathbb{N}_3(1)\}$. We have $|\mathbb{N}_3^+(1)| = 1$, otherwise $T \simeq U_7$, a contradiction by Theorem 5. Thus, T is isomorphic to a tournament of the class $\mathcal{K}$ with $V(T) \setminus W_5(T) = \{0,2\}$. Second, suppose that $|\mathbb{N}_3(0)| = 1$ and $|\mathbb{N}_3(1)| = 2$. Again by Theorem 2 and Lemma 4, $\mathcal{C}(G^T_{\mathbb{N}_3}) = \{\mathbb{N}_3^+(1) \cup \mathbb{N}_3^-, \mathbb{N}_3^-(1) \cup \mathbb{N}_3^-(0)\}$, so that $T \in \mathcal{J}$. Lastly, suppose that $|\mathbb{N}_3(0)| = |\mathbb{N}_3(1)| = 1$. By Theorem 2 and Lemma 4, we can suppose that $\mathcal{C}(G^T_{\mathbb{N}_3}) = \{\mathbb{N}_3^+(1) \cup \mathbb{N}_3^-, \mathbb{N}_3(0) \cup \mathbb{N}_3(2)\}$. By Lemma 4, we distinguish only three cases. If $|\mathbb{N}_3^-(2)| = |\mathbb{N}_3^-(0)| = 1$, then $T - \{0,1\} \simeq T - \{1,2\} \simeq W_5$, a contradiction. If $|\mathbb{N}_3^+(0)| = |\mathbb{N}_3^+(2)| = 1$, then $T \simeq U_7$, which contradicts Theorem 5. If $|\mathbb{N}_3^-(2)| = |\mathbb{N}_3^+(0)| = 1$, then $T \in \mathcal{K}$.

□

We complete our structural study of the tournaments of the class $\mathcal{T}$ by the following two corollaries.

Corollary 2 *Let T be a C_3-critical tournament such that $V(T) \setminus W_5(T) = \{0,1\}$. Then there exist $Q \neq Q' \in \mathcal{C}(G^T_{\mathbb{N}_3})$ and a tournament R on 7 vertices of the class $\mathcal{M}$ such that for all $e \in E(G^T_{\mathbb{N}_3}[Q])$ and for all $e' \in E(G^T_{\mathbb{N}_3}[Q'])$, there exists an isomorphism f from R onto $T[\mathbb{N}_3 \cup e \cup e']$. Moreover, $f(0) = 0$, $f(1) = 1$ and we have:*

1. *If $R \in \mathcal{H} \cup \mathcal{J} \cup \mathcal{J}^\star$, then $f(2) = 2$;*

2. *If $R \in \mathcal{I} \cup \mathcal{K} \cup \mathcal{K}^\star$, then $f(2) = 2$ or $\mathbb{N}_3(2) = \{f(2)\}$.*

Proof To begin, notice the following remark: given a $D[X]$-critical tournament D, for any edges a and b belonging to a same connected component of G^D_X, we have $D[X \cup a] \simeq D[X \cup b]$. Therefore, by Fact 1, Lemma 5, and Theorem 5, there exists $Q \in \mathcal{C}(G^T_{\mathbb{N}_3})$ such that for all $a \in E(G^T_{\mathbb{N}_3}[Q])$, $T[\mathbb{N}_3 \cup a] \simeq U_5$. By Lemma 4 and Remark 2, the tournament $T[\mathbb{N}_3 \cup Q]$ is isomorphic to U_{2n+1}, for some $n \geq 2$, and does not admit a prime subtournament on 7 vertices other than U_7. Therefore, by Lemma 6, Theorem 5, and the remark above, there exists $Q' \in \mathcal{C}(G^T_{\mathbb{N}_3}) \setminus \{Q\}$ such that for all $e \in E(G^T_{\mathbb{N}_3}[Q])$ and for all $e' \in E(G^T_{\mathbb{N}_3}[Q'])$, $T[\mathbb{N}_3 \cup e \cup e']$ is prime and not isomorphic to U_7. Moreover, $T[\mathbb{N}_3 \cup e \cup e'] \not\simeq P_7$ because the vertices of P_7 are all noncritical.

Likewise, $T[\mathbb{N}_3 \cup e \cup e'] \not\simeq T_7$ by Remark 2. It follows from Theorem 5 and Proposition 2 that there exists an isomorphism f from a tournament R on 7 vertices of the class $\mathcal{M}$ onto $T[\mathbb{N}_3 \cup e \cup e']$. As $(0,1) \in A(R) \cap A(T)$ and $V(R) \setminus W_5(R) = V(T) \setminus W_5(T) = \{0,1\}$ by Proposition 2, then f fixes 0 and 1. If $R \in \mathcal{H} \cup \mathcal{J} \cup \mathcal{J}^\star$, then f fixes 2 because 2 is the unique vertex x of R such that $R[\{0,1,x\}] \simeq C_3$. If $R \in \mathcal{I} \cup \mathcal{K} \cup \mathcal{K}^\star$, then $|\{x \in V(R) : R[\{0,1,x\}] \simeq C_3\}| = 2$. Therefore, $f(2) = 2$ or α, where α is the unique vertex of $\mathbb{N}_3(2)$ in the tournament $T[\mathbb{N}_3 \cup e \cup e']$. □

Corollary 3 *For all $T \in \mathcal{T}$, we have $V(T) \setminus W_5(T) = \sigma(T)$.*

Proof Let T be a tournament of the class $\mathcal{T}$ such that $V(T) \setminus W_5(T) = \{0,1\}$. By Proposition 1, we can assume that T is C_3-critical. By the same proposition, it suffices to prove that $\{0,1\} \subseteq \sigma(T)$. By Corollary 2, there is a subset X of $V(T)$ such that $\mathbb{N}_3 \subset X$ and $T[X]$ is isomorphic to a tournament on 7 vertices of the class $\mathcal{M}$. Suppose for a contradiction that T admits a critical vertex $i \in \{0,1\}$, and let $Y = X \setminus \{i\}$. By Proposition 2, $T[Y]$ is prime. As T is $T[Y]$-critical, then $i \notin \text{Ext}(Y)$ by Theorem 2. This is a contradiction because $T[X]$ is prime. □

Now, we prove that $\mathcal{M} \subseteq \mathcal{T}$. More precisely:

Proposition 3 *For all tournament T of the class $\mathcal{M}$, we have $V(T) \setminus W_5(T) = \sigma(T) = \{0,1\}$.*

Proof Let T be a tournament on $(2n+1)$ vertices of the class $\mathcal{M}$ for some $n \geq 3$. By Corollary 3, it suffices to prove that $V(T) \setminus W_5(T) = \{0,1\}$. We proceed by induction on n. By Proposition 2, the statement is satisfied for $n = 3$. Let now $n \geq 4$. Therefore, either T is a tournament on 9 vertices of the class $\mathcal{L} \cup \mathcal{L}^\star$ or there is $Q \in \mathcal{C}(G^T_{\mathbb{N}_3})$ such that $|Q| \geq 4$. In the first case, for all $e \in E(G^T_{\mathbb{N}_3})$, $T - e$ is isomorphic to U_7 or to a tournament on 7 vertices of the class $\mathcal{K} \cup \mathcal{K}^\star$. Therefore, if there exists a subset Z of $V(T)$ such that $Z \cap \{0,1\} \neq \emptyset$ and $T[Z] \simeq W_5$, then, for all $e \in E(G^T_{\mathbb{N}_3})$, $e \subset Z$ by Lemma 7. Thus, $V(T) \setminus \mathbb{N}_3 \subset Z$, a contradiction. As, furthermore, W_5 embeds into T, then $V(T) \setminus W_5(T) = \{0,1\}$ by Theorem 1. In the second case, let $Q \in \mathcal{C}(G^T_{\mathbb{N}_3})$ such that $|Q| \geq 4$. Let $\mathcal{X} = \mathcal{H}$, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$, or $\mathcal{L}$. For $T \in \mathcal{X}$, by Lemma 7, there is $e \in E(G^T_{\mathbb{N}_3}[Q])$ such that $T - e$ is C_3-critical. Moreover, $T - e$ is isomorphic to a tournament of the class $\mathcal{X}$ because $\mathcal{C}(G^{T-e}_{\mathbb{N}_3})$ is as described in the same class. By induction hypothesis, W_5 embeds into $T - e$, and thus into T. By Theorem 1, it suffices to verify that $\{0,1\} \subseteq V(T) \setminus W_5(T)$. Therefore, suppose that there exists $Z \subset V(T)$ such that $Z \cap \{0,1\} \neq \emptyset$ and $T[Z] \simeq W_5$. By induction hypothesis and by Lemma 7, $Q \subset Z$, so that $Z \subset Q \cup \mathbb{N}_3$. This is a contradiction by Theorem 5, because $T[\mathbb{N}_3 \cup Q] \simeq U_{|Q|+3}$ or $T_{|Q|+3}$ by Lemma 4. □

We are now ready to construct the tournaments of the class $\mathcal{T}$. We partition these tournaments T according to the following invariant $c(T)$. For $T \in \mathcal{T}$, $c(T)$ is the minimum of $|\mathcal{C}(G^T_{\sigma(T)\cup\{x\}})|$, the minimum being taken over all the vertices x of $W_5(T)$ such that $T[\sigma(T) \cup \{x\}] \simeq C_3$. Notice that $c(T) = c(T^\star)$. As T is $T[\sigma(T) \cup \{x\}]$-critical by Proposition 1, then $c(T) \leq 4$. Moreover, $c(T) \geq 2$ by Lemma 4. Proposition 1 leads us to classify the tournaments T of the class $\mathcal{T}$ according to the different values of $c(T)$. We will see that $c(T) = 2$ or 3. Theorem 3 results from Propositions 3, 4, 5, and 6.

Proposition 4 *Up to isomorphy, the tournaments T of the class $\mathcal{T}$ such that $c(T) = 2$ are those of the class $\mathcal{M} \setminus (\mathcal{L} \cup \mathcal{L}^\star)$.*

Proof For all $T \in \mathcal{M} \setminus (\mathcal{L} \cup \mathcal{L}^\star)$, we have $T \in \mathcal{T}$ by Proposition 3, and $c(T) = 2$ by Lemma 4. Now let T be a tournament on $(2n+1)$ vertices of the class $\mathcal{T}$ such that $c(T) = 2$. By Proposition 1, we can assume that T is C_3-critical with $V(T) \setminus W_5(T) = \{0,1\}$ and $\mid \mathcal{C}(G^T_{\mathbb{N}_3}) \mid = 2$. By Corollary 2 and by interchanging T and $T^\star$, there is a tournament R on 7 vertices of the class $\mathcal{H} \cup \mathcal{I} \cup \mathcal{J} \cup \mathcal{K}$ such that for all $e \in E(G^T_{\mathbb{N}_3}[Q])$ and for all $e' \in E(G^T_{\mathbb{N}_3}[Q'])$, there exists an isomorphism f, fixing 0 and 1, from R onto $T[\mathbb{N}_3 \cup e \cup e']$, where Q and Q' are the two different connected components of $G^T_{\mathbb{N}_3}$. If $f(2) = 2$, then, by Theorem 2, T and R are in the same class $\mathcal{H}$, $\mathcal{I}$, $\mathcal{J}$, or $\mathcal{K}$. Suppose now that $f(2) \neq 2$. By Corollary 2, $R \in \mathcal{I} \cup \mathcal{K}$. If $R \in \mathcal{I}$ (resp. $\mathcal{K}$), then $T[\mathbb{N}_3 \cup e \cup e']$ is a tournament on 7 vertices of the class $\mathcal{I}'$ (resp. $\mathcal{K}'$) of the C_3-critical tournaments Z such that $\mathcal{C}(G^Z_{\mathbb{N}_3}) = \{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^+(1), \mathbb{N}_3^-(1) \cup \mathbb{N}_3^-(2)\}$ (resp. $\mathcal{C}(G^Z_{\mathbb{N}_3}) = \{\mathbb{N}_3^+(1) \cup \mathbb{N}_3^-, \mathbb{N}_3^-(1) \cup \mathbb{N}_3^+(2)\}$). By Theorem 2, $T \in \mathcal{I}'$ (resp. $\mathcal{K}'$). Moreover, by considering the vertex $\alpha = \min T[\mathbb{N}_3^-(2)]$ (resp. $\max T[\mathbb{N}_3^+(2)]$) and by using Corollary 3, T is also $T[\{0,1,\alpha\}]$-critical with $\mathcal{C}(G^T_{\{0,1,\alpha\}}) = \{\{0,1,\alpha\}^-(0) \cup \{0,1,\alpha\}^+(1), \{0,1,\alpha\}^+(0) \cup \{0,1,\alpha\}^+(\alpha)\}$ (resp. $\mathcal{C}(G^T_{\{0,1,\alpha\}}) = \{\{0,1,\alpha\}^+(1) \cup \{0,1,\alpha\}^-, \{0,1,\alpha\}^+(0) \cup \{0,1,\alpha\}^-(\alpha)\}$). It follows that T is isomorphic to a tournament of the class $\mathcal{I}$ (resp. $\mathcal{K}$). □

Proposition 5 *Up to isomorphy, the tournaments T of the class $\mathcal{T}$ such that $c(T) = 3$ are those of the class $\mathcal{L} \cup \mathcal{L}^\star$.*

Proof Let T be a tournament of the class $\mathcal{L} \cup \mathcal{L}^\star$. $T \in \mathcal{T}$ by Proposition 3. Moreover, $c(T) = 3$ by Theorem 2. Indeed, it suffices to observe that for all $x \in \{i \in V(T) \setminus \mathbb{N}_3 : T[\{0,1,i\}] \simeq C_3\} = \mathbb{N}_3^-(2)$, we have $\max T[\mathbb{N}_3^+(1)] \in X^+(1)$, $\min T[\mathbb{N}_3^-] \in X^-$, $\min T[\mathbb{N}_3^+] \in X^+$, $\max T[\mathbb{N}_3^-(0)] \in X^-(0)$ and $2 \in X^+(x)$, where $X = \{0,1,x\}$.

Now let T be a tournament on $(2n+1)$ vertices of $\mathcal{T}$ such that $c(T) = 3$. By Proposition 1, we can assume that T is C_3-critical with $V(T) \setminus W_5(T) = \{0,1\}$ and $\mid \mathcal{C}(G^T_{\mathbb{N}_3}) \mid = 3$. By Corollary 2 and by interchanging T and $T^\star$, there is a tournament R on 7 vertices of the class $\mathcal{H} \cup \mathcal{I} \cup \mathcal{J} \cup \mathcal{K}$ such that for all $e \in E(G^T_{\mathbb{N}_3}[Q])$ and $e' \in E(G^T_{\mathbb{N}_3}[Q'])$, there exists an isomorphism f, which fixes 0 and 1, from R onto $T[\mathbb{N}_3 \cup e \cup e']$, where $Q \neq Q' \in \mathcal{C}(G^T_{\mathbb{N}_3})$. Take $e'' \in (G^T_{\mathbb{N}_3}[Q''])$, where $Q'' = \mathcal{C}(G^T_{\mathbb{N}_3}) \setminus \{Q, Q'\}$. Suppose, toward a contradiction, that $R \in \mathcal{H} \cup \mathcal{J}$. By Theorem 2 and by Corollary 2, if $R \in \mathcal{H}$ (resp. $R \in \mathcal{J}$), then $\{Q, Q'\} = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^-, \mathbb{N}_3^-(1) \cup \mathbb{N}_3^+\}$ (resp. $\{\mathbb{N}_3^+(1) \cup \mathbb{N}_3^-, \mathbb{N}_3^-(0) \cup \mathbb{N}_3^-(1)\}$). Therefore, by Lemma 4, $Q'' = \{\mathbb{N}_3^+(1) \cup \mathbb{N}_3^+(2)\}$, $\{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^+(1)\}$ or $\{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^-(2)\}$ (resp. $\{\mathbb{N}_3^+(0) \cup \mathbb{N}_3(2)\}$ or $\{\mathbb{N}_3^+ \cup \mathbb{N}_3^-(2)\}$). We verify that in each of these cases, either $T[\{0\} \cup e \cup e'']$, $T[\{0\} \cup e' \cup e'']$, $T[\{1\} \cup e \cup e'']$ or $T[\{1\} \cup e' \cup e'']$ is isomorphic to W_5, a contradiction. Therefore, $R \in \mathcal{I} \cup \mathcal{K}$. By Corollary 2, $f(2) = 2$ or α, where α is the unique vertex of $\mathbb{N}_3(2)$ in $T[\mathbb{N}_3 \cup e \cup e']$.

Suppose, again by contradiction, that $R \in \mathcal{I}$. We begin by the case where $f(2) = 2$. By Theorem 2, we can suppose that $Q = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^+(2)\}$ and $Q' = \{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^+(1)\}$. By Lemma 4, $Q'' = \{\mathbb{N}_3^-(1) \cup \mathbb{N}_3^+\}$, $\{\mathbb{N}_3^-(2) \cup \mathbb{N}_3^+\}$, or $\{\mathbb{N}_3^-(1) \cup \mathbb{N}_3^-(2)\}$. If $Q'' = \{\mathbb{N}_3^-(1) \cup \mathbb{N}_3^+\}$ (resp. $\{\mathbb{N}_3^-(2) \cup \mathbb{N}_3^+\}$), then $T[\{0\} \cup e \cup e''] \simeq W_5$ (resp. $T[\{1\} \cup e' \cup e''] \simeq W_5$), a contradiction. If $Q'' = \{\mathbb{N}_3^-(1) \cup \mathbb{N}_3^-(2)\}$, then, by taking $X = \{0,1,x\}$, where $x = \min T[\mathbb{N}_3^-(2)]$, we obtain a contradiction because, by Corollary 3, T is $T[X]$-critical with $|\mathcal{C}(G^T_X)| = 2$. Indeed, $\mathcal{C}(G^T_X) = \{X^-(0) \cup X^+(1), X^+(0) \cup X^+(x)\}$, with $X^-(0) = \mathbb{N}_3^-(0)$, $X^+(1) = \mathbb{N}_3^+(1)$, $X^+(0) = \mathbb{N}_3^+(0) \cup \mathbb{N}_3^-(1)$, and $X^+(x) = \mathbb{N}_3^+(2) \cup \{2\} \cup (\mathbb{N}_3^-(2) \setminus \{x\})$. Now if $f(2) = \alpha$, then we obtain again a contradiction. Indeed, by replacing T by $T^\star$ and by interchanging the vertices 0 and 1, $\{Q, Q'\} = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^+(2), \mathbb{N}_3^-(0) \cup \mathbb{N}_3^+(1)\}$ as in the case where $f(2) = 2$.

At present, $R \in \mathcal{K}$. We begin by the case where $f(2) = 2$. By Theorem 2, we can suppose that $Q = \{\mathbb{N}_3^- \cup \mathbb{N}_3^+(1)\}$ and $Q' = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^-(2)\}$. By Lemma 4, $Q'' = \{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^+\}$, $\{\mathbb{N}_3^-(1) \cup \mathbb{N}_3^+\}$, $\{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^-(1)\}$ or $\{\mathbb{N}_3^-(1) \cup \mathbb{N}_3^+(2)\}$. If $Q'' = \{\mathbb{N}_3^-(1) \cup \mathbb{N}_3^+\}$ (resp. $\{\mathbb{N}_3^-(1) \cup \mathbb{N}_3^-(0)\}$), then $T[\{0\} \cup e \cup e''] \simeq W_5$ (resp. $T[\{1\} \cup e' \cup e''] \simeq W_5$), a contradiction. If $Q'' = \{\mathbb{N}_3^-(1) \cup \mathbb{N}_3^+(2)\}$, then, by taking $X = \{0, 1, x\}$, where $x = \max T[\mathbb{N}_3^+(2)]$, we have a contradiction because, by Corollary 3, T is $T[X]$-critical with $|\mathcal{C}(G_X^T)| = 2$. Indeed, $\mathcal{C}(G_X^T) = \{X^- \cup X^+(1), X^+(0) \cup X^-(x)\}$, with $X^- = \mathbb{N}_3^-$, $X^+(1) = \mathbb{N}_3^+(1)$, $X^+(0) = \mathbb{N}_3^-(1) \cup \mathbb{N}_3^+(0)$ and $X^-(x) = \mathbb{N}_3^-(2) \cup \{2\} \cup (\mathbb{N}_3^+(2) \setminus \{x\})$. If $Q'' = \{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^+\}$, then $T \in \mathcal{L}$. Now suppose that $f(2) = \alpha$. By Theorem 2, we can suppose that $Q = \{\mathbb{N}_2^+(1) \cup \mathbb{N}_2^-\}$ and $Q' = \{\mathbb{N}_2^-(1) \cup \mathbb{N}_3^+(2)\}$. By Lemma 4, $Q'' = \{\mathbb{N}_3^-(2) \cup \mathbb{N}_3^+\}$, $\{\mathbb{N}_3^-(2) \cup \mathbb{N}_3(0)\}$, or $\{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^+\}$. If $Q'' = \{\mathbb{N}_3^-(2) \cup \mathbb{N}_3^+\}$ or $\{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^-(2)\}$, then $T[\{0\} \cup e \cup e''] \simeq W_5$, a contradiction. If $Q'' = \{\mathbb{N}_3^+(0) \cup \mathbb{N}_3^-(2)\}$, then we obtain the same configuration giving $|\mathcal{C}(G_X^T)| = 2$ in the case where $f(2) = 2$. If $Q'' = \{\mathbb{N}_3^-(0) \cup \mathbb{N}_3^+\}$, then T is isomorphic to a tournament of the class $\mathcal{L}^\star$. □

Proposition 6 *For any tournament T of the class $\mathcal{T}$, we have $c(T) = 2$ or 3.*

Proof Let T be a tournament on $(2n+1)$ vertices of the class $\mathcal{T}$ for some $n \geq 3$. We proceed by induction on n. By Propositions 4 and 5, the statement is satisfied for $n = 3$ and for $n = 4$. Let $n \geq 5$. By Proposition 1, we can assume that T is C_3-critical with $V(T) \setminus W_5(T) = \{0, 1\}$. By Theorem 2 and Lemma 4, $2 \leq c(T) \leq 4$. Therefore, we only consider the case where $|\mathcal{C}(G_{\mathbb{N}_3}^T)| = 4$. By Corollary 2, there exist $Q \neq Q' \in \mathcal{C}(G_{\mathbb{N}_3}^T)$ and a tournament R on 7 vertices of the class $\mathcal{M}$, such that for all $e \in E(G_{\mathbb{N}_3}^T[Q])$ and for all $e' \in E(G_{\mathbb{N}_3}^T[Q'])$, $T[\mathbb{N}_3 \cup e \cup e'] \simeq R$. By Lemma 7, there exists $e'' \in E(G_{\mathbb{N}_3}^T[Q''])$, where $Q'' \in \mathcal{C}(G_{\mathbb{N}_3}^T) \setminus \{Q, Q'\}$, such that $T - e''$ is C_3-critical. As W_5 embeds into $T - e''$, then $V(T - e'') \setminus W_5(T - e'') = \{0, 1\}$ by Theorem 1. Therefore, $T - e'' \in \mathcal{T}$. By induction hypothesis, $c(T - e'') = 2$ or 3. By Theorem 2, if $c(T - e'') = 2$, then $c(T) = 2$ or 3. Therefore, suppose that $c(T - e'') = 3$. By Proposition 5 and by interchanging T and $T^\star$, we can assume that $T - e'' \in \mathcal{L}$. By Theorem 2 and by taking $e'' = \{x, x'\}$, we can assume that $x \in \mathbb{N}_3^-(1)$ and $x' \in \mathbb{N}_3^+(2)$. Thus, for $X = \{0, 1, x'\}$, we have $T[X] \simeq C_3$ and $X^+(x') = \emptyset$. It follows from Theorem 2 that $c(T) < 4$. □

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