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A note on m-embedded subgroups of finite groups

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Abstract: Let A be a subgroup of G. A is m-embedded in G if G has a subnormal subgroup T and a $\{1 \leq G\}$-embedded subgroup C such that $G = AT$ and $T \cap A \leq C \leq A$. In this paper, we study the structure of finite groups by using m-embedded subgroups and obtain some new results about p-supersolvability and p-nilpotency of finite groups.

Key words: Sylow subgroup, $\{1 \leq G\}$-embedded, m-embedded subgroup, saturated formation, finite groups

1. Introduction

Throughout the paper, all groups are finite. Most of the notation is standard and can be found in [3, 6, 10, 11]. Let $\mathcal{F}$ be a class of groups. $\mathcal{F}$ is said to be a formation provided that (1) if $G \in \mathcal{F}$ and $H \trianglelefteq G$, then $G/H \in \mathcal{F}$, and (2) if G/M and G/N are in $\mathcal{F}$, then $G/M \cap N$ is in $\mathcal{F}$. A formation $\mathcal{F}$ is said to be saturated if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$. It is well known that the class of all p-supersolvable groups and the class of all p-nilpotent groups are saturated formations. Let A be a subgroup of G, $K \leq H \leq G$ and p a prime. Then: (1) A covers the pair (K, H) if $AH = AK$; (2) A avoids (K, H) if $A \cap H = A \cap K$. Recall that a subgroup A of G is called a CAP-subgroup [3, A, Definition 10.8] if A either covers or avoids each pair (K, H), where H/K is a chief factor of G. A subgroup A is called a partial CAP-subgroup [1] or a semicover-avoiding subgroup [8] of G if A either covers or avoids each pair (K, H), where H/K is a factor of some fixed chief series of G. By using the CAP-subgroups and the semicover-avoiding subgroups, group theorists have obtained many interesting results (see, for example, [2, 4, 9]). Furthermore, if E is a quasinormal subgroup of G, then for every maximal pair of G, that is, a pair (K, H), where K is a maximal subgroup of H, E either covers or avoids (K, H). Based on the definitions and properties above, Guo and Skiba presented a new concept as follows:

Definition 1.1 (7) *Let A be a subgroup of G and $\Sigma = G_0 \leq G_1 \leq \ldots \leq G_n$ some subgroup series of G. Then A is Σ-embedded in G if A either covers or avoids every maximal pair (K, H) such that $G_{i-1} \leq K < H \leq G_i$, for some i.*

Here we improve Theorem 4.1 of [7], and present a result of p-nilpotency of group G with some "extra hypothesis", where p is an odd prime divisor of $|G|$. Meanwhile, we study the structure of G under the

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assumption of G is p-solvable, where p is a prime divisor of $|G|$.

Theorem 1.2 *Let p be an odd prime divisor of $|G|$ and P be a Sylow p-subgroup of G. Suppose that every maximal subgroup P_1 of P is m-embedded in G. Then G is p-nilpotent if one of the following conditions holds:*
(1) *$N_G(P_1)$ is p-nilpotent for every maximal subgroup P_1 of P.*
(2) *$N_G(P)$ is p-nilpotent.*

Theorem 1.3 *Let G be a p-solvable group and P a Sylow p-subgroup of G. If every maximal subgroup of P is m-embedded in G, then G is p-supersolvable.*

Theorem 1.4 *Let G be a p-solvable group and p a prime divisor of $|G|$. If every maximal subgroup of $F_p(G)$ containing $O_{p'}(G)$ is m-embedded in G, then G is p-supersolvable.*

2. Preliminaries

For the sake of convenience, we first list here some known results that will be useful in the sequel.

Lemma 2.1 (7, Lemma 2.13) *Let K and H be subgroups of G. Suppose that K is m-embedded in G and H is normal in G. Then*
(1) If $H \leq K$, then K/H is m-embedded in G/H .
(2) If $K \leq E \leq G$, then K is m-embedded in E.
(3) If $(|H|, |K|) = 1$, then HK/H is m-embedded in G/H.
(4) Suppose that K is a p-subgroup for some prime p, K is m-embedded in G, and K is not $\{1 \leq G\}$-embedded in G. Then G has a normal subgroup M such that $|G : M| = p$ and $G = KM$.

Lemma 2.2 (7, Lemma 2.14) *Let P be a normal nonidentity p-subgroup of G with $|P| = p^n$ and $P \cap \Phi(G) = 1$. Suppose that there is an integer k such that $1 \leq k < n$ and the subgroups of P of order p^k are m-embedded in G, then some maximal subgroup of P is normal in G.*

Lemma 2.3 (7, Lemma 2.5) *Every $\{1 \leq G\}$-embedded subgroup of G is subnormal in G.*

3. The proofs

Proof of Theorem 1.1 Assume that the assertion is false and choose G to be a counterexample of minimal order. We will divide the proof into the following steps.

(1) $O_{p'}(G) = 1$.

In fact, if $O_{p'}(G) \neq 1$, then we consider the quotient group $G/O_{p'}(G)$. If $N_G(P_1)$ is p-nilpotent, then

$$N_{G/O_{p'}(G)}(P_1O_{p'}(G)/O_{p'}(G)) = N_G(P_1)O_{p'}(G)/O_{p'}(G)$$

is p-nilpotent. By Lemma 2.1(3), $G/O_{p'}(G)$ satisfies the conditions of the theorem, and the minimal choice of G implies that $G/O_{p'}(G)$ is p-nilpotent. Hence G is p-nilpotent, a contradiction. Similarly, if $N_G(P)$ is p-nilpotent, then we have $G/O_{p'}(G)$ is p-nilpotent also, a contradiction.

(2) If S is a proper subgroup of G containing P, then S is p-nilpotent.

If $N_G(P_1)$ is p-nilpotent, clearly, $N_S(P_1) \leq N_G(P_1)$ and then $N_S(P_1)$ is p-nilpotent. Applying Lemma 2.1(2), we find that S satisfies the hypothesis of our theorem. Now, the minimal choice of G implies that S is p-nilpotent. If $N_G(P)$ is p-nilpotent, then we still obtain that S is p-nilpotent since $N_S(P) \leq N_G(P)$.

(3) $O_p(G) \neq 1$ and G/N is p-nilpotent, where $N = O_p(G)$ is the unique minimal normal subgroup of G.

Case I. $N_G(P_1)$ is p-nilpotent.

Since G is not p-nilpotent, $N_G(Z(J(P)))$ is not p-nilpotent by the Glauberman–Thompson Theorem, where $J(P)$ is the Thompson subgroup of P. Then $P \leq N_G(Z(J(P)))$. By (2), we have $N_G(Z(J(P))) = G$ and hence $O_p(G) \neq 1$. Let N be a minimal normal subgroup of G contained in $O_p(G)$.

If $N = P$, then G/N is p-nilpotent. If $|P : N| = p$, then $G = N_G(N)$ is p-nilpotent, a contradiction. Now we may assume that $|P : N| > p$. For every maximal subgroup P_1/N of P/N,

$$N_{G/N}(P_1/N) = N_G(P_1N)/N = N_G(P_1)/N$$

is p-nilpotent and P_1/N is m-embedded in G/N by Lemma 2.1(1). Therefore G/N satisfies the hypothesis of the theorem, and hence G/N is p-nilpotent. Obviously, N is the unique minimal normal subgroup of G contained in $O_p(G)$ and $\Phi(G) = 1$. Then we obtain that $N = O_p(G)$ is an elementary abelian p-group.

Case II. $N_G(P)$ is p-nilpotent.

Since G is not p-nilpotent, by Corollary of [12], there exists a characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent. Since $N_G(P)$ is p-nilpotent, we may choose a characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent, but $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P with $H < K \leq P$. Since $P \leq N_G(H)$ and $N_G(H)$ is not p-nilpotent, we have $N_G(H) = G$ by (2). This leads to $O_p(G) \neq 1$ and $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P such that $O_p(G) < K \leq P$. Now by using Corollary of [12] again, we see that $G/O_p(G)$ is p-nilpotent and $|P : O_p(G)| > p$. Let N be a minimal normal subgroup of G contained in $O_p(G)$.

Since $|P : N| > p$, P/N is a Sylow p-subgroup of G/N, and

$$N_{G/N}(P/N) = N_G(PN)/N = N_G(P)/N$$

is p-nilpotent and every maximal subgroup P_1/N of P/N is m-embedded in G/N by Lemma 2.1(1). Therefore G/N satisfies the hypothesis of the theorem, and hence G/N is p-nilpotent. Obviously, N is the unique minimal normal subgroup of G contained in $O_p(G)$ and $\Phi(G) = 1$. Then we obtain that $N = O_p(G)$ is an elementary abelian p-group.

(4) $G = PQ$, where Q is a Sylow q-subgroup of G and $q \neq p$ is a prime divisor of $|G|$.

By (3), immediately we obtain that G is p-solvable, and then by (1) $C_G(N) = N$ since $N \leq C_G(N) \leq N$. For any $q \in \pi(G)$ with $q \neq p$, Theorem 6.3.5 of [5] implies that there exists a Sylow q-subgroup Q of G such that $G_1 = PQ$ is a subgroup of G. If $G_1 < G$, then G_1 is p-nilpotent by (2). This leads to $Q \leq C_G(N) \leq N$, a contradiction. Thus $G = PQ$.

(5) The final contradiction.

Since $N \nleq \Phi(G)$, there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Let M_p be Sylow p-subgroup of M. Firstly, we may assume that $M_p \neq 1$. Otherwise, $M_p = 1$ and then $P = N$. If $N_G(P)$ is p-nilpotent, then G is p-nilpotent, a contradiction. If $N_G(P_1)$ is p-nilpotent, then there exists a maximal subgroup P_1 of P such that P_1 is normal in G by Lemma 2.2. Therefore $G = N_G(P_1)$ is p-nilpotent, a contradiction. Now we may obtain the final contradiction as follows.

Now we pick a maximal subgroup P_1 of P such that $M_p \leq P_1$. By hypothesis, P_1 is m-embedded in G, that is, G has a subnormal subgroup T and a $\{1 \leq G\}$-embedded subgroup C such that $G = P_1T$ and $P_1 \cap T \leq C \leq P_1$. Applying Lemma 2.3, we obtain that $C \leq O_p(G) = N$.

Assume that $C \neq 1$. If $C < N$, then for $N \cap M = 1$, we obtain C neither covers nor avoids maximal pair (M, G), a contradiction. Hence we may assume that $C = N$, i.e. $N \leq P_1$ and then $P = NM_p \leq P_1 < P$, a contradiction.

Assume that $C = 1$. The Sylow p-subgroup of T is cyclic with order p. It follows from $N \leq O^p(G) \leq T$ that $|N| = p$. Therefore $M \cong G/N = N_G(N)/C_G(N)$ is isomorphic to a subgroup of $Aut(N)$, and then M is cyclic with order q^α by (4), that is, $M_p = 1$, a contradiction.

The final contradiction completes our proof.

Proof of Theorem 1.2 Assume that the assertion is false and choose G to be a counterexample of minimal order. Furthermore, we have that

(1) $O_{p'}(G) = 1$.

If $L = O_{p'}(G) \neq 1$, we consider G/L. Clearly, P_1L/L is a maximal subgroup of Sylow p-subgroup of G/L where P_1 is a maximal subgroup of P. Since P_1 is m-embedded in G, we have P_1L/L is also m-embedded in G/L by Lemma 2.1(3). Therefore G/L satisfies the condition of the theorem. The minimal choice of G implies that G/L is p-supersolvable, and hence G is p-supersolvable, a contradiction.

(2) $O_p(G) \neq 1$.

Since G is p-solvable and $O_{p'}(G) = 1$, we have that a minimal normal subgroup of G is an abelian p-group and hence $O_p(G) \neq 1$.

(3) Final contradiction.

By (2), we may pick a minimal normal subgroup N of G contained in $O_p(G)$. If $N = P$ then G/N is p-supersolvable. If $N = P_1$, where P_1 is a maximal subgroup of P, then G/N is p-supersolvable. Now we may assume that $|P : N| > p$. By Lemma 2.1(1), we know that G/N satisfies the condition of the theorem, and hence the minimality of G implies that G/N is p-supersolvable; on the other hand, since the class of all p-supersolvable groups is a saturated formation, we have N is the unique minimal normal subgroup of G and $O_p(G) = N \nleq \Phi(G)$. If $O_p(G) = P$, then by Lemma 2.2, some maximal subgroup of P is normal in G, a contradiction. Now we may assume that $N < P$.

Clearly, there exists a maximal subgroup M of G such that $G = NM$ with $N \cap M = 1$ and $P = NM_p$ with $M_p \neq 1$. Now we choose a maximal subgroup P_1 with $M_p \leq P_1$. By hypothesis, P_1 is m-embedded in G. Therefore G has a subnormal subgroup T and a $\{1 \leq G\}$-embedded subgroup C such that $G = P_1T$ and $P_1 \cap T \leq C \leq P_1$. On the other hand, we know that $C \leq O_p(G)$. Therefore $C \leq N$. If $1 < C < N$, then for $N \cap M = 1$, we have C neither covers nor avoids maximal pair (M, G). Now we may assume that either $C = N$ or $C = 1$. By the choice of P_1, we immediately have $P_1 \cap T = 1$ and then the Sylow p-subgroup of T is cyclic with order p. It follows from $N \leq O^p(G) \leq T$ that $|N| = p$. Therefore G is p-supersolvable since G/N p-supersolvable, a contradiction.

The final contradiction completes our proof.

Proof of Theorem 1.3. Assume that the assertion is false and choose G to be a counterexample of minimal order. Furthermore, we have that

(1) $O_{p'}(G) = 1$.

If $T = O_{p'}(G) \neq 1$, we consider G/T. Firstly, $F_p(G/T) = F_p(G)/T$. Let M/T be a maximal subgroup of $F_p(G/T)$. Then M is a maximal subgroup of $F_p(G)$ containing $O_{p'}(G)$. Since M is m-embedded in G, then M/T is m-embedded in G/T by Lemma 2.1(3). Thus G/T satisfies the hypothesis of the theorem. The minimality of G implies that G/T is p-supersolvable and so is G, a contradiction.

(2) $\Phi(G) = 1$ and $F_p(G) = F(G) = O_p(G)$.

If not, then $L = \Phi(G) \neq 1$. We consider G/L. Since $O_{p'}(G) = 1$, it is easy to show that $F_p(G) = F(G) = O_p(G)$. This implies that $F_p(G/L) = O_p(G/L) = O_p(G)/L = F_p(G)/L$. If P_1/L is a maximal subgroup of $F_p(G/L)$, then P_1 is a maximal subgroup of $F_p(G)$. Since P_1 is m-embedded in G and hence P_1/L is m-embedded in G/L by Lemma 2.1(1). Thus G/L satisfies the hypothesis of the theorem. The minimal choice of G implies that G/L is p-supersolvable and so is G, since the class of all p-supersolvable groups is a saturated formation, a contradiction.

(3) Every minimal normal subgroup of G contained in $F(G)$ is cyclic of order p.

By (2), $P = F(G) = R_1 \times \cdots \times R_t$, where R_i $(i = 1, 2, \cdots, t)$ is a minimal normal subgroup of G contained in $F(G)$. At the same time, Lemma 2.2 implies that $t \geq 2$. Since G is p-solvable and $O_{p'}(G) = 1$, we have $C_G(O_p(G)) \leq O_p(G)$. Thus $C_G(F(G)) = F(G)$. Suppose that there exists R_i such that $|R_i| > p$. Without loss of generality, let $i = 1$ and $R = R_2 \times \cdots \times R_t$. Obviously, we may assume that $P/R \cap \Phi(G/R) = 1$, in fact, if $P/R \cap \Phi(G/R) \neq 1$, then $P/R \leq \Phi(G/R)$ since $R_1 \cong P/R$ is a chief factor of G. Therefore $P \leq \Phi(G)R$ and then $P = P \cap \Phi(G)R = R(P \cap \Phi(G)) = R$, a contradiction. Applying Lemma 2.1(1), G/R satisfies the hypothesis of the theorem and we have that some maximal subgroup of P/R is normal in G/R by Lemma 2.2, which contradicts the minimality of R_1. Therefore every R_i is of order p.

(4) The final contradiction.

By (3), $P = F(G) = R_1 \times \cdots \times R_t$, where R_i is a minimal normal subgroup of G of order p. For each i the quotient $G/C_G(R_i)$ is a subgroup of $\mathrm{Aut}(R_i)$ and hence is abelian. Since the class of all p-supersolvable groups is a formation, we have $G/\bigcap_{i=1}^{t}(C_G(R_i))$ is p-supersolvable, and thus $G/F(G)$ is p-supersolvable because $\bigcap_{i=1}^{t}(C_G(R_i)) = C_G(F(G)) = F(G)$. Actually, all chief factors of G below $F(G)$ are cyclic groups of order p; therefore G is p-supersolvable.

The final contradiction completes our proof.

4. Applications

Obviously, if H is $\{1 \leq G\}$-embedded in G, then H is m-embedded in G. Therefore we have the following corollaries.

Corollary 4.1 *Let p be an odd prime divisor of $|G|$ and P be a Sylow p-subgroup of G. If every maximal subgroup P_1 of P is $\{1 \leq G\}$-embedded in G and $N_G(P_1)$ is p-nilpotent, then G is p-nilpotent.*

Corollary 4.2 *Let p be an odd prime divisor of $|G|$ and P be a Sylow p-subgroup of G. If every maximal subgroup P_1 of P is $\{1 \leq G\}$-embedded in G and $N_G(P)$ is p-nilpotent, then G is p-nilpotent.*

Corollary 4.3 *Let G be a p-solvable group. If every maximal subgroup of a Sylow subgroup of G is $\{1 \leq G\}$-embedded in G, then G is p-supersolvable.*

Corollary 4.4 *Let G be a p-solvable group and p a prime divisor of $|G|$. If every maximal subgroup of $F_p(G)$ containing $O_{p'}(G)$ is $\{1 \leq G\}$-embedded in G, then G is p-supersolvable.*

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