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On zeros of certain Dirichlet polynomials

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Abstract: In this article we establish the zero-free region of certain Dirichlet polynomials $L_{F,X}$ arising in approximate functional equation for functions in the Selberg class and we prove an asymptotic formula for the number of zeros of $L_{F,X}$.

Key words: Selberg class, Dirichlet polynomial, approximate functional equation, distribution of zeros

1. Introduction

In view of plenty of examples of Dirichlet series in arithmetic it might be reasonable to ask for a classification and to search for common patterns in their analytic properties. There were several notable attempts to define classes of relevant Dirichlet series (e.g. [9, 10]); however, these classes were in some sense lacking algebraic structure. In 1989, Selberg [14] defined a general class of Dirichlet series having an Euler product, analytic continuation, and a functional equation of Riemann type (plus some side conditions), and formulated some fundamental conjectures concerning them. Especially these conjectures give this class of Dirichlet series a certain structure that applies to central problems in number theory.

The *Selberg class* of L-functions, denoted by $\mathcal{S}$, consists of the Dirichlet series

$$L(s) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}, \quad (a_1 = 1, a_m \in \mathbb{C}, \text{ for } m = 2, 3, 4, \ldots)$$

which satisfy the following axioms.

(1) (*ordinary Dirichlet series*) The Dirichet series converges absolutely for $\sigma > 1$.

(2) (*Analytic continuation*) There exists an integer $l \geq 0$ such that the function $(s-1)^l L(s)$ is an entire function of finite order.

(3) (*Functional equation*) L satisfies the following functional equation

$$\phi(s) = \epsilon \bar{\phi}(1-s)$$

where

$$\phi(s) = A^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) L(s) = A^s G(s) L(s),$$

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$\bar{\phi}(s) = \overline{\phi(\bar{s})}$ and $r \geq 0, A > 0, \lambda_j > 0$, $\mu_j \in \mathbb{C}$ with $\Re(\mu_j) \geq 0, |\epsilon| = 1$ are parameters depending on L.

(4) (*Ramanujan hypothesis*) For every $\epsilon > 0$ we have $a_m \ll m^\epsilon$.

(5) (*Euler product*) For $\sigma > 1$ we have

$$\log L(s) = \sum_{m=1}^{\infty} \frac{b_m}{m^s},$$

where $b_m = 0$ unless $m = p^n$ with $n \geq 1$, and $b_m \ll m^\theta$ for some $\theta < 1/2$.

In order to classify the Dirichlet series $L(s)$ in the Selberg class, it is convenient to introduce the degree d_L of $L \in \mathcal{S}$ as

$$d_L = 2\sum_{j=1}^{r} \lambda_j.$$

For more information on properties of the Selberg class see e.g. [1, 6, 11, 12].

Representation of the function $L(s) \in \mathcal{S}$ in terms of the sum of two Dirichlet series valid for $0 \leq \Re(s) \leq 1$ is given by the approximate functional equation in the next theorem.

Theorem 1.1 *[3, Theorem 8.3.3] Let $L(s) \in \mathcal{S}$ be entire and satisfy the axiom (3) of the Selberg class with $\lambda_j = \lambda$ for every $j = 1, 2, \ldots r$. Then there exists a smooth function $F : (0, \infty) \to \mathbb{C}$ such that for every $w \in \mathbb{C}$ with $0 \leq \Re(w) \leq 1$, we have*

$$L(w) = \sum_{m=1}^{\infty} \frac{a_m}{m^w} F\left(\frac{m}{R_w}\right) + \epsilon\lambda_w \sum_{m=1}^{\infty} \frac{\overline{a_m}}{m^{1-w}} \overline{F}\left(\frac{m}{R_{1-w}}\right), \tag{1.1}$$

where $\lambda_w = A^{1-w}G(1-w)/A^w G(w)$, $R_w = A \cdot \prod_{j=1}^{r}(3 + |\lambda w + \mu_j|)^\lambda$.

Furthermore, the function F and its partial derivatives $F^{(k)}, (k = 1, 2, \ldots)$ satisfy, for any $\sigma > 0$, the following uniform growth estimates at 0 and ∞:

$$F(x) = \begin{cases} 1 + O_\sigma(x^\sigma) \\ O_\sigma(x^{-\sigma}) \end{cases} \qquad F^{(k)}(x) = O_\sigma(x^{-\sigma}). \tag{1.2}$$

The implied O_σ-constants depend only on σ, k, r.

The approximate functional equation (1.1) motivates the study of the properties of the Dirichlet polynomials $L_{F,X}(s)$ defined by

$$L_{F,X}(s) = \sum_{m \leq X} \frac{a_m}{m^s} F\left(\frac{m}{q_s}\right),$$

where F is a function satisfying properties stated in Theorem 1.1.

In this article we prove that there exists $\alpha, \beta \in \mathbb{R}$, such that all complex zeros of Dirichlet polynomial $L_{F,X}(s)$, where $s \in \mathbb{C}$ and $s = \sigma + it$ lie inside the strip $\alpha < \sigma < \beta$, and prove that the number of zeros of polynomial $L_{F,X}$ with imaginary part in $[-T, T]$ is

$$N_{F,X}(T) = \frac{T}{\pi} \log N + O(X), \quad \text{as} \quad T \to \infty,$$

and N is the largest integer less or the equal to X for which $a_N \neq 0$.

2. Approximate functional equation

The method of approximate functional equation is applied in a very general setting corresponding to the Selberg class in [2]. Lavrik [7] obtained an explicit approximate functional equation for a very wide class of L-functions (see [5] for a detailed exposition of more developments on these lines).

With careful analysis of the proof of Theorem 1.1 of approximate functional equation for the case when λ_j not all equal for $j = 1, 2, \ldots, r$ and $L(s) \in \mathcal{S}$ entire function or possesses a pole at $s = 1$ we can prove the following

Theorem 2.1 *Let $L(s) \in \mathcal{S}$. Define*

$$Q_w = A \cdot \prod_{j=1}^{r} (3 + |\lambda_j w + \mu_j|)^{\lambda_j}, \quad P_w = 3 + |w|. \tag{2.1}$$

Then there exists a smooth function $F : (0, \infty) \to \mathbb{C}$ such that for every $w \in \mathbb{C}$ with $0 \leq \Re(w) \leq 1$, we have

$$L(w) = \sum_{m=1}^{\infty} \frac{a_m}{m^w} F\left(\frac{m}{P_w Q_w}\right) + \epsilon \lambda_w \sum_{m=1}^{\infty} \frac{\overline{a_m}}{m^{1-w}} \overline{F}\left(\frac{m}{P_{1-w} Q_{1-w}}\right),$$

where $\lambda_w = (-1)^l A^{1-w} G(1-w)/A^w G(w)$.

The function F and its partial derivatives $F^{(k)}, (k = 1, 2, \ldots)$ satisfy condition (1.2).

Proof First, we suppose that $L(s)$ has a pole at $s = 1$ of order l.

The proof of the theorem is very similar to the proof of Theorem 1.1 and so we will give only a sketch of the proof. Let $h(s)$ be a holomorphic function satisfying

$$h(s) = h(-s) = \overline{h(\overline{s})}, \quad h(0) = 1,$$

and which is bounded in vertical strip $-2 < \sigma < 2$. For every $w \in \mathbb{C}$, and $x > 0$, we define

$$H_w(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{(s+w-1)^l G(s+w)}{(w-1)^l G(w)} h(s) x^{-s} \frac{ds}{s}. \tag{2.2}$$

We first derive an approximate functional equation in terms of the function H_w. Proceeding analogously to [3, Theorem 8.3.3], consider the integral

$$I_L(w) = \frac{1}{2\pi i} \int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty} \frac{A^{s+w} G(s+w)(s+w-1)^l L(s+w)}{A^w (w-1)^l G(w)} h(s) \frac{ds}{s},$$

and shift the line of integration to the line $\Re s = -1-\epsilon$ picking up a residue of the pole of the integrand at $s = 0$. Applying the functional equation (axiom (3) of Selberg class), and then transforming $s \to -s$, we get

$$L(w) = I_L(w) + \epsilon \lambda_w I_{\tilde{L}}(1-w), \tag{2.3}$$

where $\tilde{L}(s) = \sum_{m=1}^{\infty} \frac{\overline{a_m}}{m^s}$ denotes the dual L-function. Substituting the Dirichlet series for $L(s)$ and $\tilde{L}(s)$ and integrating term by term in (2.3), it follows that

$$L(w) = \sum_{m=1}^{\infty} \frac{a_m}{m^w} H_w\left(\frac{m}{A}\right) + \epsilon \lambda_w \sum_{m=1}^{\infty} \frac{\overline{a_m}}{m^{1-w}} H_{1-w}\left(\frac{m}{A}\right).$$

The Stirling formula for the Gamma function (see, e.g. [3, p. 243]) yields

$$\left|\frac{G(s+w)}{G(w)}\right| \ll Q_w^{\sigma} e^{\frac{\pi}{4} d_L |s|}$$

and

$$\left|\frac{s+w-1}{w-1}\right|^{l} \ll (|w|+3)^{\sigma} = P_w^{\sigma}.$$

Setting

$$F(x) = H_w\Big(Q_w P_w x\Big) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{(s+w-1)^l G(s+w)}{(w-1)^l G(w)} h(s)(Q_w P_w x)^{-s}\frac{ds}{s},$$

following proof of [3, Theorem 8.3.3], we obtain

$$\left(\frac{d}{dx}\right)^{k} F(x) = \delta_{\sigma,k} + O_{\sigma,k}\left(\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\frac{\pi}{4} d_L |s|}(1+|s|)^k |h(s)| \cdot x^{-\sigma} |ds|\right),$$

where

$$\delta_{\sigma,k} = \begin{cases} 1, & \sigma < 0,\ k = 0 \\ 0, & \text{otherwise.} \end{cases}$$

To complete the proof it is enough to choose a test function h with sufficient decay properties in the same way as in [3, p. 244].

If $L(s)$ is an entire function where λ_j are not all equal for $j = 1, 2, \ldots, r$ the proof is analogous to the proof of [3, Theorem 8.3.3], where we take $\lambda = \max\{\lambda_j\}$, $l = 0, Q_w = R_w$, and $P_w = 1$. □

3. Zero-free regions

Let $X \geq 2$, $L \in \mathcal{S}$ and let

$$L_{F,X}(s) = \sum_{m \leq X} \frac{a_m}{m^s} F\left(\frac{m}{q_s}\right), \tag{3.1}$$

where F is an arbitrary but fixed function satisfying conditions (1.2) and $q_s = Q_s P_s$, where Q_s and P_s are defined by (2.1).

In this section we derive a zero-free region for the function $L_{F,X}(s)$.

Theorem 3.1 *Let $L_{F,X}(s)$ be given by (3.1). Then there exists α depending on X and β, such that $|L_{F,X}(s)| > 0$ for $\Re s \geq \beta$ and $\Re s \leq \alpha$. In other words, we can find a rectilinear strip of the complex plane given by the inequality $\alpha < \Re s < \beta$ such that the zeros of $L_{F,X}(s)$ all lie in it.*

Proof Let $s = \sigma + it$. We show separately that $|L_{F,X}(s)| > 0$ in the right-half plane $\sigma \geq \beta$ and in the left-half plane $\sigma \leq \alpha$. Since

$$|L_{F,X}(s)| \geq |F(q_s^{-1})| - \left|\sum_{2 \leq m \leq X} \frac{a_m}{m^s} F\left(\frac{m}{q_s}\right)\right|,$$

in order to show that $|L_{F,X}(s)| > 0$ for $\sigma \geq \beta$ it is enough to find β such that

$$\left| \sum_{2\leq m\leq X} \frac{a_m}{m^s} F\left(\frac{m}{q_s}\right) \right| < |F({q_s}^{-1})|. \tag{3.2}$$

Toward this end, from conditions (1.2) there exists δ_F such that for every $x \in (0, \delta_F)$, one has $|F(x)| \geq 1 - C_\sigma x^\sigma$. Since q_s tends to infinity as $|s| \to \infty$ there exists A_1 such that $|s| \geq A_1$ implies

$$\min_{|s|\geq A_1} q_s \geq \max\{X, \delta_F^{-1}\}. \tag{3.3}$$

Let $|s| \geq A_1$, then

$$\frac{X}{q_s} \leq 1.$$

Since F has continuous derivatives F is continuous and bounded in $[0,1]$ by some constant C_F; therefore

$$\left| F\left(\frac{X}{q_s}\right) \right| \leq C_F. \tag{3.4}$$

Furthermore, $|s| \geq A_1$ implies that

$$q_s^{-1} \in (0, \delta_F),$$

yielding

$$|F(q_s^{-1})| \geq 1 - D_\sigma q_s^{-\sigma} > 1 - D_\sigma \delta_F^\sigma = d_{\sigma,F} > 0. \tag{3.5}$$

Using the axiom (4) of Selberg class with some $\epsilon \in (0,1)$ and (3.4) we have

$$\begin{aligned} \left| \sum_{m=2}^{X} \frac{a_m}{m^s} F\left(\frac{m}{q_s}\right) \right| &\leq \sum_{m=2}^{X} \frac{|a_m|}{m^\sigma} \left| F\left(\frac{m}{q_s}\right) \right| \\ &\leq C_\epsilon C_F \sum_{m=2}^{X} \frac{m^\epsilon}{m^\sigma} \leq C_\epsilon C_F \sum_{m=2}^{\infty} \frac{1}{m^{\sigma-\epsilon}}. \end{aligned}$$

Now, for $\sigma \geq \beta_1 > 2$, we have

$$\begin{aligned} \sum_{m=2}^{\infty} \frac{1}{m^{\sigma-\epsilon}} \leq \sum_{m=2}^{\infty} \frac{1}{m^{\beta_1-\epsilon}} &= \frac{1}{2^{\beta_1}} \sum_{m=2}^{\infty} \frac{4 \cdot 2^{\beta_1} \cdot 4^{-1}}{m^{2-\epsilon} m^{\beta_1-2}} \\ &= \frac{1}{2^{\beta_1}} \sum_{m=2}^{\infty} \left(\frac{4}{m^{2-\epsilon}} \left(\frac{2}{m}\right)^{\beta_1-2} \right) \\ &\leq \frac{1}{2^{\beta_1}} \sum_{m=2}^{\infty} \left(\frac{4}{m^{2-\epsilon}} \right) = \frac{1}{2^{\beta_1}} E_\epsilon, \end{aligned}$$

where

$$E_\epsilon = \sum_{m=2}^{\infty} \frac{4}{m^{2-\epsilon}}.$$

We get

$$\left|\sum_{m=2}^{X}\frac{a_m}{m^s}F\left(\frac{m}{q_s}\right)\right| \le \frac{1}{2^{\beta_1}}C_\epsilon C_F E_\epsilon. \tag{3.6}$$

From (3.2), (3.5), and (3.6) taking

$$\beta > \max\left\{2, A_1, \log_2\frac{C_\epsilon E_\epsilon C_F}{d_{\sigma,F}}\right\},$$

we get $|L_{F,X}(s)| > 0$ in the right-half plane $\sigma \ge \beta$.

Next, let N be the largest integer less than or equal to X such that $a_N \neq 0$. We start with

$$|L_{F,X}(s)| = \left|\sum_{m=1}^{N-1}\frac{a_m}{m^s}F\left(\frac{m}{q_s}\right) + \frac{a_N}{N^s}F\left(\frac{N}{q_s}\right)\right| \ge \frac{|a_N|}{N^\sigma}\left|F\left(\frac{N}{q_s}\right)\right| - \left|\sum_{m=1}^{N-1}\frac{a_m}{m^\sigma}F\left(\frac{m}{q_s}\right)\right|.$$

Since $\frac{X}{q_s}$ tends to 0 when $|s| \to \infty$ there exists $A_2 > 0$ such that $|s| \ge A_2$ implies that $\frac{X}{q_s} < \delta_F$; hence $\frac{m}{q_s} \in (0, \delta_F)$ for all $m = 1, 2, \ldots, N$. Therefore

$$\left|F\left(\frac{N}{q_s}\right)\right| > d_{\sigma,F}$$

and

$$\left|F\left(\frac{m}{q_s}\right)\right| \le 1 + C_\sigma q_s^{-\sigma} m^\sigma \quad \text{for all} \quad m = 1, 2, \ldots, N-1.$$

Assume that $|s| \ge A_2$. Then

$$|L_{F,X}(s)| > \frac{|a_N|}{N^\sigma}d_{\sigma,F} - \sum_{m=1}^{N-1}\frac{|a_m|}{m^\sigma}(1 + C_\sigma q_s^{-\sigma} m^\sigma),$$

and hence applying axiom (4) of Selberg class it is sufficient to find α_1 such that

$$\frac{1}{N^\sigma} > C_{\epsilon,\sigma}\sum_{m=1}^{N-1}\frac{m^\epsilon}{m^\sigma} + C_\sigma q_s^{-\sigma} C_{\epsilon,\sigma}\sum_{m=1}^{N-1}\frac{m^\epsilon}{m^\sigma}m^\sigma, \quad \text{for} \quad \sigma \le \alpha_1,$$

where

$$C_{\epsilon,\sigma} = \frac{C_\epsilon}{|a_N| d_{\sigma,F}}.$$

This would follow from the inequality

$$\frac{1}{N^{\alpha_1}} > C_{\epsilon,\sigma}\sum_{m=1}^{N-1}\frac{m^\epsilon}{m^{\alpha_1}} + C_\sigma q_s^{-\sigma} C_{\epsilon,\sigma}\sum_{m=1}^{N-1} m^\epsilon.$$

Since

$$C_\sigma q_s^{-\sigma} C_{\epsilon,\sigma}\sum_{m=1}^{N-1} m^\epsilon \le C_\sigma q_s^{-\sigma} C_{\epsilon,\sigma}(N-1)^{\epsilon+1}$$

and

$$C_{\epsilon,\sigma}\sum_{m=1}^{N-1}\frac{m^{\epsilon}}{m^{\alpha_1}}\leq C_{\epsilon,\sigma}(N-1)^{\epsilon}\sum_{m=1}^{N-1}\frac{1}{m^{\alpha_1}},$$

it suffices to show that

$$\frac{1}{N^{\alpha_1}}>(N-1)^{\epsilon}\sum_{m=1}^{N-1}\frac{1}{m^{\alpha_1}}. \tag{3.7}$$

For $\alpha_1<0$, we get

$$\begin{aligned}\sum_{m=1}^{N-1}\frac{1}{m^{\alpha_1}}&\leq(N-1)^{-\alpha_1}+\int_1^{N-1}\frac{dy}{y^{\alpha_1}}\\&=(N-1)^{-\alpha_1}+\frac{(N-1)^{1-\alpha_1}}{1-\alpha_1}-\frac{1^{1-\alpha_1}}{1-\alpha_1}\\&<(N-1)^{-\alpha_1}+\frac{(N-1)^{1-\alpha_1}}{1-\alpha_1}=(N-1)^{-\alpha_1}\left(1+\frac{N-1}{1-\alpha_1}\right)\\&=(N-1)^{-\alpha_1}\left(\frac{N-\alpha_1}{1-\alpha_1}\right).\end{aligned}$$

Putting this in inequality (3.7) we get

$$\left(\frac{N}{N-1}\right)^{-\alpha_1}>(N-1)^{\epsilon}\left(\frac{N-\alpha_1}{1-\alpha_1}\right).$$

Letting $N\to\infty$ the left-hand side tends to $e^{\frac{-\alpha_1}{N-1}}$. Taking the logarithm in both sides, for $\epsilon<1/2$ we have admissible choice of α_1, which is given by

$$\alpha_1=-2(N-1)\log N.$$

Finally, taking $\alpha=\min\{-A_2,\alpha_1\}$ we get $|L_{F,X}(s)|>0$ in the half plane $\sigma\leq\alpha$. This completes the proof of Theorem 3.1. □

In the special case when $F\equiv 1$ we get the following proposition.

Proposition 3.2 *Let*

$$L_X(s)=\sum_{m\leq X}\frac{a_m}{m^s}$$

be the partial sum of $L(s)\in S$, *where* $X\geq 2$. *Then we can find* $\alpha,\beta\in\mathbb{R}$, *and* α *depending on* X *such that* $|L_X(s)|>0$, *for* $\Re s\geq\beta$ *and* $\Re s\leq\alpha$.

Proof Let $s=\sigma+it$. Analogously as in Theorem 3.1 we show that $|L_X(s)|>0$ in the right-half plane $\sigma\geq\beta$ and in the left-half plane $\sigma\leq\alpha$. Since

$$|L_X(s)|=\left|\sum_{m\leq X}\frac{a_m}{m^s}\right|\geq 1-\left|\sum_{2\leq m\leq X}\frac{a_m}{m^s}\right|,$$

in order to show that $|L_X(s)| > 0$ for $\sigma \geq \beta$ it sufficient to find β so that

$$\left| \sum_{2 \leq m \leq X} \frac{a_m}{m^s} \right| < 1. \tag{3.8}$$

Proceeding as in the proof of Theorem 3.1 we see that it suffices to take

$$\beta > \max\{2, \log_2 C_\epsilon E_\epsilon\}.$$

Therefore, $L_X(s) \neq 0$ in the half plane $\sigma \geq \beta$. Next, let N be the largest positive integer less than or equal to X for which $a_N \neq 0$. Since

$$|L_X(s)| = \left| \sum_{m=1}^{N-1} \frac{a_m}{m^s} + \frac{a_N}{N^s} \right| \geq \frac{|a_N|}{N^\sigma} - \sum_{m=1}^{N-1} \frac{|a_m|}{m^\sigma},$$

it is sufficient to show that

$$\frac{|a_N|}{N^\sigma} > \sum_{m=1}^{N-1} \frac{|a_m|}{m^\sigma}, \quad \text{for} \quad \sigma \leq \alpha.$$

Analogously as in Theorem 3.1 it suffices to prove

$$\frac{|a_N|}{N^\sigma} > 1 + C_\epsilon \cdot \sum_{m=2}^{N-1} \frac{m^\epsilon}{m^\sigma} \quad \text{for} \quad \sigma \leq \alpha.$$

For $\alpha < 0$,

$$\sum_{m=2}^{N-1} \frac{m^\epsilon}{m^\alpha} \leq \sum_{m=2}^{N-1} \frac{N^\epsilon}{m^\alpha} < N^\epsilon \sum_{m=2}^{N-1} \frac{1}{m^\alpha} < N^\epsilon (N-1)^{-\alpha} \left(\frac{N-\alpha}{1-\alpha} \right),$$

and hence taking

$$\alpha = -2(N-1)\log N,$$

we get $L_X(s) \neq 0$ in the half plane $\sigma \leq \alpha$, which completes the proof. □

4. Distribution of zeros

Gonek and Ledoan [4] and Ledoan et al. [8] studied the distribution of zeros of partial sums of the Riemann zeta function

$$\zeta_X(s) = \sum_{n \leq X} n^{-s}$$

and the Dedekind zeta function of a cyclotomic field K

$$\zeta_{K,X}(s) = \sum_{\|a\| \leq X} \frac{1}{\|a\|^s}$$

and proved asymptotic formula for the number of zeros with an imaginary part in interval $[0, T]$, as $T \to \infty$.

In this section we prove analogous result for the Dirichlet polynomial $L_{F,X}(s)$ defined by (3.1).

In the proof of our main theorem, we will need the following lemma.

Lemma 1 *[13, Part V, Ch.I, No. 77. Generalization of Descartes' Rule of Signs] Let* $a_1, a_2, \ldots, a_n, \lambda_1, \lambda_2, \ldots, \lambda_n$ *be real constants,* $\lambda_1 < \lambda_2 < \ldots < \lambda_n$*. Denote by* Z *the number of real zeros of the entire function*

$$F(x) = a_1 e^{\lambda_1 x} + a_2 e^{\lambda_2 x} + \cdots + a_n e^{\lambda_n x}$$

and by C *the number of changes of sign in the sequence of numbers* $a_1, a_2, \ldots, a_n$*. Then* $C - Z$ *is a nonnegative even integer.*

Let us denote by $\rho_{F,X} = \beta_{F,X} + i\gamma_{F,X}$ the complex zero, and by $N_{F,X}(T)$ the number of zeros of $L_{F,X}(s)$ with ordinates $-T \leq \gamma_{F,X} \leq T$. If T is the ordinate of a zero, then the number of zeros is to be defined as $\lim_{\epsilon \to 0^+} N_{F,X}(T+\epsilon)$, respectively as $\lim_{\epsilon \to 0^+} N_X(T+\epsilon)$.

Theorem 4.1 *Let* $L_{F,X}(s)$ *be as in (3.1), and let* $X, T \geq 2$*. Let further* N *be the largest integer less than or equal to* X *such that* $a_N \neq 0$*. We have*

$$N_{F,X}(T) = \frac{T}{\pi} \log N + O(X), \ \ as\ T \to \infty.$$

The implied constants depend on σ, k*, and* r*.*

Proof Assuming that T does not coincide with the ordinate of any zero, we have

$$N_{F,X}(T) = \frac{1}{2\pi i} \int_R \frac{L'_{F,X}(s)}{L_{F,X}(s)} ds,$$

where R is the rectangle with vertices at $\alpha - iT, \beta - iT, \beta + iT$, and $\alpha + iT$. Thus by the argument principle

$$2\pi N_{F,X}(T) = \int_R \Im\left(\frac{L'_{F,X}(s)}{L_{F,X}(s)}\right) ds = \Delta_R \arg L_{F,X}(s), \tag{4.1}$$

where Δ_R denotes the change in $\arg L_{F,X}(s)$ around R in the positive direction.

To estimate the change in argument along the top edge of R we decompose $L_{F,X}(s)$ into its real part and its imaginary part. For $a_m = \sigma_m + it_m$ and $s = \sigma + it$, we get

$$\begin{aligned} L_{F,X}(s) &= \sum_{m=1}^{N} \frac{a_m}{m^s} F\left(\frac{m}{q_s}\right) = \sum_{m=1}^{N} (\sigma_m + it_m) F\left(\frac{m}{q_s}\right) e^{-(\sigma+it)\log m} \\ &= \sum_{m=1}^{N} F\left(\frac{m}{q_s}\right)\left[\frac{\sigma_m \cos(t\log m) + t_m \sin(t \log m)}{m^\sigma}\right] \\ &- i \sum_{m=1}^{N} F\left(\frac{m}{q_s}\right)\left[\frac{\sigma_m \sin(t\log m) - t_m \cos(t \log m)}{m^\sigma}\right], \end{aligned}$$

and hence

$$\Im L_{F,X}(\sigma + iT) = -\sum_{m=1}^{N} F\left(\frac{m}{q_s}\right) \frac{\sigma_m \sin(T \log m) - t_m \cos(T \log m)]}{m^\sigma} = -\sum_{m=1}^{N} \frac{b_m}{m^\sigma},$$

where we put

$$b_m = F\left(\frac{m}{q_s}\right)[\sigma_m \sin(T \log m) - t_m \cos(T \log m)].$$

By Lemma 1, the number of zeros of $\Im L_{F,X}(s)$ in the interval $\alpha \le \sigma \le \beta$ is at most the number of changes of sign in the sequence $\{b_m\}_{m=1}^{N}$; hence it is $\ll X$.

Since the change in argument of $L_{F,X}(\sigma + iT)$ between two consecutive zeros of $\Im L_{F,X}(\sigma + iT)$ is at most π, it follows that

$$\Delta_{[\alpha,\beta]} \arg L_{F,X}(\sigma + iT) = O(X).$$

Similarly, at the bottom edge of R we get

$$\Delta_{[\alpha,\beta]} \arg L_{F,X}(\sigma - iT) = O(X).$$

As s describes the right edge of R, equation (3.2) yields

$$|L_{F,X}(s) - F(q_s^{-1})| < |F(q_s^{-1})|, \quad \text{for} \quad \sigma \ge \beta.$$

Since

$$\Re(L_{F,X}(\beta + it) - F(q_s^{-1})) \le |L_{F,X}(\beta + it) - F(q_s^{-1})| < |F(q_s^{-1})|,$$

we get

$$-|F(q_s^{-1})| \le \Re(L_{F,X}(\beta + it) - F(q_s^{-1})) \le |F(q_s^{-1})|,$$

or, equivalently

$$F(q_s^{-1}) - |F(q_s^{-1})| \le \Re(L_{F,X}(\beta + it)) \le F(q_s^{-1}) + |F(q_s^{-1})|.$$

Now, if $F(q_s^{-1}) < 0$ it follows that $\Re L_{F,X}(\beta + it) < 0$ and $F(q_s^{-1}) \ge 0$ yields $\Re L_{F,X}(\beta + it) > 0$, for $-T \le t \le T$; hence

$$\Delta_{[-T,T]} \arg L_{F,X}(\beta + it) = O(1).$$

Finally, along the left edge of R, since N is the largest integer less than or equal to X such that $a_N \ne 0$, we have

$$\begin{aligned} L_{F,X}(\alpha + it) &= \sum_{1 \le m \le N} \frac{a_m}{m^{\alpha+it}} F\left(\frac{m}{q_s}\right) \\ &= \sum_{1 \le m \le N} \frac{a_m N^{\alpha+it} F\left(\frac{m}{q_s}\right)}{a_N m^{\alpha+it} F\left(\frac{N}{q_s}\right)} \cdot \frac{a_N F\left(\frac{N}{q_s}\right)}{N^{\alpha+it}}, \end{aligned}$$

and therefore

$$\begin{aligned} \Delta_{[-T,T]} \arg L_{F,X}(\alpha + it) = \Delta_{[-T,T]} \arg &\left(1 + \sum_{1 \le m \le N-1} \frac{a_m N^{\alpha+it} F\left(\frac{m}{q_s}\right)}{a_N m^{\alpha+it} F\left(\frac{N}{q_s}\right)}\right) \\ &+ \Delta_{[-T,T]} \arg \left(\frac{a_N F\left(\frac{N}{q_s}\right)}{N^{\alpha+it}}\right). \end{aligned}$$

In the proof of Theorem 3.1 we noticed that

$$\frac{|a_N|\left|F\left(\frac{N}{q_s}\right)\right|}{N^\alpha} > \sum_{1\le m\le N-1} \frac{|a_m|}{m^\alpha}\left|F\left(\frac{m}{q_s}\right)\right|.$$

Thus for any t, we get

$$\begin{aligned}\left|\sum_{1\le m\le N-1} \frac{a_m N^{\alpha+it} F\left(\frac{m}{q_s}\right)}{a_N m^{\alpha+it} F\left(\frac{N}{q_s}\right)}\right| &\le \sum_{1\le m\le N-1} \frac{|a_m| N^\alpha \left|F\left(\frac{m}{q_s}\right)\right|}{|a_N| m^\alpha \left|F\left(\frac{N}{q_s}\right)\right|} \\ &\le \frac{N^\alpha}{|a_N|\left|F\left(\frac{N}{q_s}\right)\right|} \sum_{1\le m\le N-1} \frac{|a_m|}{m^\alpha}\left|F\left(\frac{m}{q_s}\right)\right| < 1,\end{aligned}$$

and hence

$$\Delta_{[-T,T]} \arg\left(1 + \sum_{1\le m\le N-1} \frac{a_m N^{\alpha+it} F\left(\frac{m}{q_s}\right)}{a_N m^{\alpha+it} F\left(\frac{N}{q_s}\right)}\right) = O(1).$$

Furthermore

$$\begin{aligned}\frac{a_N F\left(\frac{N}{q_s}\right)}{N^{\alpha+it}} &= a_N F\left(\frac{N}{q_s}\right) e^{(-\alpha-it)\log N} \\ &= a_N F\left(\frac{N}{q_s}\right) e^{-\alpha\log N + t\arg N} e^{-i(t\log N+\alpha\arg N)},\end{aligned}$$

and hence for $-T \le t \le T$ we have

$$\Delta_{[-T,T]} \arg\left(\frac{a_N F\left(\frac{N}{q_s}\right)}{N^{\alpha+it}}\right) = -2T\log N.$$

This proves that

$$\Delta_{[-T,T]} \arg L_{F,X}(\alpha+it) = -2T\log N + O(1).$$

Finally, since

$$\begin{aligned}\Delta_R \arg L_{F,X}(s) &= \Delta_{[\alpha,\beta]} \arg L_{F,X}(\sigma - iT) + \Delta_{[-T,T]} \arg L_{F,X}(\beta+it) \\ &\quad - \Delta_{[\alpha,\beta]} \arg L_{F,X}(\sigma+iT) - \Delta_{[-T,T]} \arg L_{F,X}(\alpha+it) \\ &= O(X) + O(1) + O(X) + 2T\log N + O(1), \\ &= 2T\log N + O(X),\end{aligned}$$

substituting in (4.1) we obtain the Theorem. □

We now have the following proposition as a special case when $F \equiv 1$.

Proposition 4.2 *Let*

$$L_X(s) = \sum_{m \le X} \frac{a_m}{m^s}$$

be the partial sum of $L(s) \in \mathcal{S}$*, where* $X, T \ge 2$*. Let further* N *be the largest integer less than or equal to* X *such that* $a_N \neq 0$*. We have*

$$N_X(T) = \frac{T}{\pi} \log N + O(X), \ \text{as } T \to \infty.$$

Proof Assuming that T does not coincide with the ordinate of any zero, by the argument principle

$$2\pi N_X(T) = \Delta_R \arg L_X(s), \tag{4.2}$$

where R is the rectangle with vertices at $\alpha - iT, \beta - iT, \beta + iT$ and $\alpha + iT$, and

$$\begin{aligned} \Delta_R \arg L_X(s) &= \Delta_{[\alpha,\beta]} \arg L_X(\sigma - iT) + \Delta_{[0,T]} \arg L_X(\beta + it) \\ &\quad - \Delta_{[\alpha,\beta]} \arg L_X(\sigma + iT) - \Delta_{[0,T]} \arg L_X(\alpha + it). \end{aligned}$$

Similarly as in Theorem 4.1 to estimate the change in argument along the top and the bottom edge of R we decompose $L_X(s)$ into its real part and its imaginary part, using Lemma 1 we get

$$\Delta_{[\alpha,\beta]} \arg L_X(\sigma + iT) = O(X), \tag{4.3}$$

and

$$\Delta_{[\alpha,\beta]} \arg L_X(\sigma - iT) = O(X), \tag{4.4}$$

As s describes the right edge of R, (3.8) yields

$$|L_X(s) - 1| < 1.$$

It follows that $\Re L_X(\beta + it) > 0$ for $-T \le t \le T$. Hence,

$$\Delta_{[-T,T]} \arg L_X(\beta + it) = O(1). \tag{4.5}$$

Finally, along the left edge of R, analogously as in Theorem 4.1 letting N be the largest integer less than or equal to X such that $a_N \neq 0$, we get

$$L_X(\alpha + it) = \sum_{1 \le m \le N} \frac{a_m}{m^{\alpha+it}} = \sum_{1 \le m \le N} \frac{a_m N^{\alpha+it}}{a_N m^{\alpha+it}} \cdot \frac{a_N}{N^{\alpha+it}},$$

and therefore

$$\begin{aligned} \Delta_{[-T,T]} \arg L_X(\alpha + it) &= \Delta_{[-T,T]} \arg \left(1 + \sum_{1 \le m \le N-1} \frac{a_m N^{\alpha+it}}{a_N m^{\alpha+it}} \right) \\ &\quad + \Delta_{[-T,T]} \arg \left(\frac{a_N}{N^{\alpha+it}} \right). \end{aligned}$$

Proceeding as in the proof of Theorem 4.1 it follows that

$$\Delta_{[-T,T]} \arg L_X(\alpha + it) = -2T \log N + O(1). \tag{4.6}$$

We may now substitute (4.3), (4.4), (4.5), and (4.6) into (4.2) to obtain our claim. □

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