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Groups with the given set of the lengths of conjugacy classes

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Abstract: We study the structures of some finite groups such that the conjugacy class size of every noncentral element of them is divisible by a prime p.

Key words: Conjugacy class sizes, F-groups

1. Introduction

Let G be a finite group and $Z(G)$ be its center. For $x \in G$, suppose that $cl_G(x)$ denotes the conjugacy class in G containing x and $C_G(x)$ denotes the centralizer of x in G. We will use $cs(G)$ for the set $\{n : G$ has a conjugacy class of size $n\}$. It is known that some results on character degrees of finite groups and their conjugacy class sizes are parallel. Thompson in 1970 (see [6]) proved that if the degree of every nonlinear irreducible character of the finite group G is divisible by a prime p, then G has a normal p-complement. Along with this question, Caminas posed the following question:

Question. [1, Question 8.] If the conjugacy class size of every noncentral element of a group G is divisible by a prime p, what can be said about G?

It is known that $cs(GL_2(q^n)) = \{1, q^{2n}-1, q^n(q^n+1), q^n(q^n-1)\}$. Thus, if q is an odd prime, then

$$cs(GL_2(q^n)) = \{1, 2.n_1, 2^{e_2}.n_2, 2^{e_3}.n_3\},$$

where $1 < e_2 < e_3$ and $n_1 > n_2 > n_3$ are odd natural numbers. This example shows the existence of the finite groups where the conjugacy class size of their noncentral elements is divisible by a prime p but contains no normal p-complements. Thus, Thompson's result and the answer to the above question are not necessarily parallel. This example motivates us to find the structure of the finite group G with

$$cs(G) = \{1, p^{e_1}n_1, p^{e_2}n_2, \dots, p^{e_k}n_k\},$$

where $k \in \mathbb{N}$, $n_1, \dots, n_k$ are positive integers coprime to p such that $n_1 > n_2 > \cdots > n_k$ and $e_1 = 1 < e_2 < \cdots < e_k$. Throughout this paper, we say that the nonabelian finite group G and the prime p satisfy $(*)$ when

$$cs(G) = \{1, p^{e_1}n_1, p^{e_2}n_2, \dots, p^{e_k}n_k\},$$

where $k \in \mathbb{N}$, $n_1, \dots, n_k$ are positive integers coprime to the prime p such that $n_1 > n_2 > \cdots > n_k$ and $e_1 = 1 < e_2 < \cdots < e_k$. In this paper, we find the structures of the nonabelian finite groups satisfying $(*)$. More precisely, we prove the following theorem:

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Main Theorem *Let the nonabelian finite group G and the prime p satisfy $(*)$. Then G has one of the following structures:*

(i) $|cs(G)| = 2$ *and* $G \cong P \times A$, *where* A *is abelian and* $P \in \mathrm{Syl}_p(G)$ *with* $|cs(P)| = 2$;

(ii) $|cs(G)| = 3$ *and* G *is a quasi-Frobenius group with a normal* p*-Sylow subgroup;*

(iii) $|cs(G)| = 4$, $p = 2$, *and* $G/Z(G) \cong PGL_2(q^n)$, *where* $G' \cong SL_2(q^n)$ *and* q *is an odd prime.*

According to the main theorem, if the nonabelian finite group G and the prime p satisfy $(*)$, then either $p = 2$ or G is a solvable group with a normal p-Sylow subgroup.

For proving the main theorem, we show that for the nonabelian finite group G and the prime p satisfying $(*)$, either G is nilpotent or the p-part of $|G/Z(G)|$ is p^{e_2} or p^{e_2+1}, and $|cs(G)| \leq 4$. Thus, we have to consider the cases when $|cs(G)| = 2$, $|cs(G)| = 3$, and $|cs(G)| = 4$ separately and rule out the extra possibilities in these cases.

In this paper, all groups are finite. By $\gcd(c, b)$ and $\mathrm{lcm}(c, b)$ we denote the greatest common divisor and the least common multiple of the natural numbers c and b, respectively. For a finite group H, we denote by $\pi(H)$ the set of prime divisors of order of H. For the prime r (a set of primes π), the set of r-Sylow subgroups of H is denoted by $\mathrm{Syl}_r(H)$, $O_r(H)$ ($O_\pi(H)$) is the largest normal r-subgroup (π-subgroup) of H, and $O_{r'}(H)$ is the largest normal subgroup of H, its order being coprime to r. If m is a natural number and r is prime, then the r-part of m is denoted by $|m|_r$ and $|m|_{r'} = m/|m|_r$. Throughout Sections 2 and 3, let G be a nonabelian finite group and p be a prime that satisfies $(*)$.

2. Preliminary results

In the following lemma, we collect some known facts about finite groups. From [4, Theorem 5] and [3], we obtain (i) and (ii), respectively. The proof of (iii)–(v) is straightforward.

Lemma 2.1 *Let* K *be a normal subgroup of a finite group* H *and* $\bar{H} = H/K$. *Let* $\bar{x}$ *be the image of an element* x *of* H *in* $\bar{H}$ *and* $s \in \pi(H)$.

(i) s *does not divide* $|cl_H(x)|$ *for every* s'*-element* $x \in H$ *of a prime power order if and only if* H *is* s*-decomposable, i.e.* $H = O_s(H) \times O_{s'}(H)$;

(ii) *if* 1 *and* $m > 1$ *are the lengths of conjugacy classes of* H, *then for some* $r \in \pi(H)$, m *is a power of* r *and* $H = R \times A$, *where* $R \in \mathrm{Syl}_r(H)$ *and* A *is abelian;*

(iii) *assume that* $x, y \in H$ *with* $xy = yx$ *and* $\gcd(O(x), O(y)) = 1$. *Then* $C_H(xy) = C_H(x) \cap C_H(y)$. *In particular,* $C_H(xy) = C_{C_H(x)}(y)$ *is a subgroup of* $C_H(x)$ *and* $|cl_H(x)|$ *divides* $|cl_H(xy)|$;

(iv) *if* $x = yz$, *where* $y \in H$ *and* $z \in Z(H)$, *then* $C_H(x) = C_H(y)$;

(v) $|cl_{\bar{H}}(\bar{x})|$ *divides* $|cl_H(x)|$.

In the proof of the main theorem, we need to know about $cs(G'Z(G))$. The following lemma shows that $cs(G'Z(G)) = cs(G')$:

Lemma 2.2 *If K is a subgroup of H, then $cs(KZ(H)) = cs(K)$.*

Proof If $x \in KZ(H)$, then there exist $y \in K$ and $z \in Z(H)$ such that $x = yz$. Thus, by Lemma 2.1(iv), $C_{KZ(H)}(x) = C_{KZ(H)}(y)$. Also, $Z(H) \leq C_{KZ(H)}(y)$. Thus, by Dedekind modular law, $C_{KZ(H)}(y) = (C_{KZ(H)}(y) \cap K)Z(H) = C_K(y)Z(H)$, and hence $|cl_{KZ(H)}(x)| = |cl_{KZ(H)}(y)| = |KZ(H)|/|C_K(y)Z(H)| = |K|/|C_K(y)| = |cl_K(y)|$. Thus, $cs(K) = cs(KZ(H))$, as claimed. □

Let N be a normal subgroup of G. If xN is a p-element of G/N, then in order to study $C_{G/N}(xN)$, the following lemma allows us to assume that x is a p-element:

Lemma 2.3 *Let $s \in \pi(H)$. If N is a normal subgroup of H and $O(xN) = s^a$, then there exists an s-element $y \in G$ such that $xN = yN$.*

Proof Since $O(xN)$ divides $O(x)$, $O(x) = s^b.m$, where $b \geq a$ and $\gcd(s, m) = 1$. Thus, there exist natural numbers r and u such that $r.m + u.s^b = 1$ and hence $x = x_s.x_{s'} = x_{s'}.x_s$, where $x_s = x^{r.m}$ and $x_{s'} = x^{u.s^b}$. Obviously, $O(x_s) = s^b$, $O(x_{s'}) = m$ and $s^a = O(xN) = \text{lcm}(O(x_sN), O(x_{s'}N))$. This forces $O(x_{s'}N) = 1$ and hence $x_{s'} \in N$. Thus, $xN = x_sN$, as claimed. □

For some $x \in H$, Lemma 2.4 shows the relation between $|cl_{H/Z(H)}(xZ(H))|$ and $|cl_H(x)|$, which will be used in the proof of the main theorem:

Lemma 2.4 *Let $s \in \pi(H)$, $\bar{H} = H/Z(H)$ and $\bar{x}$ be the image of an element x of H in $\bar{H}$.*

(i) *If $x, y \in H$ such that $\gcd(O(x), O(y)) = 1$, then $\bar{y} \in C_{\bar{H}}(\bar{x})$ if and only if $y \in C_H(x)$;*

(ii) *if H is solvable and $O(\bar{x}) = s^a$, then $|cl_{\bar{H}}(\bar{x})|_{s'} = |cl_H(x)|_{s'}$.*

Proof If $y \in C_H(x)$, then it is obvious that $\bar{y} \in C_{\bar{H}}(\bar{x})$. Now let $\bar{y} \in C_{\bar{H}}(\bar{x})$. There exists $z \in Z(H)$ such that $y^{-1}xy = xz$. Thus, $O(x) = \text{lcm}(O(x), O(z))$, and hence $O(z)$ divides $O(x)$. On the other hand, $x^{-1}y^{-1}x = y^{-1}z$. Thus, $O(y) = \text{lcm}(O(y), O(z))$ and hence $O(z)$ divides $O(y)$. Therefore, $O(z)$ divides $\gcd(O(x), O(y)) = 1$. This forces $z = 1$ and hence $y^{-1}xy = x$. Therefore, $y \in C_H(x)$, as claimed in (i). Now we are going to prove (ii). Since $O(\bar{x}) = s^a$, Lemma 2.3 allows us to assume that x is an s-element, and since H is solvable, we can assume that $C_H(x)$ contains a $(\pi(H) - \{s\})$-Hall subgroup, namely K. Thus, (i) shows that $KZ(H)/Z(H)$ is a $(\pi(H) - \{s\})$-Hall subgroup of $C_{\bar{H}}(\bar{x})$ and hence (ii) follows. □

A group H is called quasi-Frobenius if $H/Z(H)$ is Frobenius.

The following lemma will be used in the case when $|cs(G)| = 3$.

Lemma 2.5 [2] *For a finite group H, $|cs(H)| = 3$ if and only if, up to an abelian direct factor, either:*

(1) *H is an r-group for some prime r;*

(2) *$H = KL$ with $K \trianglelefteq G$, $\gcd(|K|, |L|) = 1$, and one of the following occurs:*
(a) *both K and L are abelian, $Z(H) < L$, and H is a quasi-Frobenius group;*
(b) *K is abelian, L is a nonabelian r-group for some prime r, and $O_r(H)$ is an abelian subgroup of index r in L and $H/O_r(H)$ is a Frobenius group;*
(c) *K is an r-group with $|cs(K)| = 2$ for some prime r, L is abelian, $Z(K) = Z(H) \cap K$, and H is quasi-Frobenius.*

Remark 2.6 *Since for every $x \in G$, $Z(G) \leq C_G(x)$, we deduce that $|cl_G(x)|$ divides $|G/Z(G)|$. Also, $n_1 > n_2 > \cdots > n_k$ and $1 = e_1 < e_2 < \cdots < e_k$. Thus, for every $1 \leq i \leq k$, either $n_i \neq |G/Z(G)|_{p'}$ or $n_1 = |G/Z(G)|_{p'}$. Moreover, for every $1 \leq i \leq k$, either $p^{e_i} \neq |G/Z(G)|_p$ or $p^{e_k} = |G/Z(G)|_p$.*

Applying Lemma 2.7 leads us to find the structure of $C_G(x)$ for some p-element $x \in G - Z(G)$ and the centralizers of the p'-elements of $C_G(x)$:

Lemma 2.7 *For every noncentral p-element $x \in G$,*

(i) *$C_G(x) = O_p(C_G(x)) \times O_{p'}(C_G(x))$ and $O_{p'}(C_G(x)) \leq Z(C_G(x))$;*

(ii) *either $|cl_G(x)| = pn_1$ and $n_1 = |G/Z(G)|_{p'}$ or for every noncentral element $g \in C_G(x)$, $C_G(g) = C_G(x)$. In particular, either $|cl_G(x)| = pn_1$ and $n_1 = |G/Z(G)|_{p'}$ or $C_G(x)$ is abelian.*

Proof Since x is a noncentral p-element, we deduce that $p \mid |C_G(x)|$ and $p \mid |cl_G(x)|$. Thus, $|cl_G(x)| = p^{e_i}n_i$, for some $1 \leq i \leq k$. If y is a p'-element of $C_G(x)$, then by Lemma 2.1(iii), $|cl_G(x)| \mid |cl_G(xy)|$. Thus, assumption $(*)$ shows that $|cl_G(xy)| = |cl_G(x)|$ and hence $C_G(x) = C_G(xy) = C_G(x) \cap C_G(y) = C_{G(x)}(y)$, by Lemma 2.1(iii). Thus, $y \in Z(C_G(x))$, and hence Lemma 2.1(i) forces $C_G(x) = O_p(C_G(x)) \times O_{p'}(C_G(x))$, so (i) follows. Now let $|cl_G(x)| \neq pn_1$ or $n_1 \neq |G/Z(G)|_{p'}$. Then, since $Z(G) \leq C_G(x)$, $O_{p'}(C_G(x)) \neq O_{p'}(Z(G))$, considering Remark 2.6. Also, "$x \in C_G(x)$" guarantees that $O_p(C_G(x)) \nleq Z(G)$. Thus, there exist $g \in O_p(C_G(x)) - Z(G)$ and $h \in O_{p'}(C_G(x)) - Z(G)$. Replacing y with h in the above argument shows that $C_G(h) = C_G(xh) = C_G(x)$, and now replacing x and y with h and g in the above argument shows that

$$C_G(g) = C_G(hg) = C_G(h) = C_G(x). \tag{1}$$

Let $t \in C_G(x) - Z(G)$. As mentioned in the proof of Lemma 2.3, $t = t_p.t_{p'} = t_{p'}.t_p$, where t_p is a p-element and $t_{p'}$ is a p'-element of $C_G(x)$ such that $t_p \notin Z(G)$ or $t_{p'} \notin Z(G)$. Thus, Lemma 2.1(iii) and (1) show that $C_G(t) = C_G(t_p) \cap C_G(t_{p'}) = C_G(x)$, as claimed in (ii). □

Corollary 2.8 gives us some information about the structures of the centralizers of the noncentral elements of G:

Corollary 2.8 *For every noncentral element $x \in G$, $C_G(x) = O_p(C_G(x)) \times O_{p'}(C_G(x))$ and either $|cl_G(x)| = pn_1$ and $n_1 = |G/Z(G)|_{p'}$ or $O_p(C_G(x))$ is abelian.*

Proof If x is a p-element, then Lemma 2.7 completes the proof. If $O(x) = p^a.m$, where $a \geq 1$ and $\gcd(p, m) = 1$, then as mentioned in the proof of Lemma 2.3, we can see that $x = x_p.x_{p'} = x_{p'}.x_p$, where $O(x_p) = p^a$ and $O(x_{p'}) = m$. Thus, Lemma 2.7 shows that either $C_G(x_p) = C_G(x)$ or $x_p \in Z(G)$; in the former case, Lemma 2.7 completes the proof. In the latter case, Lemma 2.1(iv) forces $C_G(x) = C_G(x_{p'})$. Thus, without loss of generality, we can assume that x is a p'-element. If $C_G(x)$ contains a noncentral p-element y, then by Lemma 2.1(iii), $|cl_G(x)|$ and $|cl_G(y)|$ divides $|cl_G(xy)|$. Thus, our assumption shows that $|cl_G(xy)| = |cl_G(x)| = |cl_G(y)|$ and hence $C_G(x) = C_G(xy) = C_G(x) \cap C_G(y) = C_G(y)$. Therefore, Lemma 2.7 completes the proof. Otherwise, $O_p(C_G(x)) \leq Z(G)$ is a p-Sylow subgroup of $C_G(x)$ and hence Lemma 2.1(i) shows that $C_G(x) = O_p(C_G(x)) \times O_{p'}(C_G(x))$ and $O_p(C_G(x))$ is abelian. □

If $|cs(G)| \geq 3$, then applying Lemma 2.9 to the proof of the main theorem allows us to see that $|G/Z(G)|_p \in \{p^{e_2}, p^{e_2+1}\}$, which will be used in proving $|cs(G)| \leq 4$.

Lemma 2.9 *If y is a noncentral element of G such that $|cl_G(y)|_p < |G/Z(G)|_p$ and either $|cl_G(y)| \neq pn_1$ or $n_1 \neq |G/Z(G)|_{p'}$, then for every noncentral element $w \in G$, either $C_G(y) = C_G(w)$ or $C_G(y) \cap C_G(w) = Z(G)$.*

Proof Since $|cl_G(y)|_p < |G/Z(G)|_p$, we deduce that $C_G(y)$ contains a noncentral p-element t. Thus, by Lemma 2.7(ii), $C_G(y) = C_G(t)$. Now let w be a noncentral element of G with $C_G(y) \cap C_G(w) \neq Z(G)$. Then there exists a noncentral element $u \in C_G(y) \cap C_G(w)$ of primary order, so Lemma 2.7(ii) forces $C_G(u) = C_G(t) = C_G(y)$ and hence $w \in C_G(u) = C_G(t)$. Therefore, Lemma 2.7(ii) gives that $C_G(w) = C_G(t)$, so $C_G(w) = C_G(y)$ and, hence the lemma follows. □

Definition 2.10 *A group H is an F-group if for given any pair $x, y \in H$ with $x, y \notin Z(H)$, we have $C_H(x) \nless G_H(y)$.*

Corollary 2.11 *G is an F-group.*

Proof It follows immediately from our assumption on $cs(G)$. □

Note that the list of F-groups was obtained in [5].

Lemma 2.12 guarantees that $|G/Z(G)|_p = |P/Z(P)|$, for some $P \in \mathrm{Syl}_p(G)$.

Lemma 2.12 *For $P \in \mathrm{Syl}_p(G)$, $Z(P) \leq Z(G)$. In particular, $Z(G) \cap P = Z(P)$.*

Proof If $x \in Z(P)$, then $|cl_G(x)|_p = 1$, so by our assumption $x \in Z(G)$. Thus, $Z(P) \leq Z(G)$, as claimed. □

In the proof of the main theorem, we will need to know the set cs of the normal subgroups of G of index 2, which have been obtained in Lemma 2.13:

Lemma 2.13 *If $N \trianglelefteq G$ with $|G/N| = 2$, then:*

(i) *if $p \neq 2$, then*

$$cs(N) \subseteq \{1, pn_{1,1}, \ldots, pn_{1,t_1}, p^{e_2}n_{2,1}, \ldots, p^{e_2}n_{2,t_2}, \ldots, p^{e_k}n_{k,1}, \ldots, p^{e_k}n_{k,t_k}\},$$

where for $i \in \{1, \ldots, k\}$, $t_i \in \mathbb{N} \cup \{0\}$ and for $j \in \{1, \ldots, t_i\}$, $n_{i,j} \mid n_i$;

(ii) *if $p = 2$, then $cs(N) \subseteq \{1, pn_1, n_1, p^{e_2}n_2, p^{e_2-1}n_2, \ldots, p^{e_k}n_k, p^{e_k-1}n_k\}$.*

Proof If $p \neq 2$, then $\mathrm{Syl}_p(G) = \mathrm{Syl}_p(N)$, and hence for every noncentral element $x \in N$, $|C_N(x)|_p = |C_G(x)|_p$ and hence $|cl_N(x)|_p = |cl_G(x)|_p \in \{p, p^{e_2}, \ldots, p^{e_k}\}$. Also, it is easy to check that $|cl_N(x)| \mid |cl_G(x)|$, so

$$cs(N) \subseteq \{1, pn_{1,1}, \ldots, pn_{1,t_1}, p^{e_2}n_{2,1}, \ldots, p^{e_2}n_{2,t_2}, \ldots, p^{e_k}n_{k,1}, \ldots, p^{e_k}n_{k,t_k}\},$$

where for $i \in \{1, \ldots, k\}$, $t_i \in \mathbb{N} \cup \{0\}$ and for $j \in \{1, \ldots, t_i\}$, $n_{i,j} \mid n_i$. Thus, (i) follows. If $p = 2$, then for every noncentral element $x \in N$, $NC_G(x) \leq G$, so $[C_G(x) : C_N(x)]$ divides $[G : N]$. Thus, $|C_N(x)| = |C_G(x)|$ or $|C_G(x)|/2$, so $|cl_N(x)| = |cl_G(x)|$ or $|cl_G(x)|/2$. Therefore, $cs(N) \subseteq \{1, pn_1, n_1, p^{e_2}n_2, p^{e_2-1}n_2, \ldots, p^{e_k}n_k, p^{e_k-1}n_k\}$, as claimed in (ii). □

3. Proof of the main theorem

If $|cs(G)| = 2$, then Lemma 2.1(ii) completes the proof, so let $|cs(G)| \geq 3$. Since $pn_1, p^{e_2}n_2 \in cs(G)$, there exist $x, y' \in G$ such that $|cl_G(x)| = pn_1$ and $|cl_G(y')| = p^{e_2}n_2$. It is known that there exist $g \in G$ and $P \in \mathrm{Syl}_p(G)$ such that $C_P(x) = C_G(x) \cap P \in \mathrm{Syl}_p(C_G(x))$ and $C_P(y) = C_G(y) \cap P \in \mathrm{Syl}_p(C_G(y))$, where $y = g^{-1}y'g$. Also, $|cl_G(x)| \neq |cl_G(y)|$ and hence $|C_G(x)| \neq |C_G(y)|$. Thus, applying Remark 2.6 and Lemma 2.9 shows that if $|cl_G(y)|_p < |G/Z(G)|_p$, then

$$C_G(x) \cap C_G(y) = Z(G).$$

Thus, if $|cl_G(y)|_p < |G/Z(G)|_p$, then Lemma 2.12 forces

$$C_P(x) \cap C_P(y) = Z(P). \tag{2}$$

We are going to complete the proof in some steps:

Step 1. $|P/Z(P)| = p^{e_2}$ or $|P/Z(P)| = p^{1+e_2}$.

Proof If $Z(P) = C_P(y)$, then we can see at once that $|P/Z(P)| = |P|/|C_P(y)| = p^{e_2}$, as claimed. Thus, let $Z(P) \neq C_P(y)$. Then since $Z(P) < C_P(y)$, Lemma 2.12 leads us to see that $|cl_G(y)|_p < |G/Z(G)|_p$. Since $[P : C_P(x)] = |cl_G(x)|_p = p$, we conclude that $C_P(x)$ is a maximal subgroup of P. Also, $C_P(y) \neq Z(P)$, and hence (2) shows that $C_P(y)$ is not a subgroup of $C_P(x)$. Therefore, $C_P(x)C_P(y) = P$. Furthermore, (2) implies that $C_P(x) \cap C_P(y) = Z(P)$. Thus, $|C_p(y)|/|Z(P)| = |P|/|C_P(x)| = p$ and hence $|P|/|Z(P)| = [P : C_P(y)]|C_P(y)|/|Z(P)| = p^{1+e_2}$, as claimed. □

Step 2. For every $m \in cs(G) - \{1\}$, $|m|_p = p$, $|m|_p = p^{e_2}$ or $|m|_p = p^{e_2+1}$.

Proof Let $t \in G - Z(G)$ such that $|cl_G(t)|_p \notin \{p, p^{e_2}\}$. Then, since $Z(G) \leq C_G(t)$, we obtain from Step 1 and Lemma 2.12 that $|cl_G(t)|_p \leq |G/Z(G)|_p \leq p^{1+e_2}$. However, $|cl_G(t)|_p = p^{e_i}$, for some $i \geq 3$. Thus, by assumption $(*)$, $|cl_G(t)|_p > p^{e_2}$, and hence $|cl_G(t)|_p = p^{e_2+1}$, as claimed. □

Step 3. $|cs(G)| \leq 4$.

Proof It follows immediately from Step 2 and our assumption on e_is. □

Step 4. If $|P/Z(P)| = p^{e_2}$, then G is a quasi-Frobenius group with a normal p-Sylow subgroup.

Proof Since for every $t \in G$, $Z(G) \leq C_G(t)$, Lemma 2.12 forces $|cl_G(t)|_p \mid |G/Z(G)|_p = |P/Z(P)| = p^{e_2}$ and hence assumption $(*)$ shows that for every $w \in G - Z(G)$, $|cl_G(w)|_p = p^{e_2}$ or $|cl_G(w)|_p = p$. Thus, $|cs(G)| = 3$, so considering Lemma 2.5 and our assumption shows that $G = A \times KL$, with abelian subgroup A, $K \trianglelefteq G$, $\gcd(|K|, |L|) = 1$, and one of the following cases occurs:

(a) $cs(G) = \{1, |K|, |L|/|Z(L)|\}$. This forces nonidentity elements of $cs(G)$ to be coprime, which is a contradiction with our assumption on $cs(G)$;

(b) K is abelian, L is a nonabelian q-group, for some prime q, and $O_q(G)$ is an abelian subgroup of index q in L and $G/O_q(G)$ is a Frobenius group. Then $O_q(G), K \trianglelefteq G$ and $\gcd(|K|, q) = 1$, so $K \cap O_q(G) = \{1\}$. This implies that for every $w \in K$, $K \times O_q(G) \leq C_G(w)$. Thus, for every $w \in K - Z(G)$, $|cl_G(w)| = q$, and hence our assumption on $cs(G)$ forces $q = p$ and $n_1 = 1$, which is a contradiction with our assumption;

(c) K is a q-group with $cs(K) = \{1, q^a\}$, for some prime q, L is abelian, $Z(K) = Z(G) \cap K$, and G is a quasi-Frobenius group. Then

$$cs(G) = \{1, |K/Z(K)|, q^a|LZ(G)/Z(G)|\} = \{1, q^s, q^a|LZ(G)/Z(G)|\}.$$

This forces $q = p$, $|K/Z(K)| = p^{e_2}$, and $a = 1$. Thus, K is a normal p-subgroup of G, which is the p-Sylow subgroup of G.

$\square$

Step 5. If $|P/Z(P)| = p^{e_2+1}$, then $p = 2$ and $G/Z(G) \cong PGL_2(q^n)$, where $G' \cong SL_2(q^n)$ and q is an odd prime.

Proof If $|cs(G)| = 3$, then repeating the argument given in Step 4 shows that $cs(G) = \{1, |K/Z(K)|, p|LZ(G)/Z(G)|\}$, where $K \in \mathrm{Syl}_p(G)$, $K \cap Z(G) = Z(K)$ and $|K/Z(K)| = p^{e_2}$. Thus, by Lemma 2.12, $|P/Z(P)| = |G/Z(G)|_p = p^{e_2}$, which is a contradiction. Now let $|cs(G)| = 4$. By Step 2, $cs(G) = \{1, pn_1, p^{e_2}n_2, p^{e_2+1}n_3\}$, but by Corollary 2.11, G is an F-group. Thus, [5] shows that one of the following holds:

(i) G has a normal abelian subgroup N of index q, and q is a prime, but G is not abelian. Thus, $N \not\leq Z(G)$, so there exists $z \in N - Z(G)$. Since N is abelian, we have $N \leq C_G(z)$, and hence $|cl_G(z)|$ divides $[G : N] = q$. Therefore, $|cl_G(z)| = q$ and hence our assumption on $cs(G)$ forces $q = p$ and $n_1 = 1$, which is a contradiction with our assumption;

(ii) $G/Z(G)$ is a Frobenius group with the Frobenius kernel $KZ(G)/Z(G)$ and the Frobenius complement $LZ(G)/Z(G)$, and one of the following subcases holds:
(a) K and L are abelian. Then we can see that $|cs(G)| = 3$, which is a contradiction with our assumption;
(b) L is abelian, $Z(K) = Z(G) \cap K$ and $K/Z(K)$ is a q-group, for some prime q. Then for every $x \in L$, $|cl_G(x)| = |K|/|Z(K)|$. Thus, our assumption shows that $q = p$. Since G is not abelian and $n_1 > n_2 > n_3$, Remark 2.6 and Lemma 2.7(ii) show that there exist the noncentral p'-elements $x_2, x_3 \in G$ with $|cl_G(x_2)| = p^{e_2}n_2$ and $|cl_G(x_3)| = p^{e_2+1}n_3$. Thus, we can assume that $x_2, x_3 \in L$, so $L \leq C_G(x_2) \cap C_G(x_3)$. Therefore, Lemma 2.9 forces $L \leq Z(G)$, which is a contradiction;

(iii) $G/Z(G) \cong \mathbb{S}_4$. Then G is solvable and since by Lemma 2.12 and our assumption $1 < p^{e_2+1} = |P/Z(P)| = |G/Z(G)|_p$, and $|\mathbb{S}_4| = |G/Z(G)| = 2^3.3$, we deduce that $p^{e_2+1} = 2^3$. Thus, $p = 2$ and $e_2 = 2$. Therefore, $cs(G) = \{1, 2n_1, 4n_2, 8n_3\}$. Since for every $w \in G - Z(G)$, $wZ(G) \in G/Z(G) \cong \mathbb{S}_4$, we obtain $O(wZ(G)) \in \{2, 3, 4\}$ and if $O(wZ(G)) = 3$, then $|cl_{G/Z(G)}(wZ(G))| = 8$. Since by Lemma 2.1(v), $|cl_{G/Z(G)}(wZ(G))|$ divides $|cl_G(w)|$, we deduce that $O(xZ(G)), O(yZ(G)) \in \{2, 4\}$, and hence Lemma 2.3 forces the existence of 2-elements $x_1, y_1 \in G - Z(G)$ such that $xZ(G) = x_1Z(G)$ and $yZ(G) = y_1Z(G)$. Therefore, Lemma 2.4(ii) guarantees the existence of $\alpha, \beta \in cs(G/Z(G)) = cs(\mathbb{S}_4) = \{1, 3, 6, 8\}$ such that $n_1 = |\alpha|_{p'}$ and $n_2 = |\beta|_{p'}$. Thus, $n_1 = n_2 = 3$, which is a contradiction;

(iv) $G = A \times P$, where A is abelian and P is a q-group, so $cs(G) = cs(P)$, which is a contradiction with our assumption on n_is;

(v) $G/Z(G) \cong PSL_2(q^n)$ or $PGL_2(q^n)$ and $G' \cong SL_2(q^n)$, where $q^n > 3$ and q is prime. If $G/Z(G) \cong PSL_2(q^n)$, then since $G/Z(G)$ is a simple group, we deduce that $G'Z(G)/Z(G) = G/Z(G)$, and hence $G'Z(G) = G$. Thus, by Lemma 2.2, $cs(G) = cs(G'Z(G)) = cs(G')$, but

$$cs(G') = cs(SL_2(q^n)) = \begin{cases} \{1, \frac{(q^{2n}-1)}{2}, q^n(q^n+1), q^n(q^n-1)\} & \text{if } q \text{ is odd} \\ \{1, q^{2n}-1, q^n(q^n+1), q^n(q^n-1)\} & \text{if } q \text{ is even} \end{cases}, \quad (3)$$

which is a contradiction with our assumption on e_is. Now let $G/Z(G) \cong PGL_2(q^n)$. Since if $q = 2$, then $PGL_2(q^n) \cong SL_2(q^n) = PSL_2(q^n)$, and we just need to consider the case when q is odd. Thus,

$$[G : G'Z(G)] = |G/Z(G)|/|G'Z(G)/Z(G)| = |PGL_2(q^n)|/|PSL_2(q^n)| = 2.$$

If $p \neq 2$, then Lemmas 2.2 and 2.13(i) show that

$$\begin{aligned} cs(SL_2(q^n)) &= cs(G') = cs(G'Z(G)) \\ &\subseteq \{1, pn_{1,1}, \ldots, pn_{1,t_1}, p^{e_2}n_{2,1}, \ldots, p^{e_2}n_{2,t_2}, p^{e_2+1}n_{3,1}, \ldots, p^{e_2+1}n_{3,t_3}\} \end{aligned}$$

where for $i \in \{1, \ldots, 3\}$, $t_i \in \mathbb{N} \cup \{0\}$ and for $j \in \{1, \ldots, t_i\}$, $n_{i,j} \mid n_i$. Thus, p divides $\gcd(\{n : n \in cs(SL_2(q)) - \{1\}\})$, which is a contradiction considering (3) and assumption $p \neq 2$. Thus, $p = 2$.

(vi) $G/Z(G) \cong PSL_2(9)$ or $PGL_2(9)$ and $G' \cong PSL_2(9)$. Since $G' \trianglelefteq G$, there exists $Q \in \mathrm{Syl}_p(G')$ such that $Q \trianglelefteq P$ and hence $Z(P) \cap Q \neq 1$, but Lemma 2.12 implies that $Z(P) \leq Z(G)$, so $Z(P) \cap Q \leq Z(G) \cap G' \leq Z(G') = 1$, which is a contradiction.

These steps complete the proof of the main theorem. □

Corollary 3.1 *If $p \neq 2$, then G is a nilpotent group or a quasi-Frobenius group with a normal p-Sylow subgroup.*

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