Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article
Turk J Math
(2015) 39: 583 – 594
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doi:10.3906/mat-1501-74

The strong "zero-two" law for positive contractions of Banach–Kantorovich L_p-lattices

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Received: 28.01.2015 • **Accepted/Published Online:** 25.03.2015 • **Printed:** 30.07.2015

Abstract: In the present paper we study dominated operators acting on Banach–Kantorovich L_p-lattices, constructed by a measure m with values in the ring of all measurable functions. Using methods of measurable bundles of Banach–Kantorovich lattices, we prove the strong "zero-two" law for positive contractions of Banach–Kantorovich L_p-lattices.

Key words: Banach–Kantorovich L_p-lattice, strong "zero-two" law, dominated operator, positive contraction

1. Introduction

Starting from von Neumman's [23] pioneering work, the development of the theory of Banach bundles has been stimulated by many works (see, for example, [14, 15]). There are many papers devoted to the applications of this theory to several branches of analysis [1, 17, 18, 26]. Moreover, this theory is well connected with the theory of vector-valued Banach spaces [13, 14], which has several applications (see, for example, [19]). In the present paper, we concentrate on the theory of Banach bundles of L_0-valued Banach spaces (for more details, see [7, 14]). Note that such spaces are called *Banach–Kantorovich spaces*. In [14, 15, 18] the theory of Banach–Kantorovich spaces was developed. It is known [14] that the theory of measurable bundles of Banach lattices is sufficiently well explored. Therefore, it is natural to employ methods of measurable bundles of such spaces to investigate functional properties of Banach–Kantorovich spaces. It is an effective tool that gives a good opportunity to obtain various properties of these spaces [4, 5]. For example, in [8, 7] the Banach–Kantorovich lattice $L_p(\nabla,\mu)$ was represented as a measurable bundle of classical L_p -lattices. Naturally, these functional Banach–Kantorovich spaces have many properties similar to those of the classical ones, constructed by real valued measures. In [2, 11] this allowed the establishment of several weighted ergodic theorems for positive contractions of $L_p(\nabla,\mu)$-spaces. In [5] the convergence theorems of martingales on such lattices were proved. Some other applications of the measurable bundles of Banach–Kantorovich spaces can be found in [1, 12].

In [22] Ornstein and Sucheston proved that, for any positive contraction T on an L_1-space, one has either $\|T^n - T^{n+1}\|_1 = 2$ for all n or $\lim_{n\to\infty} \|T^n - T^{n+1}\|_1 = 0$. An extension of this result to positive operators

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2010 *AMS Mathematics Subject Classification:* 37A30, 47A35, 46B42, 46E30, 46G10.

on L^{∞}-spaces was given by Foguel [3]. In [27] Zahoropol generalized these results, calling it the *"zero-two" law*, and his result can be formulated as follows:

Theorem 1.1 *Let T be a positive contraction of L_p, $p > 1, p \neq 2$. If the following relation holds, $\||T^{m+1} - T^m|\| < 2$ for some $m \in \mathbb{N} \cup \{0\}$, then*

$$\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0.$$

In [16] this theorem was established for Köthe spaces. In particular, from that result the statement of the theorem for a case $p = 2$ follows.

Furthermore, the strong "zero-two" law for positive contractions of L_p-spaces, $1 \leq p < +\infty$, was proved in [25]. This result is formulated as follows:

Theorem 1.2 *Let $1 \leq p < +\infty$ and T be a positive contraction of L_p. If $\||T^{m+1} - T^m|\| < 2$ for some $m \in \mathbb{N} \cup \{0\}$, then*

$$\lim_{n\to\infty} \||T^{n+1} - T^n|\| = 0.$$

In [10] we generalized Theorem 1.1 for the positive contractions of the Banach–Kantorovich L_p-lattices. Namely, the following result was proved.

Theorem 1.3 *Let $T : L_p(\nabla, m) \to L_p(\nabla, m)$, $p > 1, p \neq 2$ be a positive linear contraction such that $T\mathbf{1} \leq \mathbf{1}$. If one has $\||T^{m+1} - T^m|\| < 2 \cdot \mathbf{1}$ for some $m \in \mathbb{N} \cup \{0\}$, then*

$$(o) - \lim_{n\to\infty} \|T^{n+1} - T^n\| = 0.$$

The main aim of this paper is to prove the strong "zero-two" law for the positive contractions of the Banach–Kantorovich lattices $L_p(\nabla, m)$. To establish the main aim, we first study dominated operators acting on Banach–Kantorovich L_p-lattices (see Section 3). Using the methods of measurable bundles of Banach–Kantorovich lattices, in Section 4 we prove the main result of the present paper. Finally, in Section 5, we illustrate an application of the methods used in Section 4 to prove a result related to dominated operators.

2. Preliminaries

Let (Ω, Σ, μ) be a complete measure space with a finite measure μ. By $\mathcal{L}(\Omega)$ (resp. $\mathcal{L}_\infty(\Omega)$) we denote the set of all (resp. essentially bounded) measurable real functions defined on Ω a.e. In the standard way, we introduce an equivalence relation on $\mathcal{L}(\Omega)$ by putting $f \sim g$ whenever $f = g$ a.e. The set $L_0(\Omega)$ of all cosets $f^\sim = \{g \in \mathcal{L}(\Omega) : f \sim g\}$, endowed with the natural algebraic operations, is an algebra with unit $\mathbf{1}(\omega) = 1$ over the field of reals $\mathbb{R}$. Moreover, with respect to the partial order $f^\sim \leq g^\sim \Leftrightarrow f \leq g$ a.e., the algebra $L_0(\Omega)$ is a Dedekind complete Riesz space with weak unit $\mathbf{1}$, and the set $B(\Omega) := B(\Omega, \Sigma, \mu)$ of all idempotents in $L_0(\Omega)$ is a complete Boolean algebra. Furthermore, $L_\infty(\Omega) = \{f^\sim : f \in \mathcal{L}_\infty(\Omega)\}$ is an order ideal in $L_0(\Omega)$ generated by $\mathbf{1}$. In what follows, we will write $f \in L_0(\Omega)$ instead of $f^\sim \in L_0(\Omega)$ by assuming that the coset of f is considered.

Let E be a linear space over the real field $\mathbb{R}$. By $\|\cdot\|$ we denote a $L_0(\Omega)$-valued norm on E. The pair $(E, \|\cdot\|)$ is then called a *lattice-normed space (LNS) over* $L_0(\Omega)$. An LNS E is said to be *d-decomposable* if for every $x \in E$ and the decomposition $\|x\| = f + g$ with f and g disjoint positive elements in $L_0(\Omega)$ there exist $y, z \in E$ such that $x = y + z$ with $\|y\| = f$, $\|z\| = g$.

Suppose that $(E, \|\cdot\|)$ is an LNS over $L_0(\Omega)$. A net $\{x_\alpha\}$ of elements of E is said to be (bo)*-converging* to $x \in E$ (in this case we write $x = (bo)\text{-}\lim x_\alpha$), if the net $\{\|x_\alpha - x\|\}$ (o)-converges to zero (here (o)-convergence means the order convergence) in $L_0(\Omega)$ (written as $(o)\text{-}\lim \|x_\alpha - x\| = 0$). A net $\{x_\alpha\}_{\alpha \in A}$ is called (bo)*-fundamental* if $(x_\alpha - x_\beta)_{(\alpha,\beta)\in A\times A}$ (bo)-converges to zero.

An LNS in which every (bo)-fundamental net (bo)-converges is called (bo)*-complete*. A *Banach–Kantorovich space (BKS) over* $L_0(\Omega)$ is a (bo)-complete d-decomposable LNS over $L_0(\Omega)$. It is well known (see [17],[18]) that every BKS E over $L_0(\Omega)$ admits an $L_0(\Omega)$-module structure such that $\|fx\| = |f| \cdot \|x\|$ for every $x \in E$, $f \in L_0(\Omega)$, where $|f|$ is the modulus of a function $f \in L_0(\Omega)$. A BKS $(\mathcal{U}, \|\cdot\|)$ is called a *Banach–Kantorovich lattice* if $\mathcal{U}$ is a vector lattice and the norm $\|\cdot\|$ is monotone, i.e. $|u_1| \le |u_2|$ implies $\|u_1\| \le \|u_2\|$. It is known [17] that the cone $\mathcal{U}_+$ of positive elements is (bo)-closed.

Let ∇ be an arbitrary complete Boolean algebra and let $X(\nabla)$ be the Stone space of ∇. Assume that $L_0(\nabla) := C_\infty(X(\nabla))$ is the algebra of all continuous functions $x : X(\nabla) \to [-\infty, +\infty]$ that take the values $\pm\infty$ only on nowhere dense subsets of $X(\nabla)$. Finally, by $C(X(\nabla))$, we denote the subalgebra of all continuous real functions on $X(\nabla)$.

Given a complete Boolean algebra ∇, let us consider a mapping $m : \nabla \to L_0(\Omega)$. Such a mapping is called an $L_0(\Omega)$*-valued measure* if one has:

(i) $m(e) \ge 0$ for all $e \in \nabla$ and $m(e) = 0 \Leftrightarrow e = 0$;

(ii) $m(e \vee g) = m(e) + m(g)$ if $e \wedge g = 0, e, g \in \nabla$;

(iii) $m(e_\alpha) \downarrow 0$ for any net $e_\alpha \downarrow 0$.

Following the well-known scheme of the construction of L_p-spaces, a space $L_p(\nabla, m)$ can be defined by

$$L_p(\nabla, m) = \left\{ f \in L_0(\nabla) : |f|_p := \int |f|^p dm - \text{exist} \right\}, \quad p \ge 1,$$

where m is an $L_0(\Omega)$-valued measure on ∇.

An $L_0(\Omega)$-valued measure m is said to be *disjunctive decomposable (*d*-decomposable)*, if for every $e \in \nabla$ and the decomposition $m(e) = a_1 + a_2, \quad a_1 \wedge a_2 = 0, \quad a_i \in L_0(B)$ there exist $e_1, e_2 \in \nabla$ such that $e = e_1 \vee e_2$ and $m(e_i) = a_i, i = 1, 2$.

Theorem 2.1 *[7] The following statements hold:*

(i) The pair $(L_p(\nabla, m), |\cdot|_p)$ *is a* (bo)*–complete lattice. Moreover, it is an ideal linear subspace of* $L_0(\nabla)$*, i.e. from* $|x| \le |y|$*,* $y \in L_p(\nabla, m)$*,* $x \in L_0(\nabla)$ *it follows that* $x \in L_p(\nabla, m)$ *and* $|x|_p \le |y|_p$*;*

(ii) If $0 \le x_\alpha \in L_p(\nabla, m)$ *and* $x_\alpha \downarrow 0$*, then* $|x_\alpha|_p \downarrow 0$*;*

(iii) If the measure m *is* d*-decomposable, then* $|\alpha)x|_p = |\alpha||x|_p$ *for all* $\alpha \in L_0(\Omega), x \in L_p(\nabla, m)$*;*

(iv) *If the measure m is d-decomposable, then $(L_p(\nabla, m), |\cdot|_p)$ is a Banach–Kantorovich space;*

(v) *One has $L_\infty(\nabla, m) := C(X(\nabla)) \subset L_p(\nabla, m) \subset L_q(\nabla, m)$, $1 \le q \le p$. Moreover, $L_\infty(\nabla, m)$ is (bo)–dense in $(L_1(\nabla, m), \|\cdot\|_1)$.*

Now we mention the necessary facts from the theory of measurable bundles of Boolean algebras and Banach spaces (see [14] for more details).

Let (Ω, Σ, μ) be the same as above and X be a mapping assigning an L_p-space constructed by a real valued measure m_ω, i.e. $L_p(\nabla_\omega, m_\omega)$, to each point $\omega \in \Omega$ and let

$$L = \left\{ \sum_{i=1}^{n} \alpha_i e_i : \alpha_i \in \mathbb{R}, \quad e_i(\omega) \in \nabla_\omega, \ i = \overline{1, n}, \ n \in \mathbb{N} \right\}$$

be a set of sections. In [7] it was established that the pair (X, L) is a measurable bundle of Banach lattices and $L_0(\Omega, X)$ is modulo ordered isomorphic to $L_p(\nabla, \mu)$.

Let ρ be a lifting in $L_\infty(\Omega)$ (see [14]). As before, let ∇ be an arbitrary complete Boolean subalgebra of $\nabla(\Omega)$ and m be an $L_0(\Omega)$-valued measure on ∇. By $L_\infty(\nabla, m)$ we denote the set of all essentially bounded functions w.r.t. m taken from $L_0(\nabla)$.

A mapping $\ell : L_\infty(\nabla, m)(\subset L_\infty(\Omega, X)) \to \mathcal{L}_\infty(\Omega, X)$ is called a *vector-valued lifting* [14] associated with the lifting ρ if it satisfies the following conditions:

(1) $\ell(\hat{u}) \in \hat{u}$ for all $\hat{u}$ such that $dom(\hat{u}) = \Omega$;

(2) $\|\ell(\hat{u})\|_{L_p(\nabla_\omega, m_\omega)} = \rho(|\hat{u}|_p)(\omega)$;

(3) $\ell(\hat{u} + \hat{v}) = \ell(\hat{u}) + \ell(\hat{v})$ for every $\hat{u}, \hat{v} \in L_\infty(\nabla, m)$;

(4) $\ell(h \cdot \hat{u}) = \rho(h)\ell(\hat{u})$ for every $\hat{u} \in L_\infty(\nabla, m)$, $h \in L_\infty(\Omega)$;

(5) $\ell(\hat{u}) \ge 0$ whenever $\hat{u} \ge 0$;

(6) the set $\{\ell(\hat{u})(\omega) : \hat{u} \in L_\infty(\nabla, m)\}$ is dense in $X(\omega)$ for all $\omega \in \Omega$;

(7) $\ell(\hat{u} \vee \hat{v}) = \ell(\hat{u}) \vee \ell(\hat{v})$ for every $\hat{u}, \hat{v} \in L_\infty(\nabla, m)$.

In [7] the existence of the vector-valued lifting was proved.

Let $L_p(\nabla, m)$ $(p \ge 1)$ be a Banach–Kantorovich lattice. A linear mapping $T : L_p(\nabla, m) \to L_p(\nabla, m)$ is called *positive* if $T\hat{f} \ge 0$ whenever $\hat{f} \ge 0$. We say that T is a $L_0(\Omega)$-*bounded mapping* if there exists a function $k \in L_0(\Omega)$ such that $|T\hat{f}|_p \le k|\hat{f}|_p$ for all $\hat{f} \in L_p(\nabla, \mu)$. For such a mapping we can define an element of $L_0(\Omega)$ as follows:

$$\|T\| = \sup_{|\hat{f}|_p \le \mathbf{1}} |T\hat{f}|_p,$$

which is called an $L_0(\Omega)$-*valued norm* of T. A mapping T is said to be a *contraction* if one has $\|T\| \le \mathbf{1}$. Some examples of contractions can be found in [11].

In the sequel we will need the following bundle representation of $L_0(\Omega)$-linear $L_0(\Omega)$-bounded operators acting in Banach–Kantorovich lattices.

Theorem 2.2 *[10] Let $L_p(\nabla, m)$ $(p \geq 1)$ be a Banach–Kantorovich lattice and $L_p(\nabla_\omega, m_\omega)$ be the corresponding L_p-spaces constructed by real valued measures. Let $T : L_p(\nabla, m) \to L_p(\nabla, m)$ be a positive linear contraction such that $T\mathbf{1} \leq \mathbf{1}$. Then for every $\omega \in \Omega$ there exists a positive contraction $T_\omega : L_p(\nabla_\omega, \mu_\omega) \to L_p(\nabla_\omega, m_\omega)$ such that $T_\omega f(\omega) = (T\hat{f})(\omega)$ a.e. for every $\hat{f} \in L_p(\nabla, m)$.*

3. Dominated operators in Banach–Kantorovich L_p-lattices

In this section, we study dominated operators in Banach–Kantorovich L_p-lattices.

Theorem 3.1 *Let $T : L_1(\nabla, m) \to L_1(\nabla, m)$ be an $L_0(\Omega)$-bounded linear operator in Banach–Kantorovich lattice $L_1(\nabla, m)$. Then there exists a unique $|T|$-$L_0(\Omega)$-bounded linear operator in $L_1(\nabla, m)$ such that*

(a) $\|T\| = \||T|\|$;

(b) one has $|T\hat{f}| \leq |T||\hat{f}|$, for all $\hat{f} \in L_1(\nabla, m)$;

(c) for each $\hat{f} \in L_1(\nabla, m)$ with $\hat{f} \geq 0$ one has $|T|\hat{f} = \sup\{|T\hat{g}| : \hat{g} \in L_1(\nabla, m), |\hat{g}| \leq \hat{f}\ \}$;

(d) $\|T\|_\infty = \||T|\|_\infty$.

Proof Let $\mathcal{P}$ denote the family of all finite measurable partitions $\pi = \{B_1, B_2, \ldots, B_m\}$ of Ω. We partially order $\mathcal{P}$ in the usual way, i.e. for $\pi = \{B_1, B_2, \ldots, B_m\}$ and $\pi' = \{B'_1, B'_2, \ldots, B'_k\}$ we write $\pi \leq \pi^{'}$ if $\pi^{'}$ is a refinement of π, i.e. each set B_i is a union of sets $\{B'_i\}$.

Given $\pi \in \mathcal{P}$, and for every $\hat{f} \in L_1(\nabla, m), \hat{f} \geq 0$, we define

$$T_\pi \hat{f} \ = \sum_{i=1}^{m} |T(\chi_{B_i}\hat{f})|.$$

Clearly $\pi \leq \pi'$ implies $T_\pi \hat{f} \leq T_{\pi'}\hat{f}$. From $|\hat{f}|_1 = \sum\limits_{i=1}^{m} |\chi_{B_i}\hat{f}|_1$ we obtain $|T_\pi \hat{f}|_1 \leq \|T\||\hat{f}|_1$. Since $\{T_\pi \hat{f} : \pi \in \mathcal{P}\}$ is increasing on $\mathcal{P}$ and is norm bounded, one can therefore define

$$|T|\hat{f} := \lim_{\pi \in \mathcal{P}} T_\pi \hat{f}, \quad \hat{f} \geq 0.$$

We clearly have

$$||T|\hat{f}|_1 \leq \|T\||\hat{f}|_1, \hat{f} \geq 0 \tag{1}$$

and $|T|$ is linear on positive functions. Therefore, $|T|$ can be extended by the linearity to the whole $L_1(\nabla, m)$. This extension is again denoted by $|T|$.

For $\hat{f} \geq 0$ and $|\hat{g}| \leq \hat{f}$ we obtain $|T|\hat{f} \geq |T\hat{g}|$ by means of the approximation argument with simple functions. This yields (b).

(c). From (b) we have $|T|\hat{g} \geq |T\hat{g}|$, i.e. T has a positive dominant. Then by [24, Theorem VIII 1.1] T is regular. Hence, using [24, formula (10),p.,231] one finds $|T|\hat{f} = \sup\{|T\hat{g}| : \hat{g} \in L_1(\nabla, m), |\hat{g}| \leq \hat{f}\ \}$.

(a). Again from (b) we get $\|T\| \leq \||T|\|$ and by (1) one finds $\||T|\| \leq \|T\|$. Hence, $\|T\| = \||T|\|$.

(d). Let $\hat{f} \in L^{\infty}(\hat{\nabla}, \hat{\mu})$. It is then clear that from $|T\hat{f}| \leq |T||\hat{f}|$ one gets $\|T\|_{\infty}\|\hat{f}\|_{\infty} \leq \||T|\|_{\infty}\|\hat{f}\|_{\infty}$, which means $\|T\|_{\infty} \leq \||T|\|_{\infty}$.

Using (c) we obtain

$$|T||\hat{f}| = \sup_{|\hat{g}|\leq|\hat{f}|} |T\hat{g}| \leq \sup_{|\hat{g}|\leq|\hat{f}|} \|T\|_{\infty}\|\hat{g}\|_{\infty}\mathbf{1} \leq \|T\|_{\infty}\|\hat{f}\|_{\infty}\mathbf{1}.$$

Hence, $\||T|\|_{\infty} \leq \|T\|_{\infty}$ and $\||T|\|_{\infty} = \|T\|_{\infty}$. □

Definition 3.2 *A linear operator $A : L_p(\nabla, m) \to L_p(\nabla, m)$ is called dominated if there exists an $L_0(\Omega)$-bounded positive linear operator $S : L_p(\nabla, m) \to L_p(\nabla, m)$ such that*

$$|A\hat{f}| \leq S(|\hat{f}|)$$

for all $\hat{f} \in L_p(\nabla, m)$. The operator S is called dominant.

Theorem 3.3 *Let $T : L_p(\nabla, m) \to L_p(\nabla, m)$ be a dominated operator with a dominant S on Banach–Kantorovich lattice $L_p(\nabla, m)$. Then there exists a unique $|T|$-$L_0(\Omega)$-bounded linear operator on $L_p(\nabla, m)$ such that*

(a) $\||T|\| \leq \|S\|$;

(b) one has $|T\hat{f}| \leq |T||\hat{f}|$, for all $\hat{f} \in L_p(\nabla, m)$;

(c) for each $\hat{f} \in L_p(\nabla, m), \hat{f} \geq 0$ one has $|T|\hat{f} = \sup\{|T\hat{g}| : \hat{g} \in L_p(\nabla, m), |\hat{g}| \leq \hat{f}\}$.

Proof The proof of the existence of $|T|$ and (b), (c) are similar to the proof of Theorem 3.1. Now we prove (a). From

$$|T|\hat{f} = \sup\{|T\hat{g}| : \hat{g} \in L_p(\nabla, m), |\hat{g}| \leq \hat{f}\} \leq \sup\{S|\hat{g}| : \hat{g} \in L_p(\nabla, m), |\hat{g}| \leq \hat{f}\} = S\hat{f}$$

we get

$$||T|\hat{f}|_p \leq |S\hat{f}|_p \leq \|S\||\hat{f}|_p$$

and hence

$$\||T|\| \leq \|S\|.$$

This completes the proof. □

Theorem 3.4 *If $A : L_p(\nabla, m) \to L_p(\nabla, m)$ is a dominated operator, and its dominant S is a contraction with $S\mathbf{1} \leq \mathbf{1}$, then for every $\omega \in \Omega$ there exists a dominated operator $A_\omega : L_p(\nabla_\omega, m_\omega) \to L_p(\nabla_\omega, m_\omega)$ such that*

$$A_\omega f(\omega) = (A\hat{f})(\omega) \quad a.e.$$

for all $\hat{f} \in L_p(\nabla, m)$.

Proof Since S is a contraction and $S\mathbf{1} \leq \mathbf{1}$, we obtain that $A(L_\infty(\nabla, m)) \subset L_\infty(\nabla, m)$.

Now we define a linear operator φ_ω from $\{\ell(\hat{f})(\omega) : \hat{f} \in L_\infty(\nabla, m)\}$ into $L_p(\nabla_\omega, m_\omega)$ by

$$\varphi_\omega(\ell(\hat{f})(\omega)) = \ell(A\hat{f})(\omega)$$

where ℓ is the vector lifting of $L_\infty(\nabla, m)$ associated with the lifting ρ.

From the dominability of A one gets

$$|\varphi(\omega)(\ell(\hat{f})(\omega))| = |\ell(A\hat{f})(\omega)| = \ell(|A\hat{f}|)(\omega) \leq \ell(S|\hat{f}|)(\omega) = S'_\omega(\ell(|\hat{f}|)(\omega)) = S'_\omega(|\ell(|\hat{f}|)(\omega)|)$$

for any positive $\hat{f} \in L_\infty(\nabla, m)$, where S'_ω is a positive contraction on $\{\ell(\hat{f})(\omega) : \hat{f} \in L_\infty(\nabla, m)\}$. This means that $\varphi(\omega)$ is a dominated operator on $\{\ell(\hat{f})(\omega) : \hat{f} \in L_\infty(\nabla, m)\}$.

From $|S\hat{f}|_p \leq |\hat{f}|_p$ we obtain

$$\|\ell(A\hat{f})(\omega)\|_{L_p(\nabla_\omega, m_\omega)} = \rho(|A\hat{f}|_p)(\omega) \leq \rho(|S\hat{f}|_p)(\omega) \leq \rho(|\hat{f}|_p)(\omega) = \|\ell(\hat{f})(\omega)\|_{L_p(\nabla_\omega, m_\omega)}$$

which implies that φ_ω and S'_ω are well defined and bounded. Moreover, S'_ω is positive (see Theorem 2.2). Due to the density of $\{\ell(\hat{f})(\omega) : \hat{f} \in L_\infty(\nabla, m)\}$ in $L_p(\nabla_\omega, m_\omega)$, we can extend φ_ω and S'_ω, respectively, to $L_p(\nabla_\omega, m_\omega)$. We respectively denote the extensions by A_ω and S_ω. One can see that A_ω is bounded, and S_ω is positive bounded.

From

$$|\varphi(\omega)(\ell(\hat{f})(\omega))| \leq S'_\omega(|\ell(\hat{f})(\omega)|)$$

for any $\hat{f} \in L_\infty(\nabla, m)$ one finds

$$|A_\omega(f(\omega))| \leq S_\omega(|f(\omega)|)$$

i.e. A_ω is dominated.

Repeating the argument of the proof of [10, Theorem 2.1], we can prove that

$$A_\omega f(\omega) = (A\hat{f})(\omega)$$

for almost all $\omega \in \Omega$ and for all $\hat{f} \in L_p(\nabla, m)$. This completes the proof. □

Theorem 3.5 *If $A : L_p(\nabla, m) \to L_p(\nabla, m)$ is a dominated operator, and its dominant S is a contraction with $S\mathbf{1} \leq \mathbf{1}$, then*

$$\||A|_\omega\|_{p,\omega} = \||A_\omega|\|_{p,\omega}$$

for almost all $\omega \in \Omega$, where $\|\cdot\|_{p,\omega}$ is the norm of an operator from $L_p(\nabla_\omega, m_\omega)$ to $L_p(\nabla_\omega, m_\omega)$.

Proof Due to $-|A| \leq A \leq |A|$ we have $-|A|_\omega \leq A_\omega \leq |A|_\omega$, which yields $|A_\omega| \leq |A|_\omega$ for almost all $\omega \in \Omega$. Hence, $\||A|_\omega\|_{p,\omega} \geq \||A_\omega|\|_{p,\omega}$ for almost all $\omega \in \Omega$.

Let $\{\pi_n\}$ be an increasing sequence in $\mathcal{P}$ such that $|A|\hat{f} = (bo) - \lim_{n\to\infty} A_{\pi_n}\hat{f}$, for $0 \leq \hat{f} \in L_p(\nabla, m)$.

One can see that

$$(A_{\pi_n}\hat{f})(\omega) = \sum_{i=1}^{m} |A(\chi_{B_i}\hat{f})|(\omega) = \sum_{i=1}^{m} |A_\omega(\chi_{B_i}(\omega)f)(\omega)| = A_{\omega,\pi_n}f(\omega) \tag{2}$$

for almost all $\omega \in \Omega$.

Now using

$$|A|\hat{f} = (bo) - \lim_{n\to\infty} A_{\pi_n}\hat{f} \text{ in } L_p(\nabla, m),$$

with (2) we obtain $|A_{\pi_n}\hat{f}|_p \stackrel{(o)}{\to} \big||A|\hat{f}\big|_p$ or $|A_{\pi_n}\hat{f}|_p(\omega) \to \big||A|\hat{f}\big|_p(\omega)$ for almost all $\omega \in \Omega$. Hence,

$$\|A_{\pi_n,\omega}f(\omega)\|_{L_p(\nabla_\omega,m_\omega)} \to \big\||A|_\omega f(\omega)\big\|_{L_p(\nabla_\omega,m_\omega)}$$

for almost all $\omega \in \Omega$.

On the other hand, one has

$$\lim_{n\to\infty} \|A_{\pi_n,\omega}f(\omega)\|_{L_p(\nabla_\omega,m_\omega)} \leq \big\||A_\omega| f(\omega)\big\|_{L_p(\nabla_\omega,m_\omega)}$$

for almost all $\omega \in \Omega$. This means that

$$\big\||A|_\omega f(\omega)\big\|_{L_p(\nabla_\omega,m_\omega)} \leq \big\||A_\omega| f(\omega)\big\|_{L_p(\nabla_\omega,m_\omega)}$$

or

$$\big\||A|_\omega\big\|_{p,\omega} \leq \big\||A_\omega|\big\|_{p,\omega}$$

for almost all $\omega \in \Omega$. Hence,

$$\big\||A|_\omega\big\|_{p,\omega} = \big\||A_\omega|\big\|_{p,\omega}$$

for almost all $\omega \in \Omega$. This completes the proof. □

4. The strong "zero-two" law

In this section we prove an analog of the strong "zero-two" law for positive contractions in the Banach–Kantorovich L_p-lattices. Before the formulation of the main result, we need some auxiliary results.

Proposition 4.1 *Let $T, S : L_p(\nabla, m) \to L_p(\nabla, m)$ be two positive linear contractions such that $T\mathbf{1} \leq \mathbf{1}$, $S\mathbf{1} \leq \mathbf{1}$. Then*

$$\big\||T_\omega - S_\omega|\big\|_{p,\omega} \geq \big\||T - S|\big\|(\omega), \quad a.e.$$

Here $|\cdot|$ means the modulus of an operator.

Proof Due to $(T - S)(\hat{f}) \leq T(\hat{f})$ for any positive $\hat{f} \in L_p(\nabla, m)$, one gets

$$|(T - S)(\hat{f})| \leq T(|\hat{f}|)$$

for any $\hat{f} \in L_p(\nabla, m)$. Hence, $T - S$ is dominated. Since T is a contraction and $T\mathbf{1} \leq \mathbf{1}$ by Theorem 3.5, we obtain $\big\||T - S|_\omega\big\|_{p,\omega} = \big\||T_\omega - S_\omega|\big\|_{p,\omega}$ for almost all $\omega \in \Omega$. By [9, Proposition 2] for any $\varepsilon > 0$ there exists $\hat{f} \in L_p(\nabla, m)$ with $|\hat{f}|_p = \mathbf{1}$ such that

$$\big\||T - S|\big\| - \varepsilon\mathbf{1} \leq \big||T - S|\hat{f}\big|_p.$$

Then

$$\begin{aligned}\big\||T-S|\big\|(\omega)-\varepsilon\mathbf{1} &\leq \big|T-S|\hat{f}\big|_p(\omega)=\|(|T-S|\hat{f})(\omega)\|_{L_p(\nabla_\omega,m_\omega)}\\ &= \||T-S|_\omega f(\omega)\|_{L_p(\nabla_\omega,m_\omega)} \leq \big\||T-S|_\omega\big\|_{p,\omega}\\ &= \big\||T_\omega-S_\omega|\big\|_{p,\omega}\end{aligned}$$

for almost all $\omega\in\Omega$. The arbitrariness of $\varepsilon>0$ implies the statement. □

Corollary 4.2 *Let $T,S:L_p(\nabla,m)\to L_p(\nabla,m)$ be two positive linear contractions such that $T\mathbf{1}\leq\mathbf{1}$, $S\mathbf{1}\leq\mathbf{1}$. Then*

$$\big\||T_\omega-S_\omega|\big\|_{p,\omega}=\big\||T-S|\big\|(\omega),\quad a.e.$$

The proof follows from [10, Proposition 3.2] and Proposition 4.1.

The next theorem is the main result of the present paper.

Theorem 4.3 *Let $T:L_p(\nabla,m)\to L_p(\nabla,m)$ be a positive linear contraction such that $T\mathbf{1}\leq\mathbf{1}$. If one has $\big\||T^{m+1}-T^m|\big\|<2\cdot\mathbf{1}$ for some $m\in\mathbb{N}\cup\{0\}$, then*

$$(o)-\lim_{n\to\infty}\big\||T^{n+1}-T^n|\big\|=0.$$

Proof From Corollary 4.2 it follows that

$$\big\||T_\omega^{m+1}-T_\omega^m|\big\|_{p,\omega}=\big\||T^{m+1}-T^m|\big\|(\omega),\quad \text{a.e.}$$

on Ω. Therefore, due to $\big\||T^{m+1}-T^m|\big\|<2\cdot\mathbf{1}$ for some $m\in\mathbb{N}\cup\{0\}$ we find $\big\||T_\omega^{m+1}-T_\omega^m|\big\|_{p,\omega}<2$ for almost all $\omega\in\Omega$. According to Theorem 2.2 we conclude that T_ω is a positive contraction on $L_p(\nabla_\omega,m_\omega)$. Hence, the contraction T_ω satisfies the conditions of Theorem 1.2 for almost all $\omega\in\Omega$, which yields that

$$\lim_{n\to\infty}\big\||T_\omega^{n+1}-T_\omega^n|\big\|=0$$

for almost all $\omega\in\Omega$. By again using Corollary 4.2, we obtain

$$\lim_{n\to\infty}\big\||T^{n+1}-T^n|\big\|(\omega)=0$$

for almost all $\omega\in\Omega$. Therefore,

$$(o)-\lim_{n\to\infty}\big\||T^{n+1}-T^n|\big\|=0.$$

This completes the proof. □

5. Dominated contractions

In this section we illustrate an application of the description of dominated operators, which may be used in the generalized "zero-two" law for positive contractions of Banach–Kantorovich L_1-lattices.

The following theorem is the vector-valued analog of [20, Theorem 2.1].

Theorem 5.1 *Let $T_1, T_2 : L_1(\nabla, m) \to L_p(\nabla, m)$ be positive linear operators with dominants S_1, S_1 correspondingly such that S_1, S_2 are contractions with $S_i\mathbf{1} \leq \mathbf{1}$, $i = 1, 2$ and $S_1S_2 = S_2S_1$. If there is an $n_0 \in \mathbb{N}$ such that $\|S_1S_2^{n_0} - T_1T_2^{n_0}\| < \mathbf{1}$, then*

$$\|S_1S_2^n - T_1T_2^n\| < \mathbf{1}$$

for every $n \geq n_0$.

Proof For every $\hat{f} \in L_1(\nabla, m)$ by Theorem 3.3 we infer that $(T_i\hat{f})(\omega) = (T_i)_\omega(f(\omega))$ and according to Theorem 2.2 $(S_i\hat{f})(\omega) = (S_i)_\omega(f(\omega))$ for almost all $\omega \in \Omega$. Obviously one has $(T_i)_\omega \leq (S_i)_\omega$ for almost all $\omega \in \Omega$ (see the proof of Theorem 3.3). From the assumption $S_1S_2 = S_2S_1$ we get $(S_1)_\omega(S_2)_\omega = (S_2)_\omega(S_1)_\omega$ for almost all $\omega \in \Omega$. It follows from Corollary 4.2 that

$$\|(S_1)_\omega(S_2)_\omega^{n_0} - (T_1)_\omega(T_2)_\omega^{n_0}\|_{1,\omega} = \|S_1S_2^{n_0} - T_1T_2^{n_0}\|(\omega).$$

The last equality with $\|S_1S_2^{n_0} - T_1T_2^{n_0}\| < \mathbf{1}$, for some $n_0 \in \mathbb{N}$, implies that

$$\|(S_1)_\omega(S_2)_\omega^{n_0} - (T_1)_\omega(T_2)_\omega^{n_0}\|_{1,\omega} < 1$$

for almost all $\omega \in \Omega$. Due to [20, Theorem 2.1] we then obtain that

$$\|(S_1)_\omega(S_2)_\omega^{n} - (T_1)_\omega(T_2)_\omega^{n}\|_{1,\omega} < 1$$

for every $n \geq n_0$ and for almost all $\omega \in \Omega$. Hence,

$$\|S_1S_2^n - T_1T_2^n\|(\omega) = \|(S_1)_\omega(S_2)_\omega^{n} - (T_1)_\omega(T_2)_\omega^{n}\|_{1,\omega} < 1$$

for every $n \geq n_0$ and for almost all $\omega \in \Omega$. This means that

$$\|S_1S_2^n - T_1T_2^n\| < \mathbf{1}$$

for every $n \geq n_0$, which completes the proof. □

From this theorem we immediately get Zaharopol's vector-valued result (see [27]) if one takes $S_1 = T_1 = id$. Namely, we have the following:

Corollary 5.2 *Let $T : L_1(\nabla, m) \to L_1(\nabla, m)$ be a positive linear operator with dominant S, such that S is a contraction with $S\mathbf{1} \leq \mathbf{1}$. If there is an $n_0 \in \mathbb{N}$ such that $\|S^{n_0} - T^{n_0}\| < \mathbf{1}$, then*

$$\|S^n - T^n\| < \mathbf{1}$$

for every $n \geq n_0$.

Remark 5.1 *We remark that an abstract version of the last result was given in [21]. Using the same argument of Theorem 5.1 and the results of [6, 12, 17], we can extend the mentioned abstract result to the vector-valued setting.*

Acknowledgments

The first author acknowledges MOE Grant FRGS13-071-0312. The second author acknowledges MOE grant FRGS14-135-0376 and the Junior Associate Scheme of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy. Finally, the authors would like to thank the referee for his useful suggestions, which improved the content of this paper.

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