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Stability of compact Ricci solitons under Ricci flow

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Abstract: In this paper we establish stability results for Ricci solitons under the Ricci flow, i.e. small perturbations of the Ricci soliton result in small variations in the solution under Ricci flow.

Key words: Stability, Ricci flow, Ricci soliton, compact manifolds

1. Introduction and preliminaries

Differential equations are interesting mathematical topics employed throughout the sciences for modeling dynamic processes. When differential equations are difficult to be solved, we try to obtain qualitative information about the long-term or asymptotic behavior of solutions. Ricci flow is a partial differential equation that evolves a Riemannian metric $\bar{g}$ on a manifold M under the following equation:

$$\frac{\partial}{\partial t}g(t) = -2Ric(g(t)), \quad g(0) = \bar{g}. \tag{1.1}$$

It was introduced by Hamilton in his seminal paper [14] in order to study the geometry and topology of manifolds. Ricci flow has been developed in the past several decades. In addition to being applied as a useful tool in geometry, it also has some applications in other fields such as computer science [28] and physics [18]. Therefore, it is important to study the equation of Ricci flow. There are some interesting questions about this equation, such as stability. The term "stable" means that a stated property is not destroyed when certain perturbations are made. The stability of solutions of differential equations is a quite difficult property to determine. Even though various kinds of stability may be discussed, the one we study here is dynamical stability. $\tilde{g}$ is dynamically stable if for $\bar{g}$ belonging to a neighborhood of $\tilde{g}$ and sufficiently close to $\tilde{g}$, the solution $g(t)$ of Ricci flow with initial value $\bar{g}$ stays near $\tilde{g}$ forever.

Stability under the Ricci flow was first discussed by Ye [29]. He replaced the Ricci flow equation with normalized Ricci flow

$$\frac{\partial}{\partial t}g(t) = -2Ric(g(t)) + \frac{2}{n}rg(t),$$

where $r = \frac{\int R d\mu}{\int d\mu}$, and investigated stability of constant nonzero sectional curvature metrics. After him, Guenther et al. [11] using maximal regularity theory [8] and center manifold analysis [9] studied stability of flat and Ricci-flat metrics. The behavior of Ricci flow close to Ricci-flat metrics was also investigated in [16, 17, 26]. Knopf

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[21] obtained stability of locally $\mathbb{R}^N$-invariant solutions of Ricci flow. Schnürer et al. showed the stability of Euclidean space [23] and hyperbolic space [24].

The objective of this paper is to investigate dynamical stability of the Ricci soliton $\tilde{g}$ on compact manifolds under the following modified Ricci flow:

$$\frac{\partial}{\partial t}g(t) = -2Ric(g(t)) - L_X g - 2\rho g, \quad g(0) = \bar{g}. \tag{1.2}$$

Definition 1.1 *A Ricci soliton is a fixed complete Riemannian manifold $(M^n, \tilde{g})$ that satisfies the equation*

$$-2Ric(\tilde{g}) = L_X\tilde{g} + 2\rho\tilde{g} \tag{1.3}$$

for some constant ρ and some complete vector field X on M^n, where $L_X\tilde{g}$ denotes the usual Lie derivative in the direction of the field X. A Ricci soliton is said to be a gradient Ricci soliton if its vector field X can be written as the gradient of some function $f : M^n \to \mathbb{R}$. The function f is called a potential function for $\tilde{g}$. Equation (1.3) then becomes

$$-Ric(\tilde{g}) = \nabla\nabla f + \rho\tilde{g}.$$

$\tilde{g}$ is called shrinking, steady, or expanding if $\rho < 0$, $\rho = 0$, or $\rho > 0$, respectively.

When either the vector field X is trivial, or the potential functional f is constant, $\tilde{g}$ is an Einstein metric. Thus, Ricci solitons are natural generalizations of Einstein metrics. They were introduced by Hamilton [15]. Indeed, they are equivalent to the self-similar solution of the Ricci flow [14]. That is, these solutions evolve by rescalings and diffeomorphisms of the initial metric, and so they can be regarded as fixed points of the Ricci flow on the space of Riemannian metrics modulo diffeomorphisms and scalings. They are also of great importance owing to their relationship with singularities of the Ricci flow (see [2, 15, 25]). Ricci solitons also are discussed in string theory in physics (see [1, 7, 10]).

Theorem 1.2 *(Perelman) Every compact Ricci soliton is a gradient Ricci soliton.*

For more details on Ricci solitons we refer the reader to [3]. Ricci solitons are stationary points of the modified Ricci flow (1.2) and so remain unchanged throughout the flow. For analytic reasons, we study the following flow, which is similar to Ricci–DeTurck flow:

$$\frac{\partial}{\partial t}g(t) = -2Ric(g(t)) + \nabla_i W_j + \nabla_j W_i - L_X g - 2\rho g, \quad g(0) = \bar{g}, \tag{1.4}$$

where $W_i = g_{ik}(\Gamma^k_{rs} - \tilde{\Gamma}^k_{rs})g^{rs}$. We use $\tilde{\Gamma}$, $\tilde{\nabla}$ to denote respectively the Christoffel symbol and the covariant derivative with respect to $\tilde{g}$. Here $\tilde{g}$ denotes the Ricci soliton.

Variational stability of gradient Ricci solitons was studied for the first time by Cao et al. [4]. Since Ricci solitons are critical points of Perelman's λ-entropy and ν-entropy, they displayed the second variation of λ and ν functionals and, according to the second variation, explored the linear stability of some examples. For more results on stability of Ricci solitons with respect to the second variation of Perelman's ν-functional see [6, 5, 13]. Kröncke [22] considered a modified τ-flow with $X(t) = -grad_{g(t)}f_{g(t)}$ where $f_{g(t)}$ is a smooth function produced by minimizing Perelman's entropy functional and examined the stability of compact shrinking Ricci solitons under the flow. Guenther et al. [12] demonstrated linear stability of some nongradient homogeneous expanding

Ricci solitons. Jablonski et al. studied the linear stability of algebric Ricci solitons on simply connected solvable Lie groups [19] and expanding Ricci solitons with bounded curvature [20]. In the linear stability of Ricci solitons after normalizing Ricci flow, one applies the DeTurck trick and linearizes the flow at a fixed point:

$$\frac{\partial}{\partial t}h = Lh := \Delta_L h + 2\lambda h + L_X h,$$

where Δ_L is the Lichnerowicz Laplacian. A stationary solution of the previous equation is strictly (resp. weakly) linearly stable if the operator L has a negative (resp. nonpositive) spectrum. Following Guenther et al. [11], one applies Simonett's stability theorem to deduce dynamical stability from linear stability.

2. Equivalency of flows

In this section we show that (1.1) and (1.2) are equivalent.

Lemma 2.1 *Let* $(M^n, \bar{g}(\bar{t}))_{\bar{t}\in[0,\bar{T})}$ *be a solution to (1.2). Define* $(M^n, g(t))_{t\in[0,T)}$ *by*

$$g(t) = (1+2\rho t)\varphi_t^*\left(\bar{g}\left(\bar{t}\right)\right),$$

where φ_t *denotes the diffeomorphisms generated by the family of vector fields* $Y_t(x) = \frac{1}{1+2\rho t}X(x)$ *on* M^n*,* $\bar{t}(t) = \frac{\ln(1+2\rho t)}{2\rho}$*, and* $T = \frac{e^{2\rho\bar{T}}-1}{2\rho}$*. Then* $g(t)$ *is a solution of the Ricci flow.*

Proof We prove it by calculating.

$$\begin{aligned}
\frac{\partial}{\partial t}g(t) &= 2\rho\varphi_t^*(\bar{g}(\bar{t})) + (1+2\rho t)\frac{\partial}{\partial t}\varphi_t^*(\bar{g}(\bar{t})) \\
&= 2\rho\varphi_t^*(\bar{g}(\bar{t})) + (1+2\rho t)\varphi_t^*(L_{Y_t}\bar{g}) \\
&\quad +(1+2\rho t)\varphi_t^*\Big(\frac{d\bar{t}}{dt}(-2Ric(\bar{g}) - L_X\bar{g} - 2\rho\bar{g})\Big) \\
&= 2\rho\varphi_t^*(\bar{g}(\bar{t})) + \varphi_t^*(L_X\bar{g}) - 2Ric(g) - \varphi_t^*(L_X\bar{g}) - 2\rho\varphi_t^*(\bar{g}(\bar{t})) \\
&= -2Ric(g(t)),
\end{aligned}$$

and hence $g(t)$ is a solution of the Ricci flow. □

Lemma 2.2 *Let* $(M^n, \bar{g}(\bar{t}))_{\bar{t}\in[0,\bar{T})}$ *be a solution to (1.1). Define* $(M^n, g(t))_{t\in[0,T)}$ *by*

$$g(t) = e^{-2\rho t}\varphi_t^*(\bar{g}(\bar{t})),$$

where $\bar{t}(t) = \frac{e^{2\rho t}-1}{2\rho}$*,* $T = \frac{\ln(1+2\rho\bar{T})}{2\rho}$*, and* φ_t *denotes the 1-parameter family of maps* $\varphi_t : M^n \longrightarrow M^n$ *satisfying*

$$\frac{\partial}{\partial t}\varphi_t(p) = (\varphi_t)_* X_p, \quad \varphi_0(p) = p$$

for all $p \in M^n$*. Then* $g(t)$ *solves the modified Ricci flow.*

Proof By calculating we have

$$\begin{aligned}\frac{\partial}{\partial t}g(t) &= -2\rho e^{-2\rho t}\varphi_t^*(\bar{g}(\bar{t})) + e^{-2\rho t}\frac{\partial}{\partial t}\varphi_t^*(\bar{g}(\bar{t}))\\ &= -2\rho e^{-2\rho t}\varphi_t^*(\bar{g}(\bar{t})) + e^{-2\rho t}L_{[(\varphi_t^{-1})_*(\varphi_t)_*X]}(\varphi_t^*\bar{g}(\bar{t})) + e^{-2\rho t}\varphi_t^*(e^{2\rho t}(-2Ric(\bar{g}(\bar{t})))\\ &= -2\rho g(t) + L_X g(t) - 2Ric(g(t)).\end{aligned}$$

We show how to go from a solution of (1.4) back to a solution of the modified Ricci flow. □

Lemma 2.3 *If $g(t)$ is a solution of (1.4), we claim that $\bar{g}(t) = \varphi_t^* g(t)$ is a solution to the modified Ricci flow, where $\varphi_t : M^n \longrightarrow M^n$ defined by*

$$\frac{\partial}{\partial t}\varphi_t(p) = -W(\varphi_t(p)) + X_{\varphi_t(p)} - \varphi_{t_*}(X_p), \quad \varphi_0(p) = p$$

is a 1-parameter family of maps.

Proof We have

$$\begin{aligned}\frac{\partial}{\partial t}\bar{g} &= \frac{\partial}{\partial t}\varphi_t^* g(t)\\ &= \varphi_t^*(-2Ric(g(t)) + L_{W(t)}g(t) - L_X g - 2\rho g)\\ &\quad + L_{[(\varphi_t^{-1})_*(-W(t)+X-\varphi_{t_*}X)]}(\varphi_t^* g(t))\\ &= -2Ric(\bar{g}(t)) + \varphi_t^*(L_{W(t)}g(t)) - \varphi_t^*(L_X g) - 2\rho\bar{g} - L_{(\varphi_t^{-1})_* W(t)}(\varphi_t^* g(t))\\ &\quad + L_{(\varphi_t^{-1})_* X}(\varphi_t^* g(t)) - L_{(\varphi_t^{-1})_*(\varphi_t)_* X}(\varphi_t^* g(t))\\ &= -2Ric(\bar{g}(t)) - L_X\bar{g} - 2\rho\bar{g}.\end{aligned}$$

□

3. Stability and main results

In this section we shall focus on investigating the stability of Ricci solitons.

Lemma 3.1 *Let $g(t)$ be a solution to (1.4). The evolution equations for g_{ij} and g^{ij} in coordinate form satisfy the following equation:*

$$\begin{aligned}\frac{\partial}{\partial t}g_{ij} &= g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta g_{ij} - g^{\alpha\beta}g_{ip}\tilde{g}^{pq}\tilde{R}_{j\alpha q\beta} - g^{\alpha\beta}g_{jp}\tilde{g}^{pq}\tilde{R}_{i\alpha q\beta}\\ &\quad + \frac{1}{2}g^{\alpha\beta}g^{pq}(\tilde{\nabla}_i g_{p\alpha}\cdot\tilde{\nabla}_j g_{q\beta} + 2\tilde{\nabla}_\alpha g_{jp}\cdot\tilde{\nabla}_q g_{i\beta} - 2\tilde{\nabla}_\alpha g_{jp}\cdot\tilde{\nabla}_\beta g_{iq}\\ &\quad - 2\tilde{\nabla}_j g_{p\alpha}\cdot\tilde{\nabla}_\beta g_{iq} - 2\tilde{\nabla}_i g_{p\alpha}\cdot\tilde{\nabla}_\beta g_{jq})\\ &\quad - \tilde{\nabla}_i X_j - \tilde{\nabla}_j X_i + g^{\alpha\beta}(\tilde{\nabla}_i g_{j\beta} + \tilde{\nabla}_j g_{i\beta} - \tilde{\nabla}_\beta g_{ij})X_\alpha - 2\rho g_{ij}.\end{aligned} \tag{3.1}$$

By this equation we also have

$$\begin{aligned}
\frac{\partial}{\partial t} g^{ij} &= g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta g^{ij} + g^{\alpha\beta}g^{ik}g^{jl}g_{kp}\tilde{g}^{pq}\tilde{R}_{l\alpha q\beta} \\
&+ g^{\alpha\beta}g^{ik}g^{jl}g_{pl}\tilde{g}^{pq}\tilde{R}_{k\alpha q\beta} + g^{\alpha\beta}g^{ip}\tilde{\nabla}_\alpha g^{jq}\cdot\tilde{\nabla}_\beta g_{pq} + g^{\alpha\beta}g^{jq}\tilde{\nabla}_\alpha g^{ip}\cdot\tilde{\nabla}_\beta g_{pq} \\
&+ \frac{1}{2}g^{\alpha\beta}g^{pq}g^{ik}g^{jl}\cdot(2\tilde{\nabla}_\alpha g_{pl}\cdot\tilde{\nabla}_\beta g_{qk} + 2\tilde{\nabla}_l g_{p\alpha}\cdot\tilde{\nabla}_\beta g_{qk} + 2\tilde{\nabla}_k g_{p\alpha}\cdot\tilde{\nabla}_\beta g_{ql} \\
&- 2\tilde{\nabla}_\alpha g_{pl}\cdot\tilde{\nabla}_q g_{\beta k} - \tilde{\nabla}_k g_{p\alpha}\cdot\tilde{\nabla}_l g_{q\beta}) \\
&+ g^{ik}g^{jl}(\tilde{\nabla}_k X_l + \tilde{\nabla}_l X_k) + g^{ik}g^{jl}g^{\alpha\beta}(\tilde{\nabla}_\beta g_{kl} - \tilde{\nabla}_k g_{l\beta} - \tilde{\nabla}_l g_{k\beta})X_\alpha + \rho g^{ij}.
\end{aligned}$$

Proof In Shi's paper [27], the evolution equation was obtained for solutions to Ricci–DeTurck flow in coordinate form as follows.

$$\begin{aligned}
\frac{\partial}{\partial t} g^{ij} &= g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta g_{ij} - g^{\alpha\beta}g_{ip}\tilde{g}^{pq}\tilde{R}_{j\alpha q\beta} - g^{\alpha\beta}g_{jp}\tilde{g}^{pq}\tilde{R}_{i\alpha q\beta} \\
&+ \frac{1}{2}g^{\alpha\beta}g^{pq}(\tilde{\nabla}_i g_{p\alpha}\cdot\tilde{\nabla}_j g_{q\beta} + 2\tilde{\nabla}_\alpha g_{jp}\cdot\tilde{\nabla}_q g_{i\beta} - 2\tilde{\nabla}_\alpha g_{jp}\cdot\tilde{\nabla}_\beta g_{iq} \\
&- 2\tilde{\nabla}_j g_{p\alpha}\cdot\tilde{\nabla}_\beta g_{iq} - 2\tilde{\nabla}_i g_{p\alpha}\cdot\tilde{\nabla}_\beta g_{jq})
\end{aligned}$$

Using Shi's results, the lemma is clear. □

Remark 3.2 *One writes* $A < B$ *for symmetric* 2*-tensor* A *and* B *if* $B - A$ *is a nonnegative definite quadratic form, that is, if* $(B-A)(V,V) \geq 0$ *for all vectors* V.

Definition 3.3 *Let* g *be a metric on* M^n. *Let* $\varepsilon > 0$. *Then we say that* g *is* ε*-close to* $\bar{g}$ *if*

$$(1+\varepsilon)^{-1}\bar{g} \leq g \leq (1+\varepsilon)\bar{g}.$$

We prove the following theorem.

Theorem 3.4 *Suppose* $g_{ij}(x,t) > 0$ *is a solution of (1.4). Then for any* $\delta > 0$ *there exists a constant* T *such that*

$$(1-\delta)\bar{g}_{ij}(x) \leq g_{ij}(x,t) \leq (1+\delta)\bar{g}_{ij}(x), \quad x \in M,\ 0 \leq t \leq T.$$

First, we prove the following lemmas. We may always choose a local coordinate around a fixed point p, so that at p we have

$$\bar{g}_{ij}(p) = \delta_{ij}, \quad g_{ij}(p) = \delta_{ij}\lambda_i(p). \tag{3.2}$$

Lemma 3.5 *Suppose* $g_{ij}(x,t)$ *is a solution of (1.4). Then for any* $\delta > 0$ *there exists a constant* T *such that*

$$g_{ij}(x,t) \geq (1-\delta)\bar{g}_{ij}(x), \quad x \in M,\ 0 \leq t \leq T.$$

Proof Define

$$\varphi(x,t) = g^{\alpha_1\beta_1}\bar{g}_{\beta_1\alpha_2}g^{\alpha_2\beta_2}\bar{g}_{\beta_2\alpha_3}g^{\alpha_3\beta_3}\bar{g}_{\beta_3\alpha_4}\dots g^{\alpha_m\beta_m}\bar{g}_{\beta_m\alpha_1}.$$

In the selected local coordinate we have

$$\varphi(x,t) = \sum_{k=1}^{n}(\frac{1}{\lambda_k})^m,$$

and thus

$$\frac{\partial\varphi}{\partial t} = m(\frac{1}{\lambda_i})^{m-1}\frac{\partial}{\partial t}g^{ii}.$$

Using Lemma 3.1 and as in [27] Lemma 2.2, we see that

$$\frac{\partial\varphi}{\partial t} \leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\varphi + \frac{2m}{\lambda_i^m\lambda_q}\tilde{g}^{ip}\tilde{R}_{iqpq} + m(\frac{1}{\lambda_i})^{m-1}\Big((\frac{1}{\lambda_i})^2(L_Xg)_{ii} + \frac{1}{\lambda_i}\rho\Big).$$

Since M is compact, there is a constant c such that

$$|\frac{2m}{\lambda_q}\tilde{g}^{ip}\tilde{R}_{iqpq} + m(\frac{1}{\lambda_i})(L_Xg)_{ii} + m\rho| \leq c.$$

It is easy to see that

$$\frac{\partial\varphi}{\partial t} \leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\varphi + c\varphi.$$

We define

$$\varphi(t) = \max_{x\in M}\varphi(x,t).$$

Using the maximal principle, we get

$$\frac{d\varphi}{dt} \leq c\varphi, \quad \varphi(0) = n.$$

Thus, we have

$$\varphi(x,t) \leq ne^{ct} \quad \forall x \in M.$$

If $0 \leq t \leq \frac{1}{c}\ln 2$, then $\varphi(x,t) \leq 2n$; that is, $\sum_{k=1}^{n}(\frac{1}{\lambda_k})^m \leq 2n$. Then $(\frac{1}{\lambda_k})^m \leq 2n\ \forall k$ and $\lambda_k \geq (\frac{1}{2n})^{1/m}\ \forall k$. According to the selected coordinate it follows that

$$g_{ij}(x,t) \geq (\frac{1}{2n})^{1/m}\bar{g}_{ij}(x).$$

Let m be an integer so that $\frac{\log 2n}{\log(1/(1-\delta))} \leq m \leq \frac{\log 2n}{\log(1/(1-\delta))} + 1$. Therefore, $(1/2n)^{1/m} \geq 1-\delta$ and

$$g_{ij}(x,t) \geq (1-\delta)\bar{g}_{ij}(x), \quad x \in M,\ 0 \leq t \leq \frac{1}{c}\ln 2.$$

□

Lemma 3.6 *Suppose $g_{ij}(x,t)$ is a solution of (1.4). Then for any $\delta > 0$ there exists a constant T such that*

$$g_{ij}(x,t) \leq (1+\delta)\bar{g}_{ij}(x), \quad x \in M,\ 0 \leq t \leq T.$$

Proof Define

$$\psi(x,t) = \bar{g}^{\alpha_1\beta_1} g_{\beta_1\alpha_2} \bar{g}^{\alpha_2\beta_2} g_{\beta_2\alpha_3} \bar{g}^{\alpha_3\beta_3} g_{\beta_3\alpha_4} \dots \bar{g}^{\alpha_m\beta_m} g_{\beta_m\alpha_1}.$$

By (3.2) we have $\psi(x,t) = \sum_{k=1}^{n} (\lambda_k)^m$. Using Lemma 3.1,

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= m(\lambda_i)^{m-1} \frac{\partial}{\partial t} g_{ii} \\ &= m(\lambda_i)^{m-1} \Big(g^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta g_{ii} - 2\frac{\lambda_i}{\lambda_\alpha} \tilde{g}^{pq} \tilde{R}_{i\alpha q\alpha} + \frac{1}{2\lambda_q\lambda_p} (\tilde{\nabla}_i g_{pq} \cdot \tilde{\nabla}_i g_{pq} + 2 \\ &\quad \tilde{\nabla}_q g_{ip} \cdot \tilde{\nabla}_p g_{iq} - 2\tilde{\nabla}_q g_{ip} \cdot \tilde{\nabla}_q g_{ip} - 4\tilde{\nabla}_i g_{pq} \cdot \tilde{\nabla}_q g_{ip}) - (\lambda_i)^2 (L_X g)^{ii} - 2\rho\lambda_i \Big) \end{aligned}$$

and then

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= g^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \psi - \frac{m}{\lambda_\alpha} (\tilde{\nabla}_\alpha g_{ij})^2 \Big[(\lambda)_i^{m-2} + (\lambda_i)^{m-3}(\lambda_j) + \dots + (\lambda_j)^{m-2} \Big] \\ &\quad + (\lambda_i)^m \Big(-2\frac{m}{\lambda_\alpha} \tilde{g}^{pq} \tilde{R}_{i\alpha q\alpha} + \frac{m}{2\lambda_i\lambda_q\lambda_p} (\tilde{\nabla}_i g_{pq} \cdot \tilde{\nabla}_i g_{pq} + 2\tilde{\nabla}_q g_{ip} \cdot \tilde{\nabla}_p g_{iq} \\ &\quad -2\tilde{\nabla}_q g_{ip} \cdot \tilde{\nabla}_q g_{ip} - 4\tilde{\nabla}_i g_{pq} \cdot \tilde{\nabla}_q g_{ip}) - \lambda_i (L_X g)^{ii} - 2\rho \Big). \end{aligned}$$

Since M is compact, there exists a constant c such that

$$\frac{\partial \psi}{\partial t} \le g^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \psi + c\psi, \quad \psi(x,0) = n.$$

Using the maximal principle, it follows that

$$\psi(x,t) \le n e^{ct}.$$

If we let $0 \le t \le \frac{1}{c} \ln 2$, we have $\psi(x,t) \le 2n$; that is, $\sum_{k=1}^{n} (\lambda_k)^m \le 2n$. Thus, $\lambda_k \le (2n)^{1/m}\ \forall k$. By (3.2) we have

$$g_{ij}(x,t) \le (\frac{1}{2n})^{1/m} \bar{g}_{ij}(x).$$

Let m be an integer so that $\frac{\log 2n}{\log(1/(1+\delta))} \le m \le \frac{\log 2n}{\log(1/(1+\delta))} + 1$. Therefore, $(1/2n)^{1/m} \le 1 + \delta$ and

$$g_{ij}(x,t) \le (1+\delta)\bar{g}_{ij}(x), \quad x \in M,\ 0 \le t \le \frac{1}{c} \ln 2.$$

A combination of Lemmas 3.5 and 3.6 easily gives Theorem 3.4. □

Lemma 3.7 *Let $g \in \mathcal{M}^\infty(M^n, [0,T))$, $0 < T < \infty$, be a solution to (1.4), which is ε-close to the $\tilde{g}$. If ε is sufficiently small, then*

$$\frac{\partial}{\partial t} |g - \tilde{g}|^2 \le g^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta |g - \tilde{g}|^2 + C|g - \tilde{g}|^2.$$

Proof We choose a local coordinate around a fixed point p, so that at p we have

$$\tilde{g}_{ij}(p) = \delta_{ij}, \quad g_{ij}(p) = \delta_{ij}\lambda_i(p). \tag{3.3}$$

Then by (3.3) we find

$$|g - \tilde{g}|^2 = (g_{ij} - \tilde{g}_{ij})(g_{kl} - \tilde{g}_{kl})\tilde{g}^{ik}\tilde{g}^{jl} = (g_{ii} - \tilde{g}_{ii})^2. \tag{3.4}$$

Lemma 3.1 yields

$$\begin{aligned} \frac{\partial}{\partial t}|g - \tilde{g}|^2 &= 2\sum_i (g_{ii} - \tilde{g}_{ii})(\frac{\partial}{\partial t} g_{ii}) \\ &= 2\sum_i (g_{ii} - \tilde{g}_{ii})\Big(g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta g_{ii} - 2\frac{\lambda_i}{\lambda_q}\tilde{R}_{iqiq} + \frac{1}{2\lambda_q\lambda_p}(\tilde{\nabla}_i g_{pq}\cdot\tilde{\nabla}_i g_{pq} \\ &\quad +2\tilde{\nabla}_q g_{ip}\cdot\tilde{\nabla}_p g_{iq} - 2\tilde{\nabla}_q g_{ip}\cdot\tilde{\nabla}_q g_{ip} - 4\tilde{\nabla}_i g_{pq}\cdot\tilde{\nabla}_q g_{ip}) \\ &\quad -(L_X g)_{ii} - 2\rho\lambda_i\Big). \end{aligned} \tag{3.5}$$

From (3.4) it follows that

$$\tilde{\nabla}_\alpha\tilde{\nabla}_\beta|g - \tilde{g}|^2 = 2(g_{ii} - \tilde{g}_{ii})\tilde{\nabla}_\alpha\tilde{\nabla}_\beta(g_{ii} - \tilde{g}_{ii}) + 2\tilde{\nabla}_\alpha(g_{ii} - \tilde{g}_{ii})\tilde{\nabla}_\beta(g_{ii} - \tilde{g}_{ii}). \tag{3.6}$$

We use $*$ as in [14]. Substituting (3.6) and (3.3) into (3.5) gives

$$\begin{aligned} \frac{\partial}{\partial t}|g - \tilde{g}|^2 &= g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta|g - \tilde{g}|^2 - 2\frac{1}{\lambda_\alpha}\tilde{\nabla}_\alpha(g_{ii} - \tilde{g}_{ii})\tilde{\nabla}_\alpha(g_{ii} - \tilde{g}_{ii}) \\ &\quad +\lambda_i(\lambda_i - 1)\Big(-4\frac{1}{\lambda_q}\tilde{R}_{iqiq} - \lambda_i(L_X g)^{ii} - 2\rho\Big) + (\lambda_i - 1)(\tilde{\nabla}g * \tilde{\nabla}g)_{ii} \\ &\leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta|g - \tilde{g}|^2 + (\lambda_i - 1)^2\Big(-4\frac{1}{\lambda_q}\tilde{R}_{iqiq} - \lambda_i(L_X g)^{ii} - 2\rho\Big) \\ &\quad +(\lambda_i - 1)\Big(-4\frac{1}{\lambda_q}\tilde{R}_{iqiq} - \lambda_i(L_X g)^{ii} - 2\rho\Big) \\ &\leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta|g - \tilde{g}|^2 + (\lambda_i - 1)^2|-4\frac{1}{\lambda_q}\tilde{R}_{iqiq} - \lambda_i(L_X g)^{ii} - 2\rho| \\ &\quad +c_1(\lambda_i - 1)^2|-4\frac{1}{\lambda_q}\tilde{R}_{iqiq} - \lambda_i(L_X g)^{ii} - 2\rho|, \end{aligned}$$

where c_1 is a fixed enough big number. Then we have

$$\frac{\partial}{\partial t}|g - \tilde{g}|^2 \leq g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta|g - \tilde{g}|^2 + C|g - \tilde{g}|^2,$$

where C is a constant. □

Theorem 3.8 *Let $g \in \mathcal{M}^\infty(M^n,[0,T))$, $0 < T < \infty$, be a solution to (1.4). Let $\delta > 0$. Then there exists $\varepsilon = \varepsilon(n,T,\delta)$ such that $\sup_{M^n} |\bar{g} - \tilde{g}| \le \varepsilon$ implies*

$$\sup_{M^n \times [0,T)} |g - \tilde{g}| \le \delta.$$

Proof Let $g \in \mathcal{M}^\infty(M^n,[0,T))$, $T > 0$, be a solution to (1.4). By Lemma 3.4 and the maximum principle we have

$$sup|g(t) - \tilde{g}|^2 \le |\bar{g} - \tilde{g}|e^{Ct}.$$

Fix $\varepsilon = \delta e^{-CT}$ where C is the constant in Lemma 3.7, and then

$$sup|g(t) - \tilde{g}|^2 e^{C(T-t)} \le sup|\bar{g} - \tilde{g}|^2 e^{CT} \le \varepsilon e^{CT} = \delta.$$

□

Theorem 3.9 *Let $g \in \mathcal{M}^\infty(M^n,[0,T))$, $T > 0$, be a solution to (1.4). If $\sup|g - \tilde{g}| < \frac{1}{3}$ then for $a = 8|\tilde{R}_m| + 8|\tilde{Ric}| + divX$ we have*

$$\int |g - \tilde{g}|^2 d\mu_{\tilde{g}} \le e^{at} \int |g(0) - \tilde{g}|^2 d\mu_{\tilde{g}}.$$

Therefore, for $a < 0$ we have L^2-norm stability.

Proof

$$\begin{aligned}
\frac{\partial}{\partial t}\int |g-\tilde{g}|^2 d\mu_{\tilde{g}} \quad &\le \quad \int g^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta|g-\tilde{g}|^2 - (2-\frac{1}{3})|\tilde{\nabla}(g-\tilde{g})|^2 - 4(g_{ii}-\tilde{g}_{ii})g^{\alpha\beta}g_{ip}\tilde{g}^{pq}\tilde{R}_{i\alpha q\beta} \\
&\quad -2(g_{ii}-\tilde{g}_{ii})((L_X g)_{ii} + 2\rho g_{ii})d\mu_{\tilde{g}} \\
&\le \quad -\int \tilde{\nabla}_\alpha g^{\alpha\beta}\tilde{\nabla}_\beta|g-\tilde{g}|^2 - (2-\frac{1}{3})|\tilde{\nabla}(g-\tilde{g})|^2 - 4(g_{ii}-\tilde{g}_{ii})g^{\alpha\beta}g_{ip}\tilde{g}^{pq}\tilde{R}_{i\alpha q\beta} \\
&\quad -2(g_{ii}-\tilde{g}_{ii})((L_X(g-\tilde{g}))_{ii} + (L_X\tilde{g})_{ii} + 2\rho\tilde{g}_{ii} + 2\rho(g_{ii}-\tilde{g}_{ii}))d\mu_{\tilde{g}} \\
&\le \quad \int -(2-3(\frac{1}{3}))|\tilde{\nabla}(g-\tilde{g})|^2 - 4(g_{ii}-\tilde{g}_{ii})g^{\alpha\beta}g_{ip}\tilde{g}^{pq}\tilde{R}_{i\alpha q\beta} + 4(g_{ii}-\tilde{g}_{ii})\tilde{R}_{ii} \\
&\quad -4\rho|g-\tilde{g}|^2 - 2(g_{ii}-\tilde{g}_{ii})(L_X(g-\tilde{g}))_{ii}d\mu_{\tilde{g}},
\end{aligned}$$

where we applied Kato's inequality $|\tilde{\nabla}|h||^2 \le |\tilde{\nabla}h|^2$, which is valid whenever $|h| \ne 0$. Furthermore, we used that $|\tilde{\nabla}_\alpha g^{\alpha\beta}\tilde{\nabla}_\beta|h|^2| \le 2(\frac{1}{3})|\tilde{\nabla}h|^2$ and that $\tilde{g}$ satisfies equation (1.3). Let $h = g - \tilde{g}$, for the last term in previous inequality; we have

$$\int (L_X h)_{ii}h_{ii}d\mu_{\tilde{g}} = \int (-\frac{1}{2}divX|h|^2 - 2\tilde{R}_i^k h_{kj}h^{ij} - 2\rho|h|^2)d\mu_{\tilde{g}}$$

and then

$$\frac{\partial}{\partial t}\int |h|^2 d\mu_{\tilde{g}} \le \int -4h_{ii}g^{\alpha\beta}g_{ip}\tilde{g}^{pq}\tilde{R}_{i\alpha q\beta} + 4h_{ii}\tilde{R}_{ii} - 4\rho|h|^2 + divX|h|^2 + 4\tilde{R}_i^k h_{kj}h^{ij} + 4\rho|h|^2.$$

In the local coordinate mentioned in Lemma 3.7 we get

$$\begin{aligned}
-(g_{ii}-\tilde{g}_{ii})g^{\alpha\beta}g_{ip}\tilde{g}^{pq}\tilde{R}_{i\alpha q\beta} &= -\frac{\lambda_i}{\lambda_\alpha}(\lambda_i-1)\tilde{R}_{i\alpha i\alpha} \\
&= \lambda_i(\lambda_i-1)(1-\frac{1}{\lambda_\alpha})\tilde{R}_{i\alpha i\alpha}-\lambda_i(\lambda_i-1)\tilde{R}_{i\alpha i\alpha} \\
&= \lambda_i(\lambda_i-1)(1-\frac{1}{\lambda_\alpha})\tilde{R}_{i\alpha i\alpha}-(\lambda_i-1)^2\tilde{R}_{i\alpha i\alpha}-(\lambda_i-1)\tilde{R}_{i\alpha i\alpha} \\
&= \frac{\lambda_i}{\lambda_\alpha}(\lambda_i-1)(\lambda_\alpha-1)\tilde{R}_{i\alpha i\alpha}+|h|^2|\tilde{Ric}|-h_{ii}\tilde{R}_{ii} \\
&\leq 2|\tilde{R}_m||h|^2+|h|^2|\tilde{Ric}|-h_{ii}\tilde{R}_{ii}
\end{aligned}$$

and then

$$\frac{\partial}{\partial t}\int|h|^2 d\mu_{\tilde{g}} \leq \int(8|\tilde{R}_m|+8|\tilde{Ric}|+divX)|h|^2,$$

and hence

$$\|g-\tilde{g}\|_{L^2} \leq e^{at}\|g(0)-\tilde{g}\|_{L^2}.$$

□

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