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Invariant distributions and holomorphic vector fields in paracontact geometry

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Abstract: Having as a model the metric contact case of V. Brînzănescu; R. Slobodeanu, we study two similar subjects in the paracontact (metric) geometry: a) distributions that are invariant with respect to the structure endomorphism φ; b) the class of vector fields of holomorphic type. As examples we consider both the 3-dimensional case and the general dimensional case through a Heisenberg-type structure inspired also by contact geometry.

Key words: Paracontact metric manifold, invariant distribution, paracontact-holomorphic vector field

1. Introduction

Paracontact geometry [7, 13] appears as a natural counterpart of the contact geometry in [9]. Compared with the huge literature in (metric) contact geometry, it seems that new studies are necessary in almost paracontact geometry; a very interesting paper connecting these fields is [5]. The present work is another step in this direction, more precisely from the point of view of some subjects of [4].

The first section deals with the distributions $\mathcal{V}$, which are invariant with respect to the structure endomorphism φ, one trivial example being the canonical distribution $\mathcal{D}$ provided by the annihilator of the paracontact 1-form η. As in the contact case, the characteristic vector field ξ must belong to $\mathcal{V}$ or $\mathcal{V}^{\perp}$. Two important tools in this study are the second fundamental form and the integrability tensor field, both satisfying important (skew)-commutation formulas in the paracontact metric and para-Sasakian geometries. Let us remark that another important class of paracontact geometries, namely the para-Kenmotsu case, was studied recently in [2] from the same points of view.

The second subject of the present paper is the class of paracontact-holomorphic vector fields that form a Lie subalgebra on a normal almost paracontact manifold; recently this type of vector fields was studied as providing the potential vector field of Ricci solitons in (3-dimensional) almost paracontact geometries in [1]. These vector fields vanish a $\bar{\partial}$-operator expressed in terms of Levi-Civita as well as the canonical paracontact connection from [14]. We also give a relationship between the paracontact-holomorphicity on the manifold M and the holomorphicity on the cone manifold $\mathcal{C}(M)$. The last result gives a characterization of paracontact-holomorphic vector fields X in terms of para-Cauchy–Riemann equations for the components of X in a paracontact-holomorphic frame.

Two types of examples are examined: firstly in dimension 3 and secondly in arbitrary dimension following the Heisenberg-type example of contact metric geometry from [3, p. 60–61]. For the former case we compute the

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fundamental functions α, β occurring in the Levi-Civita differential of φ while for the latter we use an adapted frame of $\mathcal{D}$. Let us remark that our Heisenberg-type example 2.11 is different from the hyperbolic Heisenberg group of [8, p. 85]. For the 3-dimensional example we point out the vanishing of the mixed sectional curvature of the pair $(\mathcal{D}, \xi)$ of invariant distributions in a short Appendix.

2. Invariant distributions on almost paracontact metric manifolds

Let M be a $(2n+1)$-dimensional smooth manifold, φ a $(1,1)$-tensor field called the *structure endomorphism*, ξ a vector field called the *characteristic vector field*, η a 1-form called the *paracontact form*, and g a pseudo-Riemannian metric on M of signature $(n+1, n)$. In this case, we say that (φ, ξ, η, g) defines an *almost paracontact metric structure* on M if [14]:

$$\varphi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y). \tag{2.1}$$

From the definition it follows $\varphi(\xi) = 0$, $\eta \circ \varphi = 0$, $\eta(X) = g(X, \xi)$, $g(\xi, \xi) = 1$ and the fact that φ is g-skew-symmetric: $g(\varphi X, Y) = -g(\varphi Y, X)$. The associated 2-form $\omega(X, Y) := g(X, \varphi Y)$ is skew-symmetric and is called *the fundamental form* of *the almost metric paracontact manifold* $(M, \varphi, \xi, \eta, g)$.

The $2n$-dimensional distribution $\mathcal{D} := \ker \eta$ is called the *canonical distribution* associated to the almost paracontact metric structure (φ, ξ, η, g). The vector field ξ is g-orthogonal to $\mathcal{D}$ and we have the orthogonal splitting of the tangent bundle $TM = \mathcal{D} \oplus span\{\xi\}$; let v_ξ and h_ξ be the corresponding projectors; thus $v_\xi(X) = X - \eta(X)\xi$.

We assume given a distribution $\mathcal{V}$ on M. The main hypothesis for our framework is the existence of a g-orthogonal complementary distribution $\mathcal{V}^\perp$. Let $\Gamma(\mathcal{V})$ be the $C^\infty(M)$-module of its sections. We denote with v and h the orthogonal projectors with respect to the decomposition $TM = \mathcal{V} \oplus \mathcal{V}^\perp$.

Inspired by [4] we introduce:

Definition 2.1 *The distribution $\mathcal{V}$ is called invariant if $\varphi(\mathcal{V}) \subseteq \mathcal{V}$, i.e. $h \circ \varphi \circ v = 0$.*

The first result provides an example and a characterization:

Proposition 2.2 *On $(M, \varphi, \xi, \eta, g)$ we have: i) $\mathcal{D}$ is an invariant distribution; ii) $\mathcal{V}$ is invariant if and only if $\mathcal{V}^\perp$ is invariant. Hence the invariance means $\varphi \circ v = v \circ \varphi$ respectively $\varphi \circ h = h \circ \varphi$.*

Proof i) From $\eta \circ \varphi = 0$. ii) From the skew-symmetry of φ. □

With the same proof as that of Lemma 2.1. from [4, p. 194] we have:

Proposition 2.3 *If $\mathcal{V}$ is an invariant distribution then $\xi \in \Gamma(\mathcal{V})$ or $\xi \in \Gamma(\mathcal{V}^\perp)$. Moreover, if $\xi \in \Gamma(\mathcal{V})$ then $\mathcal{V}^\perp \subseteq \mathcal{D}$.*

We consider a particular class of almost paracontact metric geometry after [14, p. 39]:

Proposition 2.4 *The almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ is a paracontact metric manifold if $\omega = d\eta$ where d is given by:*

$$2d\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]) \tag{2.2}$$

for all vector fields X, Y.

The same proof as that of Proposition 2.1 from [4, p. 195] yields:

Proposition 2.5 *Suppose that $\mathcal{V}$ is an invariant distribution in a paracontact metric manifold satisfying one of the following conditions:*
(i) $\dim(\mathcal{V}) = 2k+1$ *with* $k \leq n$,
(ii) $\mathcal{V}$ *is integrable.*
Then $\xi \in \Gamma(\mathcal{V})$. In particular, an integrable invariant distribution must be odd-dimensional.

Recall now two important tensor fields associated to a given distribution:

Definition 2.6 *If $\mathcal{V}$ is a distribution on the Riemannian manifold (M, g) then:*
i) its second fundamental form is $B^{\mathcal{V}} : \Gamma(\mathcal{V}) \times \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}^{\perp})$ given by:

$$B^{\mathcal{V}}(X,Y) = \frac{1}{2}h(\nabla_X Y + \nabla_Y X) \tag{2.3}$$

where ∇ is the Levi-Civita connection of g;
ii) its integrability tensor is $B^{\mathcal{V}} : \Gamma(\mathcal{V}) \times \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}^{\perp})$ given by:

$$I^{\mathcal{V}}(X,Y) = h([X,Y]). \tag{2.4}$$

For the class of paracontact metric structures we determine a relationship between the second fundamental form and the integrability tensor for invariant distributions transversally to the characteristic vector field:

Proposition 2.7 *Let $\mathcal{V}$ be an invariant distribution on the paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ such that $\xi \in \Gamma(\mathcal{V}^{\perp})$. If $X, Y \in \Gamma(\mathcal{V})$ then*

$$2\left[B^{\mathcal{V}}(\varphi X, Y) - B^{\mathcal{V}}(X, \varphi Y)\right] = \varphi \circ I^{\mathcal{V}}(\varphi X, \varphi Y) - \varphi \circ I^{\mathcal{V}}(X,Y). \tag{2.5}$$

In particular, for $\mathcal{V} = \mathcal{D}$ we have the symmetry

$$B^{\mathcal{D}}(\varphi X, Y) = B^{\mathcal{D}}(X, \varphi Y), \quad B^{\mathcal{D}}(\varphi X, \varphi Y) = B^{\mathcal{D}}(X,Y). \tag{2.6}$$

Proof From Lemma 2.7 of [14, p. 42] we have for all vector fields X, Y:

$$(\nabla_{\varphi X}\varphi)\varphi Y - (\nabla_X \varphi)Y = 2g(X,Y)\xi - (X - hX + \eta(X)\xi)\eta(Y) \tag{2.7}$$

where $h = \frac{1}{2}\mathcal{L}_\xi \varphi$. The Proposition 2.3 gives $\mathcal{V} \subseteq \mathcal{D}$ and then the second term in the right hand-side is zero. Hence

$$\nabla_{\varphi X}Y - \varphi(\nabla_{\varphi X}\varphi Y) - \nabla_X \varphi Y + \varphi(\nabla_X Y) = 2g(X,Y)\xi = \nabla_{\varphi Y}X - \varphi(\nabla_{\varphi Y}\varphi X) - \nabla_Y \varphi X + \varphi(\nabla_Y X)$$

gives

$$(\nabla_{\varphi X}Y + \nabla_Y \varphi X) - (\nabla_{\varphi Y}X + \nabla_X \varphi Y) = \varphi([\varphi X, \varphi Y] - [X,Y])$$

yielding

$$2\left(B^{\mathcal{V}}(\varphi X, Y) - B^{\mathcal{V}}(X, \varphi Y)\right) = h \circ \varphi([\varphi X, \varphi Y] - [X,Y]) \tag{2.8}$$

which is (2.5). For $\mathcal{V} = \mathcal{D}$ we take the g-inner product of (2.8) with ξ and use the g-skew-symmetry of φ and $\varphi(\xi) = 0$ to obtain (2.6_1). With Y replaced by φY in (2.6_1) it results (2.6_2). □

Let us study now the complementary case when $\xi \in \Gamma(\mathcal{V})$. We recall that a *para-Sasakian manifold* is a normal paracontact metric manifold; the normality means the integrability of the almost paracomplex structure J on the cone $\mathcal{C}(M) = M \times \mathbb{R}$:

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X + f\xi, \eta(X)\frac{d}{dt}\right). \tag{2.9}$$

A characterization of this case is given in [14, p. 42]

$$(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X \tag{2.10}$$

for all vector fields X, Y. In a para-Sasakian manifold we have

$$\nabla_X \xi = -\varphi X \tag{2.11}$$

which yields the commutation formula

$$\nabla_{\varphi X}\xi = \varphi(\nabla_X \xi) = -\varphi^2 X. \tag{2.12}$$

Proposition 2.8 *Let $\mathcal{V}$ be an invariant distribution with $\xi \in \Gamma(V)$ in a para-Sasakian manifold. Then for all $X, Y \in \Gamma(\mathcal{V})$ we have*

$$2\left[B^{\mathcal{V}}(X, \varphi Y) - \varphi \circ B^{\mathcal{V}}(X, Y)\right] = -\varphi \circ I^{\mathcal{V}}(X, \varphi Y) - \varphi \circ I^{\mathcal{V}}(X, Y). \tag{2.13}$$

In particular,

$$2B^{\mathcal{V}}(X, \xi) = -I^{\mathcal{V}}(X, \xi) \tag{2.14}$$

and if $\mathcal{V}$ is integrable then

$$B^{\mathcal{V}}(\varphi X, Y) = \varphi \circ B^{\mathcal{V}}(X, Y) = B^{\mathcal{V}}(X, \varphi Y). \tag{2.15}$$

Proof By using the relation (2.10) the left-hand side of (2.13) is

$$h(\nabla_X \varphi Y + \nabla_{\varphi Y} X - \varphi(\nabla_X Y) - \varphi(\nabla_Y X)) = h(\nabla_{\varphi Y} X - \varphi(\nabla_Y X)).$$

Now, using the metric character of ∇, the last term is $h(\nabla_X \varphi Y - [X, \varphi Y] - \varphi(\nabla_X Y) - \varphi([X, Y]))$ and we get the conclusion (2.13). With $Y = \xi$ in (2.13) we obtain (2.14) while (2.15) is a direct consequence of (2.13). □

Corollary 2.9 *Let N be an invariant submanifold of the para-Sasakian manifold $(M, \varphi, \xi, \eta, g)$ containing ξ and B its second fundamental form. Then for all $X, Y \in \Gamma(N)$ we have:*

$$B(X, \xi) = 0, \quad B(\varphi X, Y) = \varphi \circ B(X, Y) = B(X, \varphi Y). \tag{2.16}$$

We finish this section with some examples other than those of [8]:

Example 2.10 Suppose that $n = 1$. After [11, p. 379] we have

$$(\nabla_X \varphi)Y = g(\varphi(\nabla X\xi), Y)\xi - \eta(Y)\varphi(\nabla_X \xi) \tag{2.17}$$

and $(M, \varphi, \xi, \eta, g)$ is normal if and only if there exist smooth functions α, β on M such that

$$(\nabla_X \varphi)Y = \beta\left(g(X,Y)\xi - \eta(Y)X\right) + \alpha\left(g(\varphi X, Y)\xi - \eta(Y)\varphi(X)\right), \nabla_X \xi = \alpha(X - \eta(X)\xi) + \beta\varphi(X). \tag{2.18}$$

Hence, the para-Sasakian case is provided by $\alpha = 0$ and $\beta = -1$. $(M, \varphi, \xi, \eta, g)$ admits locally a frame $\{\xi, E, \varphi E\}$ with $g(E, E) = 1 = -g(\varphi E, \varphi E)$, which means that ξ and E are space-like vector fields while φE is a time-like vector field. We have $I^{\mathcal{D}}(E, \varphi E) = \eta([E, \varphi E])\xi$.

In order to handle a concrete example let N be an open connected subset of $\mathbb{R}^2$, (a, b) an open interval in $\mathbb{R}$, and let us consider the manifold $M = N \times (a, b)$. Let (x, y) be the coordinates on N induced from the Cartesian coordinates on $\mathbb{R}^2$ and let z be the coordinate on (a, b) induced from the Cartesian coordinate on $\mathbb{R}$. Thus (x, y, z) are the coordinates on M. Now we choose the functions

$$\omega_1, \omega_2 : N \to \mathbb{R}, \qquad \sigma, f : M \to \mathbb{R}^*_+, \tag{2.19}$$

and following the idea from [10] we define

$$g = \frac{1}{4}\begin{pmatrix} \omega_1^2 + \sigma e^{2f} & \omega_1\omega_2 & \omega_1 \\ \omega_1\omega_2 & \omega_2^2 - \sigma e^{2f} & \omega_2 \\ \omega_1 & \omega_2 & 1 \end{pmatrix} = \frac{1}{4}\sigma e^{2f}\left(dx^2 - dy^2\right) + \eta \otimes \eta, \eta = \frac{1}{2}(dz + \omega_1 dx + \omega_2 dy), \tag{2.20}$$

$$\xi = 2\frac{\partial}{\partial z}, \quad \varphi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -\omega_2 & -\omega_1 & 0 \end{pmatrix}. \tag{2.21}$$

It follows an almost paracontact metric manifold with

$$E = \frac{2e^{-f}}{\sqrt{\sigma}}\left(\frac{\partial}{\partial x} - \omega_1\frac{\partial}{\partial z}\right), \quad \varphi E = \frac{2e^{-f}}{\sqrt{\sigma}}\left(\frac{\partial}{\partial y} - \omega_2\frac{\partial}{\partial z}\right). \tag{2.22}$$

From

$$\begin{cases} [E, \xi] = \frac{2f_z\sigma + \sigma_z}{\sigma}E, \quad [\varphi E, \xi] = \frac{2f_z\sigma + \sigma_z}{\sigma}\varphi E \\ [E, \varphi E] = \frac{\sqrt{\sigma}}{e^{-f}}\left[E(\frac{e^{-f}}{\sqrt{\sigma}})\varphi E - \varphi E(\frac{e^{-f}}{\sqrt{\sigma}})E\right] + \frac{2e^{-2f}}{\sigma}\left(\frac{\partial\omega_1}{\partial y} - \frac{\partial\omega_2}{\partial x}\right)\xi \end{cases} \tag{2.23}$$

it follows that $\mathcal{D}$ is integrable if and only if the 1-form $\omega_1 dx + \omega_2 dy$ is closed; hence η is closed. We have the Levi-Civita connection

$$\begin{cases} \nabla_E E = -\frac{\sqrt{\sigma}}{e^{-f}}E(\frac{e^{-f}}{\sqrt{\sigma}})\varphi E + \frac{2f_z\sigma + \sigma_z}{\sigma}\xi \\ \nabla_E \varphi E = -\frac{4\sqrt{\sigma}}{e^{-f}}\varphi E(\frac{e^{-f}}{\sqrt{\sigma}})E + \frac{e^{-2f}}{\sigma}\left(\frac{\partial\omega_1}{\partial y} - \frac{\partial\omega_2}{\partial x}\right)\xi \\ \nabla_E \xi = \frac{2f_z\sigma + \sigma_z}{\sigma}E + \frac{e^{-2f}}{\sigma}\left(\frac{\partial\omega_1}{\partial y} - \frac{\partial\omega_2}{\partial x}\right)\varphi E \end{cases} \tag{2.24}$$

$$\begin{cases} \nabla_{\varphi E} E = -\frac{4\sqrt{\sigma}}{e^{-f}}E(\frac{e^{-f}}{\sqrt{\sigma}})\varphi E - \frac{e^{-2f}}{\sigma}\left(\frac{\partial\omega_1}{\partial y} - \frac{\partial\omega_2}{\partial x}\right)\xi \\ \nabla_{\varphi E} \varphi E = -\frac{4\sqrt{\sigma}}{e^{-f}}\varphi E(\frac{e^{-f}}{\sqrt{\sigma}})E + \frac{2f_z\sigma + \sigma_z}{\sigma}\xi \\ \nabla_{\varphi E} \xi = \frac{e^{-2f}}{\sigma}\left(\frac{\partial\omega_1}{\partial y} - \frac{\partial\omega_2}{\partial x}\right)E + \frac{2f_z\sigma + \sigma_z}{\sigma}\varphi E \end{cases} \tag{2.25}$$

$$\nabla_\xi E = \frac{e^{-2f}}{\sigma}\left(\frac{\partial\omega_1}{\partial y} - \frac{\partial\omega_2}{\partial x}\right)\varphi E, \quad \nabla_\xi \varphi E = \frac{e^{-2f}}{\sigma}\left(\frac{\partial\omega_1}{\partial y} - \frac{\partial\omega_2}{\partial x}\right)E, \quad \nabla_\xi \xi = 0 \tag{2.26}$$

and then

$$\alpha = 2f_z + \frac{\sigma_z}{\sigma}, \quad \beta = \frac{e^{-2f}}{\sigma}\left(\frac{\partial \omega_1}{\partial y} - \frac{\partial \omega_2}{\partial x}\right). \tag{2.27}$$

Hence, $(M, \varphi, \xi, \eta, g)$ is a para-Sasakian manifold if and only if

$$\sigma e^{2f} = \sigma e^{2f}(x, y), \quad \frac{\partial \omega_1}{\partial y} - \frac{\partial \omega_2}{\partial x} = -\sigma e^{2f}. \tag{2.28}$$

The first relation expresses the normality of the paracontact structure while the second condition means the metrical condition of the Definition 2.4 and yields the nonintegrability of $\mathcal{D}$ since $I^{\mathcal{D}}(E, \varphi E) = -2\xi$. Some cases when both equations hold are: i) $\omega_1 = -y$, $\omega_2 = 0 = f$, $\sigma = 1$; ii) $\omega_1 = -y$, $\omega_2 = x$, $\sigma = 2$, $f = 0$.

Other examples of 3-dimensional (almost) paracontact manifolds appear in [6, 11, 12].

Example 2.11 On $M = \mathbb{R}^{2n+1}$ with the splitting $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ we consider a Heisenberg-type structure inspired by the contact metric example from [3, p. 60-61]:

$$g = \frac{1}{4}\begin{pmatrix} \delta_{ij} + y^i y^j & 0 & -y^i \\ 0 & -\delta_{ij} & 0 \\ -y^j & 0 & 1 \end{pmatrix}, \varphi = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ \delta_{ij} & 0 & 0 \\ 0 & y^j & 0 \end{pmatrix}, \xi = 2\frac{\partial}{\partial z}, \eta = \frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i). \tag{2.29}$$

It follows that $(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g)$ is a paracontact metric manifold with

$$\mathcal{D} = span\left\{A_i = \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z}, B_i = \frac{\partial}{\partial y^i}; 1 \leq i \leq n\right\}. \tag{2.30}$$

Two classes of invariant distributions are indexed by $k \in \{1, ..., n-1\}$:

$$\mathcal{V}_k^{even} = span\{A_\alpha, B_\alpha; 1 \leq \alpha \leq k\}, \mathcal{V}_k^{odd} = \mathcal{V}_k^{even} \cup \{\xi\}. \tag{2.31}$$

Let us remark that for $n = 1$ we recover the previous Example with: $\omega_1 = -y$, $\omega_2 = 0 = f$, $\sigma = 1$. It is a para-Sasakian manifold with nonintegrable $\mathcal{D}$: $[E, \xi] = [\varphi E, \xi] = 0, \quad [E, \varphi E] = -2\xi$. The sectional curvature of the plane spanned by E and φE is

$$pK = K^M(E, \varphi E) = g(R(E, \varphi E)\varphi E, E) = g(\nabla_{\varphi E}\xi + 2\nabla_\xi \varphi E, E) = g(-E - 2E, E) = -3 \tag{2.32}$$

similar to the metric contact case.

3. Infinitesimal paracontact-holomorphicity

Definition 3.1 *The vector field $X \in \Gamma(TM)$ is called paracontact-holomorphic if*

$$v_\xi \circ \mathcal{L}_X \varphi = 0. \tag{3.1}$$

Let $\mathfrak{phol}(M)$ be the set of all paracontact-holomorphic vector fields. The distribution $\mathcal{V}$ is paracontact-holomorphic if its sections are elements of $\mathfrak{phol}(M)$.

The condition (3.1) says that for all vector fields Y we have that $(\mathcal{L}_X \varphi)Y$ is collinear with ξ; let us denote $\alpha_X(Y)$ the collinearity factor. We have

$$\alpha_X(Y) = g([X, \varphi Y] - \varphi([X, Y]), \xi) = \eta([X, \varphi Y]). \tag{3.2}$$

The next result shows the invariance of the above defined holomorphicity and its proof is exactly as in [4]:

Proposition 3.2 *Let X be a paracontact-holomorphic vector field on the normal almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$. Then φX is also a paracontact-holomorphic vector field.*

Remarks 3.3 i) Fix X a paracontact-holomorphic vector field. Then computing $\alpha_X(\xi)$ with (3.2) we get

$$\alpha_X(\xi) = 0 \tag{3.3}$$

which means that $[X, \xi]$ is collinear with ξ, i.e. $v_\xi([X, \xi]) = 0$.
ii) The vanishing of the tensor field $N^{(3)} = \mathcal{L}_\xi \varphi$ means that ξ is a paracontact-holomorphic vector field with $\alpha_\xi = 0$.
iii) The paracontact-holomorphicity of a fixed X implies for every vector field Y

$$\mathcal{L}_X Y = \eta(\mathcal{L}_X Y)\xi + \varphi(\mathcal{L}_X \varphi Y), \quad \mathcal{L}_X \varphi Y = \alpha_X(Y)\xi + \varphi([X, Y]). \tag{3.4}$$

In both relations, the first term in the right-hand side belongs to *span*ξ while the second belongs to $\mathcal{D}$.

By using these remarks we get:

Proposition 3.4 *If $(M, \varphi, \xi, \eta, g)$ is a normal almost paracontact manifold then $\mathfrak{phol}(M)$ is a Lie subalgebra in the Lie algebra of vector fields of M.*

Proof Let X and Y be paracontact-holomorphic vector fields and Z an arbitrary vector field. Then

$$(\mathcal{L}_{[X,Y]}\varphi)Z = [X, (\mathcal{L}_Y \varphi)Z] - (\mathcal{L}_Y)([X, Z]) - [Y, (\mathcal{L}_X \varphi)Z] + (\mathcal{L}_X \varphi)([Y, Z]). \tag{3.5}$$

From the property of X, Y we have that the second and fourth terms are collinear with ξ. Also

$$[X, (\mathcal{L}_Y \varphi)Z] = X(\alpha_Y(Z))\xi - \alpha_Y(Z)[X, \xi]$$

and the first relation (3.4) gives that this expression is collinear with ξ. The same fact holds for the third term of (3.5). □

As in the contact case we can express the paracontact-holomorphicity by the vanishing of some $\bar{\partial}$-operator. More precisely, we define the map $\bar{\partial} : \Gamma(TM) \to End(TM)$ given by

$$\bar{\partial}(X)(Y) = \varphi\left(\nabla_Y X - \varphi(\nabla_{\varphi Y} X) + \varphi(\nabla_X \varphi)Y\right). \tag{3.6}$$

Thus, X is a paracontact-holomorphic vector field if and only if $\bar{\partial}(X) = 0$. For a general vector field X, if $(M, \varphi, \xi, \eta, g)$ is a para-Sasakian manifold then

$$\bar{\partial}(X)(\xi) = \varphi([\xi, X]) \tag{3.7}$$

and for $Y \in \mathcal{D}$ we have

$$\bar{\partial}(X)(Y) = \varphi\left(\nabla_Y X - \varphi(\nabla_{\varphi Y} X)\right). \tag{3.8}$$

If $n = 1$ then the expression (3.6) reduces to

$$\bar{\partial}(X)(Y) = \varphi\left(\nabla_Y X - \varphi(\nabla_{\varphi Y} X) - \eta(Y)(\alpha X + \beta \varphi X)\right). \tag{3.9}$$

For the general n and using the canonical paracontact connection $\tilde{\nabla}$ of [14, p. 49] we have

$$\bar{\partial}(X)(Y) = \varphi\left(\tilde{\nabla}_Y X - \varphi(\tilde{\nabla}_{\varphi Y} X) + \varphi(\tilde{\nabla}_X \varphi)Y + 2\eta(X)(\varphi N^{(3)}Y - \varphi^2 N^{(3)}\varphi Y) - \eta(Y)\varphi N^{(3)}X\right). \quad (3.6can)$$

Recall now that on the cone $\mathcal{C}(M)$ we have

$$[(X, f\frac{d}{dt}), (Y, g\frac{d}{dt})] = \left([X,Y], (X(g) - Y(f) + f\frac{dg}{dt} - g\frac{df}{dt})\frac{d}{dt}\right) \quad (3.10)$$

which yields:

Proposition 3.5 *Fix $X \in \Gamma(TM)$ and $f \in C^\infty(M \times \mathbb{R})$. Then $(X, f\frac{d}{dt})$ is a paraholomorphic vector field on the cone $\mathcal{C}(M)$ if and only if the following three conditions hold:*
i) $(\mathcal{L}_X\varphi)Y = -Y(f)\xi$,
ii) $(\mathcal{L}_X\eta)(Y) = \varphi Y(f) + \eta(Y)\frac{df}{dt}$,
iii) $\mathcal{L}_X\xi = -\frac{df}{dt}\xi$,
where $Y \in \Gamma(TM)$ is arbitrary. Consequently, if $(X, f\frac{d}{dt})$ is a paraholomorphic vector field on $\mathcal{C}(M)$ then X is paracontact-holomorphic vector field on M and f is a first integral if ξ.

Proof By using (3.10) we get with respect to J of (2.9)

$$(\mathcal{L}_{(X, f\frac{d}{dt})}J)(Y, 0) = \left((\mathcal{L}_X\varphi)Y + Y(f)\xi, (X(\eta(Y)) - \varphi Y(g) - \eta(Y)\frac{df}{dt} - \eta([X,Y]))\frac{d}{dt}\right) \quad (3.11)$$

$$(\mathcal{L}_{(X, f\frac{d}{dt})}J)(0, \frac{d}{dt}) = \left([X,\xi] + \frac{df}{dt}, -\xi(f)\frac{d}{dt}\right). \quad (3.12)$$

The paraholomorphicity of $(X, f\frac{d}{dt})$ means the vanishing of the above left-hand sides and this is equivalent with f being first integral of ξ and the relations i)-iii). However, with $Y = \xi$ in i) and using iii) it follows that $\xi(f) = 0$. The equation i) means that X is a paracontact-holomorphic vector field. □

Corollary 3.6 *The paracontact-holomorphic vector fields on M, which come about by projection of the paraholomorphic fields on $\mathcal{C}(M)$, form a Lie subalgebra of $\mathfrak{phol}(M)$, denoted by $\mathfrak{phol}_{pr}(M)$. They are paracontact-holomorphic fields X with two additional properties:*
a) The 1-form α_X is exact: there exists a smooth function f on M such that $\alpha_X = d(-f)$,
b) $\eta([X,\xi])$ is a (locally) constant, i.e. constant on any connected component of M.

Proof a) it results by applying η to i); more precisely $Y(-f) = \eta([X, \varphi Y])$ for all vector fields Y. By applying η to iii) we get $\frac{df}{dt} = \eta([\xi, X])$ and then around a point $p_0 \in M$ we have the following expression of f:

$$f(p,t) = \eta([\xi, X])(p)t - F(p). \quad (3.13)$$

Plugging this expression in a) we get: $Y(F) + Y(\eta([X,\xi]))t = \eta([X, \varphi Y])$ and it results in b). □

Corollary 3.7 *On a normal almost paracontact metric manifold* $(M, \varphi, \xi, \eta, g)$ *we have:*
iv) $a\xi$ *is a contact-holomorphic vector field, for any function* $a \in M$*; so* $a\xi \in \mathfrak{phol}(M)$ *but it is not necessarily the case that* $a\xi \in \mathfrak{phol}_{pr}(M)$*,*
v) $(\xi, c\frac{d}{dt})$ *is a holomorphic vector field on* $\mathcal{C}(M)$ *if and only if* c *is a constant.*

Proof The first part is a direct consequence of

$$(\mathcal{L}_{a\xi}\varphi)Y = a(\mathcal{L}_{\xi}\varphi)Y - \varphi Y(a)\xi. \tag{3.14}$$

Let us remark that the normality implies that $\alpha_{a\xi}(Y) = -\varphi Y(a)$. For the second part, from iii) of Proposition 3.5 it results that $\frac{dc}{dt} = 0$ while i) gives that $Y(c) = 0$ for all vector fields Y. □

Proposition 3.8 *Let* $(M, \varphi, \xi, \eta, g)$ *be a paracontact metric manifold. Then any two of the following conditions imply the third one:*
(i) $(\mathcal{L}_X g)(Y, Z) = 0$ *for all* $Y, Z \in \Gamma(\mathcal{D})$*,*
(ii) $i_X d\eta$ *is a closed form,*
(iii) X *is a paracontact-holomorphic vector field.*

Proof It is a direct consequence of the formula

$$(\mathcal{L}_X g)(Y, \varphi Z) = (\mathcal{L}_X d\eta)(Y, Z) - g(Y, (\mathcal{L}_X\varphi)Z) \tag{3.15}$$

for all vector fields Y, Z. □

Example 3.9 *Returning to Example 2.11, let*

$$X = \alpha^i A_i + \beta^i B_i + \gamma\xi = \alpha^i \frac{\partial}{\partial x^i} + \beta^i \frac{\partial}{\partial y^i} + (2\gamma + (\sum_{j=1}^{n} y^j \alpha^j))\frac{\partial}{\partial z}. \tag{3.16}$$

Then $X \in \mathfrak{phol}(M)$ *if and only if the coefficients* α *and* β *satisfy the para-Cauchy–Riemann equations with respect to the variables* (x, y) *and are constant with respect to* z*:*

$$\begin{cases} \frac{\partial\alpha^i}{\partial x^j} = \frac{\partial\beta^i}{\partial y^j}, \quad \frac{\partial\alpha^i}{\partial y^j} = \frac{\partial\beta^i}{\partial x^j} \\ \frac{\partial\alpha^i}{\partial z} = \frac{\partial\beta^i}{\partial z} = 0. \end{cases} \tag{3.17}$$

The following analogy with the contact case shows that these computations have a general nature:

Proposition 3.10 *On a normal almost paracontact metric manifold there always exist (local) adapted frames* $(E_i, \varphi E_i, \xi)$ *consisting of contact-holomorphic vector fields. If the vector field* X *has the expression* $X = \alpha^i E_i + \beta^i \varphi E_i + \gamma\xi$ *then* X *is a paracontact-holomorphic vector field if and only if the coefficients* α, β *satisfy the generalized para-Cauchy–Riemann equations:*

$$E_j(\alpha^i) = \varphi E_j(\beta^i), \quad \varphi E_j(\alpha^i) = E_j(\beta^i) \tag{3.18}$$

and are first integrals of ξ*.*

4. Appendix: The mixed sectional curvature

The main result of [4] is the Bochner-type Theorem 5.1 stated on page 206. The technical ingredient of this result is *the mixed sectional curvature*:

$$s_{mix}(\mathcal{V}, \mathcal{V}^{\perp}) = \sum K^M(e_i \wedge f_\alpha) \tag{a.1}$$

where $\{e_i\}$ respectively $\{f_\alpha\}$ are local orthonormal frames for the given distribution. The cited Bochner-type result deals with an invariant distribution $\mathcal{V}$ of dimension $2p+1$ in the Sasakian case and concerns the case $s_{mix} \geq 2(n-p)$.

The aim of this short Appendix is to compute this quantity for our example 2.10:

$$s_{mix}(\mathcal{D}, \xi) = K^M(E \wedge \xi) + K^M(\varphi E \wedge \xi) = g(R(E, \xi)\xi, E) + g(R(\varphi E, \xi)\xi, \varphi E) \tag{a.2}$$

Since E is a space-like vector field while φE is a time-like one, a direct computation yields the vanishing: $s_{mix}(\mathcal{D}, \xi) = 0$.

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