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Approximate duals and nearly Parseval frames

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Abstract: In this paper we introduce approximate duality of g-frames in Hilbert C^*-modules and we show that approximate duals of g-frames in Hilbert C^*-modules share many useful properties with those in Hilbert spaces. Moreover, we obtain some new results for approximate duality of frames and g-frames in Hilbert spaces; in particular, we consider approximate duals of ε-nearly Parseval and ε-close frames.

Key words: Hilbert C^*-module, g-frame, frame, approximate duality, ε-nearly Parseval frame

1. Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [10] in 1952 to study some problems in nonharmonic Fourier series, and they were reintroduced in 1986 by Daubechies et al. [9]. Frames have important applications in signal and image processing, wireless communications, and many other fields. There exist various generalizations of frames. A recent and general one is called g-frame [26].

As we know, duals play an important role in frame theory, especially they are used in the reconstruction of signals. It is well known that every frame in a Hilbert space has at least one dual (see [7]), and if a dual of a frame is found, then each signal can be reconstructed easily. However, it is usually difficult to calculate a dual. Here, approximate duals can be useful. Approximate duals in frame theory have important applications (see [4, 12, 27]). Approximate duality of frames in Hilbert spaces was recently investigated in [8]. Khosravi and Mirzaee Azandaryani (the present author) also introduced approximate duality of g-frames in Hilbert spaces and obtained some properties and applications of approximate duals (see [20]). In particular, it was shown that approximate duals are stable under small perturbations and they are useful for erasures (see [20, Section 3]).

Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers.

Frank and Larson presented a general approach to the frame theory in Hilbert C^*-modules (see [11]). They showed that every countably generated Hilbert C^*-module over a unital C^*-algebra admits a frame. It was also shown in [25] that every Hilbert C^*-module that is countably generated in the set of adjointable operators admits a frame of multipliers. Furthermore, g-frames in Hilbert C^*-modules were introduced in [16].

Frames in Hilbert C^*-modules are not trivial generalizations of Hilbert space frames due to the complex structure of C^*-algebras. Since there are important differences between the theory of Hilbert C^*-modules

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and Hilbert spaces (see Chapter 1 in [21]), it is expected that problems about frames in Hilbert C^*-modules are more complicated than those in Hilbert spaces.

In this paper we generalize the concept of approximate duality of g-frames to Hilbert C^*-modules and we get some results for approximate duals of frames and g-frames in Hilbert spaces. In particular, approximate duals of ε-nearly Parseval and ε-close frames are studied.

First, in the following section, we have a brief review of the definitions and basic properties of frames and g-frames in Hilbert C^*-modules.

In this note, all index sets are finite or countable subsets of $\mathbb{Z}$.

2. Frames and g-frames in Hilbert C^*-modules

Suppose that $\mathfrak{A}$ is a unital C^*-algebra and E is a left $\mathfrak{A}$-module such that the linear structures of $\mathfrak{A}$ and E are compatible. E is a pre-Hilbert $\mathfrak{A}$-module if E is equipped with an $\mathfrak{A}$-valued inner product $\langle .,. \rangle : E \times E \longrightarrow \mathfrak{A}$, such that

(i) $\langle \alpha x + \beta y, z\rangle = \alpha \langle x, z\rangle + \beta \langle y, z\rangle$, for each $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in E$;

(ii) $\langle ax, y\rangle = a\langle x, y\rangle$, for each $a \in \mathfrak{A}$ and $x, y \in E$;

(iii) $\langle x, y\rangle = \langle y, x\rangle^*$, for each $x, y \in E$;

(iv) $\langle x, x\rangle \geq 0$, for each $x \in E$ and if $\langle x, x\rangle = 0$, then $x = 0$.

For each $x \in E$, we define $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ and $|x| = \langle x, x\rangle^{\frac{1}{2}}$. If E is complete with $\|.\|$, it is called a *Hilbert* $\mathfrak{A}$*-module* or a *Hilbert* C^**-module* over $\mathfrak{A}$. Let E and F be Hilbert $\mathfrak{A}$-modules. An operator $T : E \longrightarrow F$ is called *adjointable* if there exists an operator $T^* : F \longrightarrow E$ such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$, for each $x \in E$ and $y \in F$. Every adjointable operator T is bounded and $\mathfrak{A}$-linear (that is, $T(ax) = aT(x)$ for each $x \in E$ and $a \in \mathfrak{A}$). We denote the set of all adjointable operators from E into F by $\mathfrak{L}(E, F)$. $\mathfrak{L}(E, E)$ is a C^*-algebra and we denote it by $\mathfrak{L}(E)$. Note that if $\{E_i : i \in I\}$ is a sequence of Hilbert $\mathfrak{A}$-modules, then $\oplus_{i\in I} E_i$, which is the set

$$\oplus_{i\in I} E_i = \Big\{\{x_i\}_{i\in I} : x_i \in E_i \text{ and } \sum_{i\in I}\langle x_i, x_i\rangle \text{ is norm convergent in } \mathfrak{A}\Big\},$$

is a Hilbert $\mathfrak{A}$-module with pointwise operations and $\mathfrak{A}$-valued inner product $\langle x, y\rangle = \sum_{i\in I}\langle x_i, y_i\rangle$, where $x = \{x_i\}_{i\in I}$ and $y = \{y_i\}_{i\in I}$. For each $x = \{x_i\}_{i\in I} \in \oplus_{i\in I} E_i$, we define $\|.\|_2$ by $\|x\|_2 = \|\sum_{i\in I}\langle x_i, x_i\rangle\|^{\frac{1}{2}}$. For more details about Hilbert C^*-modules, see [21].

In this paper we focus on finitely and countably generated Hilbert C^*-modules over unital C^*-algebras. A Hilbert $\mathfrak{A}$-module E is *finitely generated* if there exists a finite set $\{x_1, \dots, x_n\} \subseteq E$ such that every element $x \in E$ can be expressed as an $\mathfrak{A}$-linear combination $x = \sum_{i=1}^n a_i x_i, a_i \in \mathfrak{A}$. A Hilbert $\mathfrak{A}$-module E is *countably generated* if there exists a countable set $\{x_i\}_{i\in I} \subseteq E$ such that E equals the norm-closure of the $\mathfrak{A}$-linear hull of $\{x_i\}_{i\in I}$.

Let E be a Hilbert $\mathfrak{A}$-module. A family $\{f_i\}_{i\in I} \subseteq E$ is a *frame* for E, if there exist real constants $0 < A \leq B < \infty$, such that for each $x \in E$,

$$A\langle x, x\rangle \leq \sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle \leq B\langle x, x\rangle. \tag{1}$$

The numbers A and B are called the *lower* and *upper bound* of the frame, respectively. In this case we call it an (A, B) *frame*. The *optimal lower frame bound* is the supremum over all lower frame bounds and the *optimal upper frame bound* is the infimum over all upper frame bounds. If $A = B$, the frame is called *tight* (A-*tight*) and if $A = B = 1$, the frame is *Parseval*. If only the second inequality is required, we call it a *Bessel sequence*. If the sum in (1) converges in norm, the frame is called *standard*.

Let $\{E_i\}_{i\in I}$ be a sequence of Hilbert $\mathfrak{A}$-modules. A sequence $\Lambda = \{\Lambda_i \in \mathfrak{L}(E, E_i) : i \in I\}$ is called a *g-frame* for E with respect to $\{E_i : i \in I\}$ if there exist real constants $A, B > 0$ such that for each $x \in E$,

$$A\langle x, x\rangle \leq \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \leq B\langle x, x\rangle.$$

A and B are g-frame bounds of Λ. In this case we call it an (A, B) *g-frame*. The *optimal bounds* and *tight* and *Parseval* g-frames are defined similarly to frames. The g-frame is *standard* if for each $x \in E$, the sum converges in norm. If only the second-hand inequality is required, then Λ is called a *g-Bessel sequence*.

For a standard g-Bessel sequence Λ, the operator $T_\Lambda : \oplus_{i\in I}E_i \longrightarrow E$ defined by $T_\Lambda(\{x_i\}_{i\in I}) = \sum_{i\in I}\Lambda_i^*(x_i)$ is called the *synthesis operator* of Λ. T_Λ is adjointable and $T_\Lambda^*(x) = \{\Lambda_i x\}_{i\in I}$. Now we define the operator S_Λ on E by $S_\Lambda x = T_\Lambda T_\Lambda^*(x) = \sum_{i\in I}\Lambda_i^*\Lambda_i(x)$. If Λ is a standard (A, B) g-frame, then $A.Id_E \leq S_\Lambda \leq B.Id_E$.

Note that $\mathcal{F} = \{f_i\}_{i\in I}$ is a standard Bessel sequence (resp. frame) if and only if $\Lambda_{\mathcal{F}} = \{\Lambda_{f_i}\}_{i\in I}$ is a standard g-Bessel sequence (resp. g-frame), where $\Lambda_{f_i}(x) = \langle x, f_i\rangle$, for each $x \in E$ (see [16, Example 3.2]). This shows that each Bessel sequence (resp. frame) generates a g-Bessel sequence (resp. g-frame). For a standard Bessel sequence $\mathcal{F} = \{f_i\}_{i\in I}$, we denote $S_{\Lambda_{\mathcal{F}}}$ by $S_{\mathcal{F}}$.

Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be an (A,B) standard g-frame. We call $\widetilde{\Lambda} = \{\Lambda_i S_\Lambda^{-1}\}_{i\in I}$ the *canonical g-dual* of Λ, which is a $(\frac{1}{B}, \frac{1}{A})$ standard g-frame. We denote the *canonical dual* of a standard frame $\mathcal{F} = \{f_i\}_{i\in I}$ by $\widetilde{\mathcal{F}} = \{\widetilde{f}_i\}_{i\in I}$, where $\widetilde{f}_i = S_{\mathcal{F}}^{-1}f_i$. Recall that if $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Gamma = \{\Gamma_i\}_{i\in I}$ are standard $g-$Bessel sequences such that $\sum_{i\in I}\Gamma_i^*\Lambda_i x = x$ or equivalently $\sum_{i\in I}\Lambda_i^*\Gamma_i x = x$, for each $x \in E$, then Γ (resp. Λ) is called a *g-dual* of Λ (resp. Γ). Also, *duals* for two standard Bessel sequences $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{G} = \{g_i\}_{i\in I}$ can be defined by using the generated g-Bessel sequences, so $\mathcal{G}$ (resp. $\mathcal{F}$) is a dual of $\mathcal{F}$ (resp. $\mathcal{G}$) if $x = \sum_{i\in I}\langle x, f_i\rangle g_i$ or equivalently $x = \sum_{i\in I}\langle x, g_i\rangle f_i$, for each $x \in E$ (see [11, 13]). For more details about frames and g-frames in Hilbert C^*-modules, see [11, 2, 16, 28].

3. Approximate duals of g-frames in Hilbert C^*-modules

In this section all C^*-algebras are unital and all Hilbert C^*-modules are finitely or countably generated. All frames, g-frames, Bessel sequences, and g-Bessel sequences are standard. Λ and Γ denote $\{\Lambda_i \in \mathfrak{L}(E, E_i) : i \in I\}$ and $\{\Gamma_i \in \mathfrak{L}(E, E_i) : i \in I\}$, respectively. Also, $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{G} = \{g_i\}_{i\in I}$ are subsets of a Hilbert C^*-module E.

For two standard g-Bessel sequences Λ and Γ, the operator $S_{\Gamma\Lambda}$ is defined on E by $S_{\Gamma\Lambda} = T_\Gamma {T_\Lambda}^*$. Since $S_{\Gamma\Lambda}^* = S_{\Lambda\Gamma}$, we have $\|Id_E - S_{\Gamma\Lambda}\| = \|(Id_E - S_{\Gamma\Lambda})^*\| = \|Id_E - S_{\Lambda\Gamma}\|$.

Now we introduce approximate duals for g-Bessel sequences (and also for Bessel sequences by using the generated g-Bessel sequences) in Hilbert C^*-modules:

Definition 3.1 (i) *Two standard g-Bessel sequences* Λ *and* Γ *are approximately dual g-frames if* $\|Id_E - S_{\Gamma\Lambda}\| < 1$ *or equivalently* $\|Id_E - S_{\Lambda\Gamma}\| < 1$. *In this case, we say that* Γ *(resp.* Λ*) is an approximate g-dual of* Λ *(resp.* Γ*).*

(ii) *Two standard Bessel sequences* $\mathcal{F}$ *and* $\mathcal{G}$ *are approximately dual frames if* $\Lambda_{\mathcal{F}}$ *and* $\Lambda_{\mathcal{G}}$ *are approximately dual g-frames, i.e.* $\|Id_E - S_{\Lambda_{\mathcal{G}}\Lambda_{\mathcal{F}}}\| < 1$ *or equivalently* $\|Id_E - S_{\Lambda_{\mathcal{F}}\Lambda_{\mathcal{G}}}\| < 1$. *In this case, we say that* $\mathcal{G}$ *(resp.* $\mathcal{F}$*) is an approximate dual of* $\mathcal{F}$ *(resp.* $\mathcal{G}$*). We denote* $S_{\Lambda_{\mathcal{G}}\Lambda_{\mathcal{F}}}$ *and* $S_{\Lambda_{\mathcal{F}}\Lambda_{\mathcal{G}}}$ *by* $S_{\mathcal{G}\mathcal{F}}$ *and* $S_{\mathcal{F}\mathcal{G}}$*, respectively.*

It is clear that $S_{\Gamma\Lambda}(x) = \sum_{i\in I} {\Gamma_i}^*\Lambda_i(x)$ and $S_{\mathcal{G}\mathcal{F}}(x) = \sum_{i\in I}\langle x, f_i\rangle g_i$, for each $x \in E$. If Λ and Γ are g-duals, then they are approximately dual g-frames because $S_{\Lambda\Gamma} = Id_E$. Using the Neumann algorithm, we can see that $S_{\Lambda\Gamma}$ is invertible with ${S_{\Lambda\Gamma}}^{-1} = \sum_{n=0}^{\infty} (Id_E - S_{\Lambda\Gamma})^n$, so each $x \in E$ can be reconstructed as

$$x = \sum_{n=0}^{\infty} S_{\Lambda\Gamma}(Id_E - S_{\Lambda\Gamma})^n x, \quad x = \sum_{n=0}^{\infty} (Id_E - S_{\Lambda\Gamma})^n S_{\Lambda\Gamma} x.$$

Recall from [17] that a standard g-frame Λ is a *modular g-Riesz basis* if it has the following property: if $\sum_{i\in\Omega} \Lambda_i^* g_i = 0$, where $g_i \in E_i$ and $\Omega \subseteq I$, then $g_i = 0$, for each $i \in \Omega$.

A standard frame $\{f_i\}_{i\in I}$ for E is a *modular Riesz basis* if it has the following property: if an $\mathfrak{A}$-linear combination $\sum_{i\in\Omega} a_i f_i$ with coefficients $\{a_i : i \in \Omega\} \subseteq \mathfrak{A}$ and $\Omega \subseteq I$ is equal to zero, then $a_i = 0$, for each $i \in \Omega$.

The following result is a generalization of Proposition 2.3 in [20] to Hilbert C^*-modules.

Theorem 3.2 *Let* Λ *and* Γ *be approximately dual g-frames with upper bounds* B *and* D*, respectively. Then:*

(i) Λ *and* Γ *are* $(\frac{\|{S_{\Gamma\Lambda}}^{-1}\|^{-2}}{D}, B)$ *and* $(\frac{\|{S_{\Lambda\Gamma}}^{-1}\|^{-2}}{B}, D)$ *g-frames, respectively.*

(ii) $\{\Gamma_i + \sum_{n=1}^{\infty}\Gamma_i(Id_E - S_{\Lambda\Gamma})^n\}_{i\in I}$ *is a g-dual of* Λ.

(iii) *For each* $N \in \mathbb{N}$*, define* ${\psi_i}^N = \Gamma_i + \sum_{n=1}^{N}\Gamma_i(Id_E - S_{\Lambda\Gamma})^n$. *Then* $\Psi_N = \{{\psi_i}^N\}_{i\in I}$ *is an approximate g-dual of* Λ *with* $\|Id_E - S_{\Lambda\Psi_N}\| \leq \|Id_E - S_{\Lambda\Gamma}\|^{N+1} < 1$.

(iv) *If* Λ *is a modular g-Riesz basis, then* $\widetilde{\Lambda_i} = \Gamma_i + \sum_{n=1}^{\infty}\Gamma_i(Id_E - S_{\Lambda\Gamma})^n = \lim_{N\to\infty}\psi_i^N$*, for each* $i \in I$.

Proof (i) Since Λ and Γ are approximately dual g-frames, $S_{\Gamma\Lambda}$ is invertible, so $\|S_{\Gamma\Lambda}^{-1}\|^{-1}\|x\| \leq \|S_{\Gamma\Lambda}x\|$, for each $x \in E$. Now by using the Cauchy–Schwarz inequality in Hilbert C^*-modules, we have

$$\begin{aligned}
\|S_{\Gamma\Lambda}^{-1}\|^{-1}\|x\| \leq \|S_{\Gamma\Lambda}x\| &= \sup_{\|y\|=1}\|\langle S_{\Gamma\Lambda}x, y\rangle\| = \sup_{\|y\|=1}\left\|\sum_{i\in I}\langle\Lambda_i x, \Gamma_i y\rangle\right\| \\
&\leq \sup_{\|y\|=1}\left\|\sum_{i\in I}\langle\Lambda_i x, \Lambda_i x\rangle\right\|^{\frac{1}{2}}\left\|\sum_{i\in I}\langle\Gamma_i y, \Gamma_i y\rangle\right\|^{\frac{1}{2}} \\
&\leq \sqrt{D}\left\|\sum_{i\in I}\langle\Lambda_i x, \Lambda_i x\rangle\right\|^{\frac{1}{2}}.
\end{aligned}$$

Hence:

$$\frac{\|S_{\Gamma\Lambda}^{-1}\|^{-2}\|x\|^2}{D} \leq \left\| \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \right\|,$$

and so by Theorem 3.1 in [28], Λ is a standard g-frame with the lower bound $\frac{\|S_{\Gamma\Lambda}{}^{-1}\|^{-2}}{D}$. Similarly, by considering $S_{\Lambda\Gamma}$ instead of $S_{\Gamma\Lambda}$ in the above conclusions, we obtain that Γ is a $(\frac{\|S_{\Lambda\Gamma}{}^{-1}\|^{-2}}{B}, D)$ standard g-frame.

(ii) Since $S_{\Lambda\Gamma}{}^{-1} = \sum_{n=0}^{\infty}(Id_E - S_{\Lambda\Gamma})^n$, we have $\Gamma_i S_{\Lambda\Gamma}^{-1} = \Gamma_i + \sum_{n=1}^{\infty}\Gamma_i(Id_E - S_{\Lambda\Gamma})^n$ and it is easy to see that $\{\Gamma_i S_{\Lambda\Gamma}^{-1}\}_{i\in I}$ is a g-dual of Λ.

(iii) For each $n = 0, \ldots, N$, we have

$$\left\| \sum_{i\in I}\langle \Gamma_i(Id_E - S_{\Lambda\Gamma})^n x, \Gamma_i(Id_E - S_{\Lambda\Gamma})^n x\rangle \right\| \leq D\|(Id_E - S_{\Lambda\Gamma})^n\|^2\|x\|^2,$$

so $\{\Gamma_i(Id_E - S_{\Lambda\Gamma})^n\}_{i\in I}$ is a standard g-Bessel sequence by Theorem 3.1 in [28] and consequently Ψ_N is a standard g-Bessel sequence. Now the result can be obtained similar to the proof of Proposition 2.3 in [20].

(iv) Since Λ is a modular g-Riesz basis, Corollary 4.1 in [17] yields that $\widetilde{\Lambda}$ is the unique g-dual of Λ. According to part (ii), $\{\Gamma_i + \sum_{n=1}^{\infty}\Gamma_i(Id_E - S_{\Lambda\Gamma})^n\}_{i\in I}$ is also a g-dual of Λ, so $\widetilde{\Lambda_i} = \Gamma_i + \sum_{n=1}^{\infty}\Gamma_i(Id_E - S_{\Lambda\Gamma})^n = \lim_{N\to\infty}\psi_i^N$. □

As a consequence of the above theorem and Example 3.2 in [16], we obtain the following result. Parts (ii) and (iii) of the following corollary are generalizations of Proposition 3.2 in [8] to Hilbert C^*-modules.

Corollary 3.3 *Let $\mathcal{F}$ and $\mathcal{G}$ be approximately dual frames with upper bounds B and D, respectively. Then:*

(i) *$\mathcal{F}$ and $\mathcal{G}$ are $(\frac{\|S_{\mathcal{G}\mathcal{F}}{}^{-1}\|^{-2}}{D}, B)$ and $(\frac{\|S_{\mathcal{F}\mathcal{G}}{}^{-1}\|^{-2}}{B}, D)$ frames, respectively.*

(ii) *$\{g_i + \sum_{n=1}^{\infty}(Id_E - S_{\mathcal{G}\mathcal{F}})^n g_i\}_{i\in I}$ is a dual of $\mathcal{F}$.*

(iii) *For each $N \in \mathbb{N}$, define $h_i^N = g_i + \sum_{n=1}^{N}(Id_E - S_{\mathcal{G}\mathcal{F}})^n g_i$. Then $h_N = \{h_i^N\}_{i\in I}$ is an approximate dual of $\mathcal{F}$ with $\|Id_E - S_{h_N\mathcal{F}}\| \leq \|Id_E - S_{\mathcal{G}\mathcal{F}}\|^{N+1} < 1$.*

(iv) *If $\mathcal{F}$ is a modular Riesz basis, then $\widetilde{f_i} = g_i + \sum_{n=1}^{\infty}(Id_E - S_{\mathcal{G}\mathcal{F}})^n g_i = \lim_{N\to\infty} h_i^N$, for each $i \in I$.*

We can get from the above theorem and corollary that a standard g-Bessel sequence (resp. Bessel sequence) is a standard g-frame (resp. frame) if and only if it has an approximate g-dual (resp. approximate dual).

Note that Theorem 2.5 in [20] shows that if Λ and Γ are two g-Bessel sequences in a Hilbert space H, then a necessary and sufficient condition for Λ and Γ to be approximately dual g-frames is that there exist two Bessel sequences $\mathcal{F}$ and $\mathcal{G}$ in H that are approximately dual frames with $S_{\Lambda\Gamma} = S_{\mathcal{F}\mathcal{G}}$. Now we have a similar result for approximate duals in Hilbert C^*–modules.

Proposition 3.4 *Let Λ and Γ be two g-Bessel sequences. Then Λ and Γ are approximately dual g-frames if and only if there exist two Bessel sequences $\mathcal{F}$ and $\mathcal{G}$ in E such that $\mathcal{F}$ and $\mathcal{G}$ are approximately dual frames with $S_{\Lambda\Gamma} = S_{\mathcal{F}\mathcal{G}}$.*

Proof Let Λ and Γ be approximately dual g-frames. As a result of Kasparov's stabilization theorem, every finitely or countably generated Hilbert C^*-module has a standard Parseval frame (see [11, 22]). Let $\{f_{ij}\}_{j\in J_i}$ be a standard Parseval frame for E_i. It follows from Corollary 3.4 in [16] that $\mathcal{F} = \{{\Lambda_i}^*(f_{ij})\}_{i\in I, j\in J_i}$ and $\mathcal{G} = \{{\Gamma_i}^*(f_{ij})\}_{i\in I, j\in J_i}$ are standard Bessel sequences. Then for each $x \in E$, we have

$$S_{\mathcal{F}\mathcal{G}}(x) = \sum_{i\in I}\sum_{j\in J_i}\langle x, {\Gamma_i}^*(f_{ij})\rangle {\Lambda_i}^*(f_{ij}) = \sum_{i\in I}{\Lambda_i}^*\Gamma_i x = S_{\Lambda\Gamma}x,$$

so $\|S_{\mathcal{F}\mathcal{G}} - Id_E\| = \|S_{\Lambda\Gamma} - Id_E\| < 1$, and the result follows. The converse is clear. □

Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two C^*-algebras. Then $\mathfrak{A}\otimes\mathfrak{A}'$ is a C^*-algebra with the spatial norm and for each $a \in \mathfrak{A}$ and $a' \in \mathfrak{A}'$, we have $\|a\otimes a'\| = \|a\|\|a'\|$. The multiplication and involution on simple tensors are defined by $(a\otimes a')(b\otimes b') = ab\otimes a'b'$ and $(a\otimes a')^* = a^*\otimes a'^*$, respectively. As we know, if $a, a' \geq 0$, then $a\otimes a' \geq 0$.

Now let E be a Hilbert $\mathfrak{A}$-module and E' be a Hilbert $\mathfrak{A}'$-module. Then the (Hilbert C^*-module) tensor product $E\otimes E'$ is a Hilbert $\mathfrak{A}\otimes\mathfrak{A}'$-module. The module action and inner product for simple tensors are defined by $(a\otimes a')(x\otimes x') = (ax)\otimes(a'x')$ and $\langle x\otimes x', y\otimes y'\rangle = \langle x, y\rangle\otimes\langle x', y'\rangle$, respectively. Let U and U' be adjointable operators on E and E', respectively. Then the tensor product $U\otimes U'$ is an adjointable operator on $E\otimes E'$. Also, $(U\otimes U')^* = U^*\otimes U'^*$ and $\|U\otimes U'\| = \|U\|\|U'\|$. For more results about tensor products of C^*-algebras and Hilbert C^*-modules, see [23, 21].

Tensor products of frames and g-frames have been studied by some authors recently; see [15, 6, 16, 18].

It was proved in Proposition 3.2 in [19] that the direct sum of a countable number of g-duals (in Hilbert spaces) is a g-dual in the direct sum space but Example 2.9 in [20] shows that this is not necessarily true for approximate g-duals.

It was also shown in [20, Proposition 2.10] and [18, Corollary 3.8] (by using resolutions of the identity) that the tensor product of two g-duals (in Hilbert spaces) gives a g-dual in the tensor product space. In the following example, we show that the result does not necessarily hold for approximate g-duals:

Example 3.5 *Let H be a separable Hilbert space (as a special case of a Hilbert C^*–module) and $\Lambda = \{\Lambda_i\}_{i\in I}$ be an A-tight g-frame with $\sqrt{2} < A < 2$. It is easy to see that Λ is an approximate g-dual of itself. Now the proof of Corollary 2.2 in [18] yields that $\Lambda\otimes\Lambda = \{\Lambda_i\otimes\Lambda_j\}_{i,j\in I}$ is an A^2-tight g-frame, so $S_{(\Lambda\otimes\Lambda)(\Lambda\otimes\Lambda)} = S_{(\Lambda\otimes\Lambda)} = A^2.Id_{(H\otimes H)}$. Thus, $\|S_{(\Lambda\otimes\Lambda)(\Lambda\otimes\Lambda)} - Id_{(H\otimes H)}\| = A^2 - 1 > 1$. This means that $\Lambda\otimes\Lambda$ is not an approximate g-dual of itself.*

Now we consider tensor products of g-duals and approximate g-duals in Hilbert C^*-modules. In the following proposition $\Lambda' = \{\Lambda'_j \in \mathfrak{L}(E', E'_j) : j \in J\}$, $\Gamma' = \{\Gamma'_j \in \mathfrak{L}(E', E'_j) : j \in J\}$, $\mathcal{F}' = \{f'_j\}_{j\in J}$ and $\mathcal{G}' = \{g'_j\}_{j\in J} \subseteq E'$, where E' and E'_j's are Hilbert $\mathfrak{A}'$-modules.

Proposition 3.6 (i) *Let Γ be an approximate g-dual (resp. a g-dual) of Λ. If Γ' is a g-dual of Λ', then $\Gamma\otimes\Gamma' = \{\Gamma_i\otimes\Gamma'_j\}_{i\in I, j\in J}$ is an approximate g-dual (resp. a g-dual) of $\Lambda\otimes\Lambda' = \{\Lambda_i\otimes\Lambda'_j\}_{i\in I, j\in J}$.*

(ii) *Let $\mathcal{G}$ be an approximate dual (resp. a dual) of $\mathcal{F}$. If $\mathcal{G}'$ is a dual of $\mathcal{F}'$, then $\mathcal{G}\otimes\mathcal{G}' = \{g_i\otimes g'_j\}_{i\in I, j\in J}$ is an approximate dual (resp. a dual) of $\mathcal{F}\otimes\mathcal{F}' = \{f_i\otimes f'_j\}_{i\in I, j\in J}$.*

Proof (i) First suppose that Γ and Γ' are approximate g-dual and g-dual of Λ and Λ', respectively. It follows from Theorem 2.2.5 in [23] that $0 \leq S_\Lambda \otimes S_{\Lambda'} \leq \|S_\Lambda \otimes S_{\Lambda'}\|.Id_{E\otimes E'} \leq BB'.Id_{E\otimes E'}$, where B and B' are upper bounds of Λ and Λ', respectively. Hence, Lemma 4.1 in [21] implies that $0 \leq \langle (S_\Lambda \otimes S_{\Lambda'})z, z\rangle \leq BB'\langle z, z\rangle$, for each $z \in E \otimes E'$. Now it is easy to obtain that $\sum_{(i,j)\in I\times J}\langle(\Lambda_i \otimes \Lambda'_j)z, (\Lambda_i \otimes \Lambda'_j)z\rangle$ converges in norm and

$$\left\| \sum_{(i,j)\in I\times J} |(\Lambda_i \otimes \Lambda'_j)z|^2 \right\| = \left\| \langle (S_\Lambda \otimes S_{\Lambda'})z, z\rangle \right\| \leq BB'\|z\|^2,$$

so $\Lambda \otimes \Lambda'$ is a standard g-Bessel sequence by Theorem 3.1 in [28] (also, see [16, Section 5]). Similarly, we can get that $\Gamma \otimes \Gamma'$ is a standard g-Bessel sequence. It is also easy to see that

$$S_{(\Gamma\otimes\Gamma')(\Lambda\otimes\Lambda')}(x \otimes x') = (S_{\Gamma\Lambda} \otimes S_{\Gamma'\Lambda'})(x \otimes x') = (S_{\Gamma\Lambda} \otimes Id_{E'})(x \otimes x'),$$

for each $x \otimes x' \in E \otimes E'$, and since the operators are bounded, we have $S_{(\Gamma\otimes\Gamma')(\Lambda\otimes\Lambda')} = S_{\Gamma\Lambda} \otimes Id_{E'}$. Therefore

$$\|S_{(\Gamma\otimes\Gamma')(\Lambda\otimes\Lambda')} - Id_{(E\otimes E')}\| = \|(S_{\Gamma\Lambda} - Id_E) \otimes Id_{E'}\| = \|S_{\Gamma\Lambda} - Id_E\| < 1.$$

This means that $\Gamma \otimes \Gamma'$ is an approximate g-dual of $\Lambda \otimes \Lambda'$. It is clear that if Γ and Γ' are g-duals of Λ and Λ', respectively, then $S_{(\Gamma\otimes\Gamma')(\Lambda\otimes\Lambda')} = Id_{(E\otimes E')}$, so $\Gamma \otimes \Gamma'$ is a g-dual of $\Lambda \otimes \Lambda'$.

(ii) We can get the result by using Example 3.2 in [16] and part (i). □

Note that Proposition 2.10, Corollary 2.11 in [20], and part (ii) of Corollary 3.8 in [18] are special cases of the above proposition.

Now we show that approximate duals in Hilbert C^*-modules are stable under small perturbations. The following result is analogous to part (i) of Theorem 3.1 in [20] that we need in the next section.

Proposition 3.7 *Let Λ be a g-Bessel sequence and $\Psi = \{\psi_i\}_{i\in I}$ be an approximate g-dual (resp. a g-dual) of Λ with upper bound C. If Γ is a sequence such that $\Gamma - \Lambda$ is a g-Bessel sequence with upper bound K and $CK < (1 - \|Id_E - S_{\Psi\Lambda}\|)^2$ (resp. $CK < 1$), then Γ and Ψ are approximately dual g-frames.*

Proof Let Ω be a finite subset of I and B be an upper bound for Λ. Then

$$\left\| \sum_{i\in\Omega} \langle \Gamma_i x, \Gamma_i x\rangle \right\|^{\frac{1}{2}} \leq \|\{\Lambda_i x\}_{i\in\Omega}\|_2 + \|\{\Gamma_i x - \Lambda_i x\}_{i\in\Omega}\|_2 \leq (\sqrt{B} + \sqrt{K})\|x\|,$$

for each $x \in E$. Thus, by Theorem 3.1 in [28], Γ is a standard g-Bessel sequence. Now by using the Cauchy–Schwarz inequality in Hilbert C^*-modules, for each $x \in E$, we have

$$\begin{aligned}
\|(Id_E - S_{\Psi\Gamma})x\| &\leq \|(Id_E - S_{\Psi\Lambda})x\| + \|(S_{\Psi\Lambda} - S_{\Psi\Gamma})x\| \\
&\leq \|(Id_E - S_{\Psi\Lambda})x\| + \sup_{\|y\|=1} \left\{ \left\| \sum_{i\in I} |(\Lambda_i - \Gamma_i)x|^2 \right\|^{\frac{1}{2}} \left\| \sum_{i\in I} |\psi_i y|^2 \right\|^{\frac{1}{2}} \right\} \\
&\leq \|(Id_E - S_{\Psi\Lambda})x\| + \sqrt{CK}\|x\| \leq (\|Id_E - S_{\Psi\Lambda}\| + \sqrt{CK})\|x\|.
\end{aligned}$$

Hence, $\|Id_E - S_{\Psi\Gamma}\| \leq \|Id_E - S_{\Psi\Lambda}\| + \sqrt{CK} < 1$. Also, if Λ and Ψ are g-duals, then $S_{\Psi\Lambda} = Id_E$ and we have $\|Id_E - S_{\Psi\Gamma}\| \leq \sqrt{CK} < 1$. □

The following result is a generalization of Proposition 3.10 in [20] to Hilbert C^*-modules.

Proposition 3.8 *Let* $0 \leq \lambda_1, \lambda_2 < 1$, $A, B, \varepsilon > 0$, *and* $K = \lambda_1 + \frac{\varepsilon}{\sqrt{A}} + \frac{\lambda_2[(1+\lambda_1)\sqrt{A}+\varepsilon]}{\sqrt{A}(1-\lambda_2)}$.

(i) *If* Λ *is an* (A, B) *g-frame and* Γ *is a sequence satisfying*

$$\Big\|\sum_{i\in\Omega}(\Lambda_i^* - \Gamma_i^*)f_i\Big\| \leq \lambda_1\Big\|\sum_{i\in\Omega}\Lambda_i^* f_i\Big\| + \lambda_2\Big\|\sum_{i\in\Omega}\Gamma_i^* f_i\Big\| + \varepsilon\Big\|\sum_{i\in\Omega}|f_i|^2\Big\|^{\frac{1}{2}}, \tag{2}$$

for each finite subset $\Omega \subseteq I$, $f_i \in E_i$ *with* $K < 1$, *then* $\widetilde{\Lambda}$ *is an approximate g-dual of* Γ *and* Γ *is a g-frame.*

(ii) *If* $\mathcal{F} = \{f_i\}_{i\in I}$ *is an* (A, B) *frame and* $\mathcal{G} = \{g_i\}_{i\in I}$ *is a sequence satisfying*

$$\Big\|\sum_{i\in\Omega}a_i f_i - \sum_{i\in\Omega}a_i g_i\Big\| \leq \lambda_1\Big\|\sum_{i\in\Omega}a_i f_i\Big\| + \lambda_2\Big\|\sum_{i\in\Omega}a_i g_i\Big\| + \varepsilon\Big\|\sum_{i\in\Omega}|a_i|^2\Big\|^{\frac{1}{2}},$$

for each finite subset $\Omega \subseteq I$, $\{a_i\}_{i\in\Omega} \subseteq \mathfrak{A}$ *with* $K < 1$, *then* $\widetilde{\mathcal{F}}$ *is an approximate dual of* $\mathcal{G}$ *and* $\mathcal{G}$ *is a frame.*

Proof (i) Suppose that $\{e_{i,k}\}_{k\in J_i}$ is a Parseval frame for E_i and $\{c_{i,k}\}_{i\in\Omega, k\in\Omega_i}$ is a finite subset of $\mathfrak{A}$, where Ω and Ω_is are finite index sets. Since Λ is an (A, B) standard g-frame, Corollary 3.4 in [16] yields that $\{u_{i,k} = \Lambda_i^*(e_{i,k}) : i \in I, k \in J_i\}$ is an (A, B) standard frame. Now for $v_{i,k} = \Gamma_i^*(e_{i,k})$, we have

$$\begin{aligned}
\Big\|\sum_{i\in\Omega}\sum_{k\in\Omega_i}c_{i,k}(u_{i,k} - v_{i,k})\Big\| &= \Big\|\sum_{i\in\Omega}(\Lambda_i^* - \Gamma_i^*)(\sum_{k\in\Omega_i}c_{i,k}e_{i,k})\Big\| \\
&\leq \lambda_1\Big\|\sum_{i\in\Omega}\sum_{k\in\Omega_i}c_{i,k}u_{i,k}\Big\| + \lambda_2\Big\|\sum_{i\in\Omega}\sum_{k\in\Omega_i}c_{i,k}v_{i,k}\Big\| \\
&+ \varepsilon\Big\|\sum_{i\in\Omega}\sum_{k\in\Omega_i}|c_{i,k}|^2\Big\|^{\frac{1}{2}}.
\end{aligned}$$

Hence, Theorem 3.2 in [14] implies that $\{v_{i,k} = \Gamma_i^*(e_{i,k}) : i \in I, k \in J_i\}$ is a standard Bessel sequence with upper bound $\frac{[(1+\lambda_1)\sqrt{B}+\varepsilon]^2}{(1-\lambda_2)^2}$, so (by [16, Theorem 3.3]) Γ is a standard g-Bessel sequence with upper bound $\frac{[(1+\lambda_1)\sqrt{B}+\varepsilon]^2}{(1-\lambda_2)^2}$. Thus, for each $\{f_i\}_{i\in I} \in \oplus_{i\in I}E_i$, the series $\sum_{i\in I}\Gamma_i^* f_i$ converges in E and from (2), we can get

$$\Big\|\sum_{i\in I}(\Lambda_i^* - \Gamma_i^*)f_i\Big\| \leq \lambda_1\Big\|\sum_{i\in I}\Lambda_i^* f_i\Big\| + \lambda_2\Big\|\sum_{i\in I}\Gamma_i^* f_i\Big\| + \varepsilon\Big\|\sum_{i\in I}|f_i|^2\Big\|^{\frac{1}{2}}. \tag{3}$$

Since for each $x \in E$, $\{f_i = \widetilde{\Lambda_i}x\}_{i\in I} \in \oplus_{i\in I}E_i$ and $\frac{1}{A}$ is an upper bound for $\widetilde{\Lambda}$, by using (3), we have

$$\|S_{\Gamma\widetilde{\Lambda}}x\| - \|x\| \leq \|S_{\Gamma\widetilde{\Lambda}}x - x\| \leq (\lambda_1 + \frac{\varepsilon}{\sqrt{A}})\|x\| + \lambda_2\|S_{\Gamma\widetilde{\Lambda}}x\|.$$

Now we can obtain a result similar to the proof of Proposition 3.10 in [20].

(ii) Since $\Lambda^*_{f_i}(a) = af_i$ and $\Lambda^*_{g_i}(a) = ag_i$, for each $a \in \mathfrak{A}$, the result follows from part (i) for $\Lambda_i = \Lambda_{f_i}$ and $\Gamma_i = \Lambda_{g_i}$. □

4. Approximate duals and ε-nearly Parseval frames

In this section, we consider ε-nearly Parseval frames in Hilbert spaces, which are useful in applications (see [5]). We obtain some results for approximate duals of ε-nearly Parseval and ε-close frames (since Hilbert spaces are special cases of Hilbert C^*-modules, we do not state the definitions of frames, g-frames, and approximate duals in Hilbert spaces separately).

ε-nearly Parseval frames were defined in [5] and we have the following definition:

Definition 4.1 *Suppose that H is a separable Hilbert space and $\{H_i\}_{i\in I}$ is a sequence of separable Hilbert spaces.*

(i) *Let Λ_i be a bounded operator from H into H_i and $\varepsilon < 1$. We say that $\Lambda = \{\Lambda_i\}_{i\in I}$ is an* ε-nearly Parseval g-frame *if for each $f \in H$*

$$(1-\varepsilon)\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq (1+\varepsilon)\|f\|^2.$$

(ii) *Let $\{f_i\}_{i\in I}$ be a sequence in H and $\varepsilon < 1$. We say that $\{f_i\}_{i\in I}$ is an* ε-nearly Parseval frame *if for each $f \in H$*

$$(1-\varepsilon)\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq (1+\varepsilon)\|f\|^2.$$

It is clear that if $\varepsilon = 0$, then an ε-nearly Parseval g-frame is a Parseval g-frame and so it is a g-dual of itself. Now we have the following result for approximate duals:

Theorem 4.2 (i) *If Λ is an ε-nearly Parseval g-frame, then it is an approximate g-dual of itself.*

(ii) *If $\{f_i\}_{i\in I}$ is an ε-nearly Parseval frame, then it is an approximate dual of itself.*

Proof (i) Since Λ is an ε-nearly Parseval g-frame, we have $(1-\varepsilon).Id_H \leq S_\Lambda \leq (1+\varepsilon).Id_H$, so $-\varepsilon.Id_H \leq (S_\Lambda - Id_H) \leq \varepsilon.Id_H$. Because $S_{\Lambda\Lambda} = S_\Lambda$, we obtain that $\|S_{\Lambda\Lambda} - Id_H\| \leq \varepsilon < 1$. This means that Λ is an approximate g-dual of itself.

(ii) Since frames are special cases of g-frames, we get the result from part (i). □

Note that if $\{\Lambda_i\}_{i\in I}$ is a g-Bessel sequence and $J \subset I$, then we denote the optimal upper bound of $\{\Lambda_i\}_{i\in J^c}$ by $B(J^c)$.

It was shown in Theorem 3.1 in [20] that if $\{\Lambda_i\}_{i\in I}$ is a Parseval g-frame and $B(J^c) < 1$, then $\{\Lambda_i\}_{i\in J}$ is an approximate g-dual of itself. Now we have the following result for ε-nearly Parseval g-frames:

Proposition 4.3 (i) *Let Λ be an ε-nearly Parseval g-frame and $J \subset I$ such that $B(J^c) < 1-\varepsilon$. Then $\{\Lambda_i\}_{i\in J}$ is an approximate g-dual of itself.*

(ii) *Let* $\{f_i\}_{i\in I}$ *be an* ε*-nearly Parseval frame and* $J\subset I$ *such that* $B(J^c)<1-\varepsilon$. *Then* $\{f_i\}_{i\in J}$ *is an approximate dual of itself.*

Proof (i) We have

$$\begin{aligned}\sum_{i\in J}\|\Lambda_i f\|^2 &= \sum_{i\in I}\|\Lambda_i f\|^2-\sum_{i\in J^c}\|\Lambda_i f\|^2\\ &\geq \sum_{i\in I}\|\Lambda_i f\|^2-B(J^c)\|f\|^2\geq\Big((1-\varepsilon)-B(J^c)\Big)\|f\|^2.\end{aligned}$$

Hence, $\Big(1-(\varepsilon+B(J^c))\Big)$ is a lower bound for $\{\Lambda_i\}_{i\in J}$. Therefore, $\{\Lambda_i\}_{i\in J}$ is an ε'–nearly Parseval g-frame, where $\varepsilon'=(\varepsilon+B(J^c))$. Now by Theorem 4.2, $\{\Lambda_i\}_{i\in J}$ is an approximate g-dual of itself.

(ii) The result follows from part (i). □

Recall that two sequences $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ in H are ε*-close* if $\sum_{i\in I}\|f_i-g_i\|^2\leq\varepsilon^2$ (see [5, Definition 2.4]).

Example 4.4 *Let* $H=\mathbb{C}^2$, $\{e_1,e_2\}$ *be the standard orthonormal basis for* H *and* $\frac{2}{3}<\varepsilon<1$. *For* $\mathcal{F}=\{\sqrt{\frac{\varepsilon}{2}}e_1,e_2\}$ *and* $\mathcal{G}=\{0,e_2\}$, *it is easy to see that* $\mathcal{F}$ *is an* ε*-nearly Parseval frame that is* ε*-close to* $\mathcal{G}$, *but* $\mathcal{F}$ *and* $\mathcal{G}$ *are not approximately dual frames because* $\mathcal{G}$ *is not a frame.*

Now we have the following result:

Proposition 4.5 (i) *Let* $\mathcal{F}=\{f_i\}_{i\in I}$ *be an* ε*-nearly Parseval frame with upper bound* A. *If* $\{g_i\}_{i\in I}$ *and* $\{f_i\}_{i\in I}$ *are* ε*-close with* $\sqrt{A}\varepsilon<1-\|Id_H-S_{\mathcal{F}}\|$, *then* $\mathcal{F}$ *is an approximate dual of* $\{g_i\}_{i\in I}$.

(ii) *If* $\mathcal{F}$ *in part* (i) *is also an* A*-tight frame with* $\sqrt{A}\varepsilon<1-|1-A|$, *then* $\mathcal{F}$ *is an approximate dual of* $\{g_i\}_{i\in I}$.

Proof (i) Since $\mathcal{F}$ is an ε-nearly Parseval frame, it is an approximate dual of itself by Theorem 4.2. Also, for each $f\in H$, we have

$$\sum_{i\in I}|\langle f,f_i-g_i\rangle|^2\leq\|f\|^2\sum_{i\in I}\|f_i-g_i\|^2\leq\varepsilon^2\|f\|^2.$$

By considering $C=A$ and $K=\varepsilon^2$, we obtain that $\{f_i-g_i\}_{i\in I}$ is a Bessel sequence with upper bound K and $\sqrt{CK}=\varepsilon\sqrt{A}<1-\|Id_H-S_{\mathcal{F}}\|$. Now the result follows from Proposition 3.7 (or [20, Theorem 3.1]).

(ii) The result follows from part (i) by considering $S_{\mathcal{F}}=A.Id_H$. □

Remark 4.6 *We can obtain from part* (ii) *of the above proposition that if* $\mathcal{F}=\{f_i\}_{i\in I}$ *is an* A*-tight frame with* $A\leq 1$, *then each sequence* $\{g_i\}_{i\in I}$ *that is* ε*-close to* $\mathcal{F}$ *with* $\varepsilon<\sqrt{A}$ *is an approximate dual of* $\mathcal{F}$. *Especially if* $\mathcal{F}$ *is a Parseval frame that is* ε*-close to* $\mathcal{G}=\{g_i\}_{i\in I}$, *then* $\mathcal{F}$ *and* $\mathcal{G}$ *are approximately dual frames.*

Let $\mathcal{F}=\{f_i\}_{i=1}^n$ be an ε-nearly Parseval frame for a d-dimensional Hilbert space H. As we know (see [7, Theorem 5.3.4] and [3]), $\{S_{\mathcal{F}}^{\frac{-1}{2}}f_i\}_{i=1}^n$ is the closest Parseval frame to $\mathcal{F}$. It was proved in [5, Proposition 3.1 and

Remark 3.2] that the relation $\sum_{i=1}^{n}\|S_{\mathcal{F}}^{\frac{-1}{2}}f_i-f_i\|^2=d(2-\varepsilon-2\sqrt{1-\varepsilon})\leq\frac{d\varepsilon^2}{4}$ holds if $\{\frac{1}{\sqrt{1-\varepsilon}}f_i\}_{i=1}^{n}$ is a Parseval frame (or equivalently $S_{\mathcal{F}}^{\frac{-1}{2}}=\frac{1}{\sqrt{1-\varepsilon}}.Id_H$, so we have $\sum_{i=1}^{n}\|\frac{1}{\sqrt{1-\varepsilon}}f_i-f_i\|^2=d(2-\varepsilon-2\sqrt{1-\varepsilon})\leq\frac{d\varepsilon^2}{4}$). It was also shown in Example 2.4 in [24] that if $\mathcal{F}=\{f_i\}_{i=1}^{n}$ is an ε-nearly Parseval frame for a finite-dimensional Hilbert space H, then $\{\frac{1}{2}(f_i+S_{\mathcal{F}}^{-1}f_i)\}_{i=1}^{n}$ is a $(1,1+\frac{\varepsilon}{4})$ frame, which is much closer to $\mathcal{F}$ than $\{S_{\mathcal{F}}^{\frac{-1}{2}}f_i\}_{i=1}^{n}$ with better frame bounds compared with the bounds of $\mathcal{F}$. Therefore, the frames of the forms $\{\frac{1}{\sqrt{1-\varepsilon}}f_i\}_{i\in I}$ and $\{\frac{1}{2}(f_i+S_{\mathcal{F}}^{-1}f_i)\}_{i\in I}$ are useful in applications (also, see [1, Section 3]). Now we have the following results:

Proposition 4.7 (i) *Let $\Lambda=\{\Lambda_i\}_{i\in I}$ be an ε-nearly Parseval g-frame with $\varepsilon<\frac{1}{3}$. Then $\{\frac{1}{\sqrt{1-\varepsilon}}\Lambda_i\}_{i\in I}$ is an approximate g-dual of itself.*

(ii) *Let $\{f_i\}_{i\in I}$ be an ε-nearly Parseval frame with $\varepsilon<\frac{1}{3}$. Then $\{\frac{1}{\sqrt{1-\varepsilon}}f_i\}_{i\in I}$ is an approximate dual of itself.*

Proof (i) We have $(1-\varepsilon).Id_H\leq S_\Lambda\leq(1+\varepsilon).Id_H$, so $0\leq\frac{S_\Lambda}{1-\varepsilon}-Id_H\leq\frac{2\varepsilon}{1-\varepsilon}.Id_H$. Therefore, $\|\frac{S_\Lambda}{1-\varepsilon}-Id_H\|\leq\frac{2\varepsilon}{1-\varepsilon}<1$. This means that $\{\frac{1}{\sqrt{1-\varepsilon}}\Lambda_i\}_{i\in I}$ and $\{\frac{1}{\sqrt{1-\varepsilon}}\Lambda_i\}_{i\in I}$ are approximately dual g-frames.

(ii) The result follows from part (i). □

Proposition 4.8 *Let $\mathcal{F}=\{f_i\}_{i\in I}$ be a frame and $\mathcal{G}=\{g_i\}_{i\in I}$, where $g_i=\frac{1}{2}(f_i+S_{\mathcal{F}}^{-1}f_i)$.*

(i) *If $\|S_{\mathcal{F}}-Id_H\|<2$, then $\mathcal{F}$ and $\mathcal{G}$ are approximately dual frames.*

(ii) *If $\mathcal{F}$ is an ε-nearly Parseval frame, then $\mathcal{F}$ and $\mathcal{G}$ are approximately dual frames.*

(iii) *If $\mathcal{F}=\{f_i\}_{i\in I}$ is an A-tight frame with $A<3$, then $\mathcal{F}$ and $\mathcal{G}$ are approximately dual frames.*

Proof (i) For each $f\in H$, we have

$$S_{\mathcal{F}\mathcal{G}}(f)=\frac{1}{2}\Big((\sum_{i\in I}\langle f,f_i\rangle f_i)+(\sum_{i\in I}\langle f,S_{\mathcal{F}}^{-1}f_i\rangle f_i)\Big)=\frac{1}{2}(S_{\mathcal{F}}f+f).$$

Hence:

$$\|S_{\mathcal{F}\mathcal{G}}-Id_H\|=\frac{1}{2}\|S_{\mathcal{F}}-Id_H\|<1.$$

This means that $\mathcal{F}$ and $\mathcal{G}$ are approximately dual frames.

(ii) Since $\mathcal{F}$ is an ε-nearly Parseval frame, $\mathcal{F}$ is an approximate dual of itself by Theorem 4.2, and so $\|S_{\mathcal{F}}-Id_H\|<1$. Now we get the result from part (i).

(iii) Since $\mathcal{F}$ is A-tight, we have $S_{\mathcal{F}}=A.Id_H$, so $\|S_{\mathcal{F}}-Id_H\|=|A-1|<2$, and the result follows from part (i). □

References

[1] Abdollahi A, Rahimi E. Some results on g-frames in Hilbert spaces. Turk J Math 2011; 35: 695–704.

[2] Arambasic L. On frames for countably generated Hilbert C^*-modules. P Am Math Soc 2007; 135: 469–478.

[3] Balan R. Equivalence relations and distances between Hilbert frames. P Am Math Soc 1999; 127: 2353–2366.

[4] Balazs P, Feichtinger HG, Hampejs M, Kracher G. Double preconditioning for Gabor frames. IEEE T Signal Process 2006; 54: 4597–4610.

[5] Bodmann BG, Casazza P. The road to equal-norm Parseval frames. J Funct Anal 2010; 258: 397–420.

[6] Bourouihiya A. The tensor product of frames. Sampling Theory in Signal and Image Processing 2008; 7: 65–76.

[7] Christensen O. Frames and Bases: An Introductory Course. Boston, MA, USA: Birkhäuser, 2008.

[8] Christensen O, Laugesen RS. Approximate dual frames in Hilbert spaces and applications to Gabor frames. Sampling Theory in Signal and Image Processing 2011; 9: 77–90.

[9] Daubechies I, Grossmann A, Meyer Y. Painless nonorthogonal expansions. J Math Phys 1986; 27: 1271–1283.

[10] Duffin RJ, Schaeffer AC. A class of nonharmonic Fourier series. T Am Math Soc 1952; 72: 341–366.

[11] Frank M, Larson DR. Frames in Hilbert C^*-modules and C^*-algebras. J Operat Theor 2002; 48: 273–314.

[12] Gilbert JE, Han YS, Hogan JA, Lakey JD, Weiland D, Weiss G. Smooth molecular decompositions of functions and singular integral operators. Mem Am Math Soc 2002; 742: 1–74.

[13] Han D, Jing W, Larson D, Mohapatra R. Riesz bases and their dual modular frames in Hilbert C^*-modules. J Math Anal Appl 2008; 343: 246–256.

[14] Han D, Jing W, Mohapatra R. Perturbations of frames and Riesz bases in Hilbert C^*-modules. Linear Algebra Appl 2009; 431: 746–759.

[15] Khosravi A, Khosravi B. Frames and bases in tensor products of Hilbert spaces and Hilbert C^*-modules. P Indian AS-Math Sci 2007; 117: 1–12.

[16] Khosravi A, Khosravi B. Fusion frames and g-frames in Hilbert C^*-modules. Int J Wavelets Multi 2008; 6: 433–446.

[17] Khosravi A, Khosravi B. G-frames and modular Riesz bases. Int J Wavelets Multi 2012; 10: 1–12.

[18] Khosravi A, Mirzaee Azandaryani M. Fusion frames and g-frames in tensor product and direct sum of Hilbert spaces. Appl Anal Discrete Math 2012; 6: 287–303.

[19] Khosravi A, Mirzaee Azandaryani M. G-frames and direct sums. B Malays Math Sci So 2013; 36: 313–323.

[20] Khosravi A, Mirzaee Azandaryani M. Approximate duality of g-frames in Hilbert spaces. Acta Math Sci 2014; 34: 639–652.

[21] Lance EC. Hilbert C^*-Modules: A Toolkit for Operator Algebraists. Cambridge, UK: Cambridge University Press, 1995.

[22] Li H. A Hilbert C^*-module admitting no frames. B Lond Math Soc 2010; 42: 388–394.

[23] Murphy GJ. C^*-Algebras and Operator Theory. San Diego, CA, USA: Academic Press, 1990.

[24] Obeidat S, Samarah S, Casazza P, Tremain JC. Sums of Hilbert space frames. J Math Anal Appl 2009; 351: 579–585.

[25] Raeburn I, Thompson S. Countably generated Hilbert modules, the Kasparov stabilisation theorem, and frames in Hilbert modules. P Am Math Soc 2003; 131: 1557–1564.

[26] Sun W. G-frames and g-Riesz bases. J Math Anal Appl 2006; 322: 437–452.

[27] Werther T, Eldar YC, Subbanna NK. Dual Gabor frames: theory and computational aspects. IEEE T Signal Process 2005; 53: 4147–4158.

[28] Xiao X, Zeng X. Some properties of g-frames in Hilbert C^*-modules. J Math Anal Appl 2010; 363: 399–408.