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Some identities for the Glasser transform and their applications[†]

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Abstract: In the present paper we consider a new integral transform, denoted by $\mathcal{G}_\nu$, which may be regarded as a generalization of the well-known transform due to Glasser. Many identities involving this transform are given. By making use of these identities, a number of new Parseval–Goldstein type identities are obtained for these and many other well-known integral transforms. The identities proven in this paper are shown to give rise to useful corollaries for evaluating infinite integrals of special functions. Some examples are also given as illustrations of the results presented here.

Key words: Laplace transforms, Widder potential transforms, Fourier sine transforms, Fourier cosine transforms, $\mathcal{H}_\nu$-transforms, $\mathcal{K}_\nu$-transforms, Parseval–Goldstein type theorems

1. Introduction, definitions, and preliminaries

Glasser [4] considered the integral transform

$$\mathcal{G}\left\{f(x);y\right\}=\int_0^\infty \frac{f(x)}{(x^2+y^2)^{1/2}}\,dx, \tag{1.1}$$

gave the following Parseval–Goldstein type theorem [4, p. 171, Eq. (4)],

$$\int_0^\infty f(x)\mathcal{G}\left\{g(y);x\right\}\,dx=\int_0^\infty g(x)\mathcal{G}\left\{f(y);x\right\}dx, \tag{1.2}$$

and evaluated a number of infinite integrals involving Bessel functions. Additional results about the Glasser transform can be found in Srivastava and Yurekli [9] and Kahramaner *et al.* [6]. The Widder potential transform is defined by

$$\mathcal{P}\left\{f(x);y\right\}=\int_0^\infty \frac{x\,f(x)}{x^2+y^2}\,dx. \tag{1.3}$$

The classical Laplace transform, the Fourier cosine transform, the Fourier sine transform, and the $\mathcal{K}_\nu$-transform are defined as follows, respectively:

$$\mathcal{L}\left\{f(x);y\right\}=\int_0^\infty \exp(-x\,y)\,f(x)\,dx, \tag{1.4}$$

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$$\mathcal{F}_C\left\{f(x);y\right\}=\int_0^\infty \cos(x\,y)\,f(x)\,dx, \tag{1.5}$$

$$\mathcal{F}_S\left\{f(x);y\right\}=\int_0^\infty \sin(x\,y)\,f(x)\,dx, \tag{1.6}$$

and

$$\mathcal{K}_\nu\left\{f(x);y\right\}=\int_0^\infty (x\,y)^{1/2}\,K_\nu\left(x\,y\right)\;f(x)\,dx. \tag{1.7}$$

The Hankel transform $\mathcal{H}_\nu$ is defined as

$$\mathcal{H}_\nu\left\{f(x);y\right\}=\int_0^\infty (x\,y)^{1/2}\,J_\nu\left(x\,y\right)\;f(x)\,dx. \tag{1.8}$$

The $\mathcal{G}_\nu$-transform is defined by

$$\mathcal{G}_\nu\left\{f(x);y\right\}=\int_0^\infty \frac{f(x)}{\left(x^2+y^2\right)^{\nu+\frac{1}{2}}}\,dx, \tag{1.9}$$

which is related to the Glasser transform (1.1) and Widder potential transform (1.3) as follows:

$$\mathcal{G}_0\left\{f(x);y\right\}=\mathcal{G}\left\{f(x);y\right\}$$

and

$$\mathcal{G}_{1/2}\left\{f(x);y\right\}=\mathcal{P}\left\{\frac{f(x)}{x};y\right\}$$

The object of this paper is first to establish a Parseval–Goldstein type theorem involving the $\mathcal{G}_\nu$-transform (1.9), the Fourier sine transform, the Fourier cosine transform, the $\mathcal{K}_\nu$-transform, and the Hankel transform. The theorem yields new identities for the integral transform introduced above. Using these identities, a number of new Parseval–Goldstein type identities are obtained for these and many other well-known integral transforms. As applications of the identities and theorems, some illustrative examples are also given.

2. Parseval–Goldstein type theorems

In the following lemma, we give identities involving the $\mathcal{G}_\nu$-transform, $\mathcal{H}_\nu$-transform, and classical $\mathcal{L}$-Laplace transform.

Lemma 2.1 *The following identities hold true:*

$$\mathcal{G}_\nu\left\{u^{\nu+\frac{1}{2}}\mathcal{H}_\nu\left\{f(x);u\right\};y\right\}=\frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}\mathcal{L}\left\{x^{\nu-1/2}f(x);y\right\}, \tag{2.1}$$

$$\mathcal{H}_\nu\left\{u^{\nu+\frac{1}{2}}\mathcal{G}_\nu\left\{f(x);u\right\};y\right\}=\frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}y^{\nu-1/2}\mathcal{L}\left\{f(x);y\right\}, \tag{2.2}$$

$$\mathcal{G}_\nu\left\{u^{\nu+\frac{1}{2}}\mathcal{H}_\nu\left\{x^{-\nu-\frac{1}{2}}f(x);u\right\};y\right\}=\frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}\mathcal{L}\left\{\frac{f(x)}{x};y\right\}, \tag{2.3}$$

provided that the integrals involved converge absolutely.

Proof We only give the proof of (2.1) because the proof of (2.2) is similar. Indeed, to prove (2.1), we start by using the definitions (1.9) and (1.8) of the $\mathcal{G}_\nu$-transform and $\mathcal{H}_\nu$-transform.

$$\mathcal{G}_\nu\left\{u^{\nu+\frac{1}{2}}\mathcal{H}_\nu\left\{f(x);u\right\};y\right\}=\int_0^\infty\frac{u^{\nu+\frac{1}{2}}}{(u^2+y^2)^{\nu+\frac{1}{2}}}\left(\int_0^\infty(x\,u)^{1/2}J_\nu\left(x\,u\right)\,f(x)dx\right)du.$$

Changing the order of integration, which is permissible by absolute convergence of the integrals involved, we find that

$$\mathcal{G}_\nu\left\{u^{\nu+\frac{1}{2}}\mathcal{H}_\nu\left\{f(x);u\right\};y\right\}=\int_0^\infty f(x)\left(\int_0^\infty x^{1/2}J_\nu\left(ux\right)\frac{u^{\nu+1}}{(u^2+y^2)^{\nu+\frac{1}{2}}}du\right)dx. \tag{2.4}$$

Evaluating the inner integral on the right-hand side of (2.4), [3, p.24, Entry (18)] and definition (1.4) of the Laplace transform, $\operatorname{Re}(\nu)>-\frac{1}{2}$, we obtain

$$\mathcal{G}_\nu\left\{u^{\nu+\frac{1}{2}}\mathcal{H}_\nu\left\{f(x);u\right\};y\right\}=\frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}\mathcal{L}\left\{x^{\nu-1/2}f(x);y\right\}$$

To prove (2.3), we start by using the definitions (1.9) and (1.8) of the $\mathcal{G}_\nu$-transform and $\mathcal{H}_\nu$-transform.

$$\mathcal{G}_\nu\left\{u^{\nu+\frac{1}{2}}\mathcal{H}_\nu\left\{x^{-\nu-\frac{1}{2}}f(x);u\right\};y\right\}=\int_0^\infty\frac{u^{\nu+\frac{1}{2}}}{(u^2+y^2)^{\nu+\frac{1}{2}}}\left(\int_0^\infty(x\,u)^{1/2}J_\nu\left(xu\right)x^{-\nu-\frac{1}{2}}f(x)dx\right)du$$

Changing the order of integration, which is permissible by absolute convergence of the integrals involved, we have

$$\mathcal{G}_\nu\left\{u^{\nu+\frac{1}{2}}\mathcal{H}_\nu\left\{x^{-\nu-\frac{1}{2}}f(x);u\right\};y\right\}=\int_0^\infty x^{-\nu-\frac{1}{2}}f(x)\left(\int_0^\infty(ux)^{1/2}J_\nu\left(ux\right)\frac{u^{\nu+\frac{1}{2}}}{(u^2+y^2)^{\nu+\frac{1}{2}}}du\right)dx \tag{2.5}$$

Evaluating the inner integral on the right-hand side of (2.5), [3, p.24, Entry (18)] and the definition Laplace transform of (1.4), $\operatorname{Re}(\nu)>-\frac{1}{2}$, we obtain

$$\mathcal{G}_\nu\left\{u^{\nu+\frac{1}{2}}\mathcal{H}_\nu\left\{x^{-\nu-\frac{1}{2}}f(x);u\right\};y\right\}=\frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}\mathcal{L}\left\{\frac{f(x)}{x};y\right\}.$$

□

Theorem 2.1 *The following Parseval–Goldstein type identity holds true:*

$$\int_0^\infty u^{\nu+\frac{1}{2}}\mathcal{G}_\nu\left\{f(x);u\right\}\mathcal{H}_\nu\left\{g(y);u\right\}du=\frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}\int_0^\infty f(x)\mathcal{L}\left\{y^{\nu-1/2}g(y);x\right\}dx, \tag{2.6}$$

provided that the integrals involved converge absolutely.

Proof Using the definition (1.9) of the $\mathcal{G}_\nu$-transform, we have

$$\int_0^\infty u^{\nu+\frac{1}{2}}\mathcal{G}_\nu\left\{f(x);u\right\}\mathcal{H}_\nu\left\{g(y);u\right\}du=\int_0^\infty u^{\nu+\frac{1}{2}}\left(\int_0^\infty\frac{f(x)}{(x^2+u^2)^{\nu+\frac{1}{2}}}\,dx\right)\mathcal{H}_\nu\left\{g(y);u\right\}du.$$

Changing the order of the integration, which is permissible by absolute convergence of the integrals involved, we find from (2.1) that

$$\int_0^\infty u^{\nu+\frac{1}{2}}\mathcal{G}_\nu\left\{f(x);u\right\}\mathcal{H}_\nu\left\{g(y);u\right\}du = \int_0^\infty f(x)\left(\int_0^\infty \frac{u^{\nu+\frac{1}{2}}}{(u^2+x^2)^{\nu+\frac{1}{2}}}\mathcal{H}_\nu\left\{g(y);u\right\}du\right)dx$$

$$= \frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}\int_0^\infty f(x)\mathcal{L}\left\{y^{\nu-1/2}g(y);x\right\}dx.$$

□

The identity (2.6) can be rewritten by using the simple property of the Laplace transform

$$\int_0^\infty f(x)\left(\int_0^\infty g(y)\,e^{-yx}dy\right)dx = \int_0^\infty g(y)\left(\int_0^\infty f(x)\,e^{-xy}dx\right)dy$$

provided absolute convergence of the integrand, as follows:

$$\int_0^\infty u^{\nu+\frac{1}{2}}\mathcal{G}_\nu\left\{f(x);u\right\}\mathcal{H}_\nu\left\{g(y);u\right\}du = \frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}\int_0^\infty y^{\nu-1/2}g(y)\mathcal{L}\left\{f(x);y\right\}dy. \tag{2.7}$$

Comparing (2.6) and (2.7) we see that

$$\int_0^\infty f(x)\mathcal{L}\left\{y^{\nu-1/2}g(y);x\right\}dx = \int_0^\infty y^{\nu-1/2}g(y)\mathcal{L}\left\{f(x);y\right\}dy$$

provided that the integrals involved converge absolutely.

As a consequence of our Theorem 2.1, we can give the following identity, which was obtained earlier in [1, Lemma 1, (18)]:

$$\mathcal{K}_\nu\left\{\mathcal{H}_\nu\left\{g(y);u\right\};a\right\} = a^{-\nu+1/2}\mathcal{P}\left\{y^{\nu-1/2}g(y);a\right\}.$$

Lemma 2.2 *The following identities hold true:*

$$\mathcal{F}_S\left\{u\mathcal{G}_\nu\left\{f(x);u\right\};y\right\} = \frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}y^{\nu-1/2}\mathcal{K}_{1-\nu}\left\{x^{-\nu+1/2}f(x);y\right\}, \tag{2.8}$$

$$\mathcal{F}_C\left\{\mathcal{G}_\nu\left\{f(x);u\right\};y\right\} = \frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}y^{\nu-1/2}\mathcal{K}_{\nu}\left\{x^{-\nu-1/2}f(x);y\right\}, \tag{2.9}$$

$$\mathcal{G}_\nu\left\{u\mathcal{F}_S\left\{f(x);u\right\};y\right\} = \frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}y^{-\nu+1/2}\mathcal{K}_{1-\nu}\left\{x^{\nu-1/2}f(x);y\right\}, \tag{2.10}$$

$$\mathcal{G}_\nu\left\{\mathcal{F}_C\left\{f(x);u\right\};y\right\} = \frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}y^{-\nu-1/2}\mathcal{K}_{\nu}\left\{x^{\nu-1/2}f(x);y\right\}, \tag{2.11}$$

provided that the integrals involved converge absolutely.

Proof Indeed, using the definitions (1.9) and (1.6) of the $\mathcal{G}_\nu$-transform and $\mathcal{F}_S$-transform, we have

$$\mathcal{F}_S\left\{u\mathcal{G}_\nu\left\{f(x);u\right\};y\right\} = \int_0^\infty u\sin(uy)\mathcal{G}_\nu\left\{f(x);u\right\}du$$

$$= \int_0^\infty u\sin(uy)\left(\int_0^\infty \frac{f(x)}{(x^2+u^2)^{\nu+\frac{1}{2}}}dx\right)du.$$

Changing the order of integration, which is permissible by absolute convergence of the integrals involved, we find

$$\mathcal{F}_S\left\{u\mathcal{G}_\nu\left\{f(x);u\right\};y\right\}=\int_0^\infty f(x)\left(\int_0^\infty u\left(u^2+x^2\right)^{-\nu-\frac{1}{2}}\sin(uy)\,du\right)dx. \tag{2.12}$$

Evaluating the inner integral on the right-hand side of (2.12), [5, p. 442, 3.771, Entry (5)] and the definition (1.7) of $\mathcal{K}_\nu$-transform, $\mathrm{Re}\,(\nu)>0$, we obtain

$$\mathcal{F}_S\left\{u\mathcal{G}_\nu\left\{f(x);u\right\};y\right\}=\frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}y^{\nu-1/2}\mathcal{K}_{1-\nu}\left\{x^{-\nu+1/2}f(x);y\right\}.$$

Similarly, using the definitions (1.9) and (1.5) of the $\mathcal{G}_\nu$-transform and $\mathcal{F}_C$-transform, we have

$$\begin{aligned}\mathcal{F}_C\left\{\mathcal{G}_\nu\left\{f(x);u\right\};y\right\}&=\int_0^\infty \cos(uy)\mathcal{G}_\nu\left\{f(x);u\right\}du\\&=\int_0^\infty \cos(uy)\left(\int_0^\infty \frac{f(x)}{(x^2+u^2)^{\nu+\frac{1}{2}}}dx\right)du,\end{aligned}$$

and changing the order of the integration, which is permissible by absolute convergence of the integrals involved, we find

$$\mathcal{F}_C\left\{\mathcal{G}_\nu\left\{f(x);u\right\};y\right\}=\int_0^\infty f(x)\left(\int_0^\infty \left(u^2+x^2\right)^{-\nu-\frac{1}{2}}\cos(uy)du\right)dx. \tag{2.13}$$

Evaluating the inner integral on the right-hand side of (2.13), [5, p. 442, 3.771, Entry (2)] and the definition (1.7) of $\mathcal{K}_\nu$-transform, $\mathrm{Re}\,(\nu)>-\frac{1}{2}$, we obtain

$$\mathcal{F}_C\left\{\mathcal{G}_\nu\left\{f(x);u\right\};y\right\}=\frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}y^{\nu-1/2}\mathcal{K}_{\nu}\left\{x^{-\nu-1/2}f(x);y\right\}.$$

The proof of the identities (2.10) and (2.11) is similar. □

Remark 1 *Setting $\nu=0$ in (2.9), we obtain the following identity:*

$$\mathcal{F}_C\left\{\mathcal{G}\left\{x^{1/2}f(x);t\right\};y\right\}=y^{-1/2}\mathcal{K}_0\left\{f(x);y\right\}. \tag{2.14}$$

We would like to note that (2.14) was obtained earlier (see [6, p.8, Lemma 2.1, (2.1)])

Theorem 2.2 *The following Parseval–Goldstein type identities hold true:*

$$\int_0^\infty u\mathcal{G}_\nu\left\{f(x);u\right\}\mathcal{F}_S\left\{g(y);u\right\}du=\frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}\int_0^\infty y^{\nu-1/2}g(y)\mathcal{K}_{1-\nu}\left\{x^{-\nu+1/2}f(x);y\right\}dy, \tag{2.15}$$

$$\int_0^\infty \mathcal{G}_\nu\left\{f(x);u\right\}\mathcal{F}_C\left\{g(y);u\right\}du=\frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}\int_0^\infty y^{\nu-1/2}g(y)\mathcal{K}_{\nu}\left\{x^{-\nu-1/2}f(x);y\right\}dy, \tag{2.16}$$

provided that the integrals involved converge absolutely.

Proof We only give the proof of the Parseval–Goldstein identity (2.15) because the proof of (2.16) is similar. Indeed,

$$\int_0^\infty u\mathcal{G}_\nu \{f(x);u\}\, \mathcal{F}_S \{g(y);u\}\, du = \int_0^\infty u\mathcal{G}_\nu \{f(x);u\} \left\{ \int_0^\infty \sin(uy) g(y)\ dy \right\} du,$$

and changing the order of the integration, which is permissible by absolute convergence of the integrals involved, we find from (2.8) that

$$\begin{aligned}\int_0^\infty u\mathcal{G}_\nu \{f(x);u\}\, \mathcal{F}_S \{g(y);u\}\, du &= \int_0^\infty g(y) \left\{ \int_0^\infty u \sin(uy) \mathcal{G}_\nu \{f(x);u\}\, du \right\} dy \\ &= \frac{\sqrt{\pi}}{2^\nu \Gamma(\nu+\frac{1}{2})} \int_0^\infty y^{\nu-1/2} g(y) \mathcal{K}_{1-\nu} \left\{ x^{-\nu+1/2} f(x); y \right\} dy.\end{aligned}$$

□

Corollary 2.1 *We have*

$$\mathcal{L}\{\mathcal{G}_\nu \{f(x);u\}\,;a\} = \frac{a}{2^{\nu-1}\sqrt{\pi}\Gamma(\nu+\dfrac{1}{2})} \mathcal{P}\left\{ y^{\nu-3/2} \mathcal{K}_\nu \left\{ x^{-\nu-1/2} f(x); y \right\}; a \right\}, \tag{2.17}$$

provided that the integrals involved converge absolutely.

Proof We put

$$g(y) = \frac{1}{y^2+a^2} \tag{2.18}$$

in (2.16) of our Theorem 2.2. Utilizing the known formula [2, p.8, Entry (11)], we have

$$\frac{\pi}{2a} \int_0^\infty \mathcal{G}_\nu \{f(x);u\}\ e^{-au}\ du = \frac{\sqrt{\pi}}{2^\nu \Gamma(\nu+\frac{1}{2})} \int_0^\infty \frac{y^{\nu-1/2}}{y^2+a^2} \mathcal{K}_\nu \left\{ x^{-\nu-1/2} f(x); y \right\} dy. \tag{2.19}$$

Using the definition of the Laplace transform (1.4) and Widder potential transform (1.3), we obtain

$$\mathcal{L}\{\mathcal{G}_\nu \{f(x);u\}\,;a\} = \frac{a}{2^{\nu-1}\sqrt{\pi}\Gamma(\nu+\dfrac{1}{2})} \mathcal{P}\left\{ y^{\nu-3/2} \mathcal{K}_\nu \left\{ x^{-\nu-1/2} f(x); y \right\}; a \right\}.$$

□

3. Illustrative examples

Example 3.1 *We show for* $\operatorname{Re}(\nu) > 1/2$

$$\int_0^\infty \frac{u}{(u^2+y^2)^{\nu+1/2}} \gamma(\nu, u^2/4a)\ du = \frac{\sqrt{\pi}}{2^{3(\nu-1)/2} a^{\nu-1/2}} \frac{\Gamma(2\nu-1)}{\Gamma(\nu+\frac{1}{2})} \exp\left(\frac{y^2}{8a}\right) D_{-2\nu+1}\left(\frac{y}{\sqrt{2a}}\right),$$

where $\gamma(\alpha, x)$ *denotes the incomplete gamma function and* $D_\alpha(x)$ *denotes the parabolic-cylinder function.*

Proof We put

$$f(x) = x^{\nu-3/2}e^{-ax^2} \tag{3.1}$$

in (2.1) of Lemma 2.1. Using the known result [3, p.30, Entry (11)], we find that

$$\mathcal{H}_\nu\left\{f(x); u\right\} = 2^{\nu-1}u^{1/2-\nu}\gamma(\nu, u^2/4a). \tag{3.2}$$

Substituting the result (3.2) into identity (2.1) of our Lemma 2.1, we obtain

$$2^{\nu-1}\int_0^\infty \frac{u}{(u^2+y^2)^{\nu+1/2}}\gamma(\nu, u^2/4a)\ du = \frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}\mathcal{L}\left\{x^{2\nu-2}e^{-ax^2}; y\right\},$$

and using the known result [8, p.29, Entry (6)], we find that

$$\int_0^\infty \frac{u}{(u^2+y^2)^{\nu+1/2}}\gamma(\nu, u^2/4a)\ du = \frac{\sqrt{\pi}}{2^{3(\nu-1)/2}a^{\nu-1/2}}\frac{\Gamma(2\nu-1)}{\Gamma(\nu+\frac{1}{2})}\exp\left(\frac{y^2}{8a}\right)D_{-2\nu+1}\left(\frac{y}{\sqrt{2a}}\right).$$

□

Example 3.2 *We show*

$$\mathcal{F}_C\left\{\frac{1}{(u^2+y^2)^{\nu+1/2}}; a\right\} = \frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{2a}{y}\right)^\nu\frac{\Gamma(2\nu)}{(y+a)^{2\nu}}\mathcal{K}_\nu(ay),$$

where $a < x < \infty$ *and* $\mathrm{Re}(\nu) > -1/2$.

Proof We put

$$f(x) = x^{-\nu+1/2}(x^2-a^2)^{\nu-1/2} \tag{3.3}$$

in (2.1) of Lemma 2.1. Using the known result [3, p.25, Entry (28)], we find that

$$\mathcal{H}_\nu\left\{f(x); u\right\} = \frac{1}{\sqrt{\pi}2^\nu}\Gamma\left(\nu+\tfrac{1}{2}\right)u^{-\nu-1/2}\cos(au), \tag{3.4}$$

and

$$\mathcal{G}_\nu\left\{u^{\nu+1/2}\mathcal{H}_\nu\left\{f(x); u\right\}; y\right\} = \frac{1}{\sqrt{\pi}2^\nu}\Gamma\left(\nu+\tfrac{1}{2}\right)\int_0^\infty \frac{1}{(u^2+y^2)^{\nu+1/2}}\cos(au)du. \tag{3.5}$$

Substituting the results (3.4) and (3.5) into identity (2.1) of our Lemma 2.1, we obtain

$$\frac{1}{\sqrt{\pi}2^\nu}\Gamma\left(\nu+\tfrac{1}{2}\right)\int_0^\infty \frac{1}{(u^2+y^2)^{\nu+1/2}}\cos(au)du = \frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}\mathcal{L}\left\{(x^2-a^2)^{\nu-1/2}; y\right\},$$

and using the known result [8, p.22, Entry (13)] and the definition (1.5) of the $\mathcal{F}_C$-transform, we find that

$$\mathcal{F}_C\left\{\frac{1}{(u^2+y^2)^{\nu+1/2}}; a\right\} = \frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{2a}{y}\right)^\nu\frac{\Gamma(2\nu)}{(y+a)^{2\nu}}\mathcal{K}_\nu(ay).$$

□

The following example is deduced from the identities given in Lemma 2.2.

Example 3.3 *We show*

$$\mathcal{G}_{\nu}\left\{(a^2-u^2)^{\nu-1/2};y\right\}=\frac{\sqrt{\pi}}{2}\Gamma\left(\nu+\tfrac{1}{2}\right)\left(\frac{a}{y}\right)^{\nu}(a^2+y^2)^{-1/2}P_{-1/2}^{-\nu}\left(\frac{y^2-a^2}{y^2+a^2}\right)$$

where $P_{\nu}^{\mu}(x)$ *denotes the associated Legendre function of the first kind and* $\mathrm{Re}\,(\nu)>-1/2$.

Proof We put

$$f(x)=x^{-\nu}J_{\nu}(ax) \tag{3.6}$$

in (2.11) of Lemma 2.2. Using the known result [2, p.44, Entry (9)], we find that

$$\mathcal{F}_C\left\{f(x);u\right\}=\frac{\sqrt{\pi}}{2^{\nu}a^{\nu}\Gamma(\nu+\frac{1}{2})}(a^2-u^2)^{\nu-1/2}, \tag{3.7}$$

and

$$\mathcal{G}_{\nu}\left\{\mathcal{F}_C\left\{f(x);u\right\};y\right\}=\frac{\sqrt{\pi}}{2^{\nu}a^{\nu}\Gamma(\nu+\frac{1}{2})}\mathcal{G}_{\nu}\left\{(a^2-u^2)^{\nu-1/2};y\right\}, \tag{3.8}$$

where $0<u<a$ and $\mathrm{Re}\,(\nu)>-1/2$. Substituting the result (3.7) and (3.8) into identity (2.11) of our Lemma 2.2, we obtain

$$a^{-\nu}\mathcal{G}_{\nu}\left\{(a^2-u^2)^{\nu-1/2};y\right\}=\frac{\sqrt{\pi}}{2^{\nu}\Gamma(\nu+\frac{1}{2})}y^{-\nu-1/2}\mathcal{K}_{\nu}\left\{x^{-1/2}J_{\nu}(ax);y\right\},$$

and using the known result [7, p.365, 2.16.21, Entry (1)], we find that

$$\mathcal{G}_{\nu}\left\{(a^2-u^2)^{\nu-1/2};y\right\}=\frac{\sqrt{\pi}}{2}\Gamma\left(\nu+\tfrac{1}{2}\right)\left(\frac{a}{y}\right)^{\nu}(a^2+y^2)^{-1/2}P_{-1/2}^{-\nu}\left(\frac{y^2-a^2}{y^2+a^2}\right).$$

□

We conclude this investigation by remarking that many other infinite integrals can be evaluated in this manner by applying the above lemmas, the above theorems, and their various corollaries and consequences considered here.

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