Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2015) 39: 719 – 732
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doi:10.3906/mat-1504-43

Fiber product preserving bundle functors on fibered-fibered manifolds

Włodzimierz M. MIKULSKI*
Institute of Mathematics, Jagiellonian University, Krakow, Poland

Received: 15.04.2015 • **Accepted/Published Online:** 05.06.2015 • **Printed:** 30.09.2015

Abstract: We introduce the concept of modified vertical Weil functors on the category $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ of fibered-fibered manifolds with (m_1, m_2)-dimensional bases and their local fibered-fibered maps with local fibered diffeomorphisms as base maps. We then describe all fiber product preserving bundle functors on $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ in terms of modified vertical Weil functors.

Key words: Weil algebra, Weil functor, vertical Weil functor, Weil algebra bundle functor, modified vertical Weil functor, bundle functor, fiber product preserving bundle functor, natural transformation

1. Introduction

We assume that any manifold considered in this paper is Hausdorff, second countable, finite dimensional, without boundary, and smooth (i.e. of class C^∞). All maps between manifolds are assumed to be smooth (of class C^∞).

Let $\mathcal{M}f$ be the category of manifolds and their local maps, $\mathcal{M}f_m$ the category of m-dimensional manifolds and their local diffeomorphisms, $\mathcal{FM}$ the category of fibered manifolds (surjective submersions between manifolds) and their local fibered maps, $\mathcal{FM}_{m_1,m_2}$ the category of (m_1, m_2)-dimensional fibered manifolds (i.e. with m_1-dimensional bases and m_2-dimensional fibers) and their local fibered diffeomorphisms, $\mathcal{FM}_m$ the category of fibered manifolds with m-dimensional bases and their local fibered maps with embeddings as base maps, $\mathcal{F}_2\mathcal{M}$ the category of fibered-fibered manifolds (surjective fibered submersions between fibered manifolds with submersions between fibers) and their local fibered-fibered maps, and $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ the category of fibered-fibered manifolds with (m_1, m_2)-dimensional bases and their $\mathcal{F}_2\mathcal{M}$-maps with base maps being $\mathcal{FM}_{m_1,m_2}$-maps.

Thus, any $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-object is of the form $Y = ((p, \underline{p}) : (q : Y \to X) \to (\underline{q} : \underline{Y} \to \underline{X}))$ (a surjective fibered submersion from an $\mathcal{FM}$-object $q : Y \to X$ onto an $\mathcal{FM}_{m_1,m_2}$-object $\underline{q} : \underline{Y} \to \underline{X}$ inducing submersions between fibers). Any $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-map $f : Y \to Y'$ is a system $f = (f, \underline{f}) = (f, f_1, \underline{f}, f_2) : Y \to Y'$ of an $\mathcal{FM}$-map $f = (f, f_1) : (q : Y \to X) \to (q' : Y' \to X')$ and an $\mathcal{FM}_{m_1,m_2}$-map $\underline{f} = (\underline{f}, f_2) : (\underline{q} : \underline{Y} \to \underline{X}) \to (\underline{q}' : \underline{Y}' \to \underline{X}')$ with $p' \circ f = \underline{f} \circ p$.

A bundle functor F on $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ in the sense of [7] is a functor $F : \mathcal{F}_2\mathcal{M}_{m_1,m_2} \to \mathcal{FM}$ such that the value FY is a fibered manifold $\pi_Y : FY \to Y$ for any $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-object Y as above, the value $Ff : FY \to FY'$

*Correspondence: wlodzimierz.mikulski@im.uj.edu.pl
2010 *AMS Mathematics Subject Classification:* 58A05, 58A20.

of $f : Y \to Y'$ is a fiber map covering f for any $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-map $f : Y \to Y'$, and $Fi_U : FU \to \pi_Y^{-1}U$ is a diffeomorphism for the inclusion map $i_U : U \to Y$ of an open subset U of Y. The definitions of bundle functors on $\mathcal{M}f$, $\mathcal{M}f_m$, $\mathcal{FM}$, $\mathcal{FM}_{m_1,m_2}$, $\mathcal{FM}_m$ are quite similar (we replace $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ by $\mathcal{M}f$ or $\mathcal{M}f_m$ or $\mathcal{FM}$ or $\mathcal{FM}_{m_1,m_2}$ or $\mathcal{FM}_m$). A bundle functor F on $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ is fiber product preserving if for any $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-objects $Y_1 = ((Y_1 \to X_1) \to (\underline{Y} \to \underline{X}))$ and $Y_2 = ((Y_2 \to X_2) \to (\underline{Y} \to \underline{X}))$ we have $F(Y_1 \times_{\underline{Y}} Y_2) = FY_1 \times_{\underline{Y}} FY_2$ modulo (Fpr_1, Fpr_2), where $pr_i : Y_1 \times_{\underline{Y}} Y_2 \to Y_i$ are the usual projections. We remark that $Y_1 \times_{\underline{Y}} Y_2 = ((Y_1 \times_{\underline{Y}} Y_2 \to (X_1 \times_{\underline{X}} X_2)^o) \to (\underline{Y} \to \underline{X}))$, where $(X_1 \times_{\underline{X}} X_2)^o$ is the (open) image of $Y_1 \times_{\underline{Y}} Y_2 \to X_1 \times_{\underline{X}} X_2$.

The vertical functor V on $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ sends any Y as above into

$$VY := \bigcup_{\underline{y} \in \underline{Y}} (Tp)^{-1}(0_{\underline{y}}) \subset TY$$

(i.e into the usual vertical bundle of the $\mathcal{FM}$-object $p : Y \to \underline{Y}$). The vertical functor V on $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ is a fiber product preserving bundle functor.

A natural transformation $\eta : F \to F^1$ between bundle functors on $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ is a family of maps $\eta_Y : FY \to F^1Y$ for any $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-manifold Y such that $F^1f \circ \eta_Y = \eta_{Y^1} \circ Ff$ for any $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-map $f : Y \to Y^1$. (One can show that then $\eta_Y : FY \to F^1Y$ is a fibered map covering the identity map id_Y for any $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-object Y [7].)

A Weil algebra is a finite dimensional real local commutative algebra A with unity (i.e. $A = \mathbf{R}.1 \oplus N_A$, where N_A is a finite dimensional ideal of nilpotent elements).

In [17], Weil introduced the concept of near A-point on a manifold M as an algebra homomorphism of the algebra $C^\infty(M, \mathbf{R})$ of smooth functions on M into a Weil algebra A. The space T^AM of all near A-points on M is called a Weil bundle. Eck (see [4]), Luciano (see [10]), and Kainz and Michor (see [5]) proved independently that product preserving bundle functors $G : \mathcal{M}f \to \mathcal{FM}$ (i.e. satisfying $G(M \times M_1) = GM \times GM_1$ for any $\mathcal{M}f$-objects M and M_1) are the Weil functors $T^A : \mathcal{M}f \to \mathcal{FM}$ for Weil algebras $A = G\mathbf{R}$ and that natural transformations $\eta : G \to G_1$ between product preserving bundle functors on $\mathcal{M}f$ are in bijection with the algebra homomorphisms $\eta_{\mathbf{R}} : G\mathbf{R} \to G_1\mathbf{R}$ between the corresponding Weil algebras.

Replacing (in the construction of V) the tangent functor T by the Weil functor T^A corresponding to a Weil algebra A and $0_{\underline{Y}}$ by the canonical section $e_{\underline{Y}}$ of $T^A\underline{Y}$, one can define (in the same way) the vertical Weil functor V^A on $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$. Functor V^A is a fiber product preserving bundle functor on $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$, too.

In [11], for any homomorphism $\mu : A \to B$ of Weil algebras, the author introduced the bundle functor $T^\mu : \mathcal{FM} \to \mathcal{FM}$ and described all product preserving bundle functors on $\mathcal{FM}$ in terms of functors T^μ. For the reader's convenience we present the construction of T^μ in Section 2.

Replacing A (in the construction of V^A) by $\mu : A \to B$ as above, we can define the functor V^μ on $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ by

$$V^\mu Y = \bigcup_{\underline{y} \in \underline{Y}} (T^\mu p)^{-1}(e_{\underline{Y}}(\underline{y})) \subset T^\mu Y \ ,$$

where Y in $T^\mu Y$ denotes the fibered manifold $Y = (q : Y \to X)$ and where $e_{\underline{Y}} : \underline{Y} \to T^\mu \underline{Y}$ is the canonical section. Then $V^\mu : \mathcal{F}_2\mathcal{M}_{m_1,m_2} \to \mathcal{FM}$ is a fiber product preserving bundle functor (the details in a more general setting can be found in Section 4).

A Weil algebra bundle functor on $\mathcal{FM}_{m_1,m_2}$ is a bundle functor A on the category $\mathcal{FM}_{m_1,m_2}$ such that A_zZ is a Weil algebra and $A_zg : A_zZ \to A_{g(z)}Z_1$ is an algebra isomorphism for any $\mathcal{FM}_{m_1,m_2}$-map $g : Z \to Z'$ and any point $z \in Z$ (or shortly and more precisely A is a bundle functor on $\mathcal{FM}_{m_1,m_2}$ into the category of all Weil algebra bundles and their algebra bundle maps).

Modifying the examples from [9], we have the following Weil algebra bundle functors on $\mathcal{FM}_{m_1,m_2}$.
— The trivial Weil algebra bundle functor A on $\mathcal{FM}_{m_1,m_2}$ given by $AZ = Z \times A$ and $Ag = g \times \mathrm{id}_A$, where A is a fixed Weil algebra.
— The Weil algebra bundle functor A on $\mathcal{FM}_{m_1,m_2}$ given by $AZ = (\bigwedge TZ)^0$ and $Ag = \bigwedge Tg_{|(\bigwedge TZ)^o}$, where $\bigwedge TZ = (\bigwedge TZ)^0 \oplus (\bigwedge TZ)^1$ is the Grassmann algebra bundle of the tangent bundle TZ and $(\bigwedge TZ)^0$ is the even degree subalgebra bundle.
— In the previous example we can replace the tangent functor T by an arbitrary vector bundle functor G on $\mathcal{FM}_{m_1,m_2}$.
— The Weil algebra bundle functor A on $\mathcal{FM}_{m_1,m_2}$ given by $AZ = J^r(Z, \mathbf{R})$ and $Ag = J^r(g, \mathrm{id}_{\mathbf{R}})$.
— We can apply a fiber-wise tensor product to the above examples of Weil algebra bundle functors on $\mathcal{FM}_{m_1,m_2}$.

A natural transformation between Weil algebra bundle functors A and A^1 on $\mathcal{FM}_{m_1,m_1}$ is an $\mathcal{FM}_{m_1,m_2}$-natural transformation $\nu : A \to A^1$ of bundle functors such that $\nu_z := (\nu_Z)_z : A_zZ \to A^1_zZ$ is an algebra homomorphism for any $\mathcal{FM}_{m_1,m_2}$-object Z and any point $z \in Z$.

In the present paper, essentially extending the technique from [9, 12], we modify the above concept of the functors V^μ on $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ as follows. First, given a natural transformation $\mu : A \to B$ between Weil algebra bundle functors A and B on $\mathcal{FM}_{m_1,m_2}$, we define the bundle functor T^μ on $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ by

$$T^\mu Y = \bigcup_{y \in Y} T_y^{\mu_{p(y)}} Y$$

for any $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-object Y, where Y on the right side denotes the fibered manifold $q : Y \to X$. By "restriction", we define $T^\mu : \mathcal{FM}_{m_1,m_2} \to \mathcal{FM}$. Next, given an $\mathcal{FM}_{m_1,m_2}$-canonical section σ of $T^\mu : \mathcal{FM}_{m_1,m_2} \to \mathcal{FM}$ (i.e. a system of sections $\sigma : \underline{Y} \to T^\mu \underline{Y}$ for any $\mathcal{FM}_{m_1,m_2}$-object $\underline{Y}$ such that $T^\mu g \circ \sigma = \sigma \circ g$ for any $\mathcal{FM}_{m_1,m_2}$-map $g : \underline{Y} \to \underline{Y}'$), we define the so-called modified vertical Weil functor $V^{\mu,\sigma} : \mathcal{F}_2\mathcal{M}_{m_1,m_2} \to \mathcal{FM}$ by

$$V^{\mu,\sigma} Y := \bigcup_{\underline{y} \in \underline{Y}} (T^\mu p)^{-1}(\sigma(\underline{y})) \subset T^\mu Y\ .$$

Then $V^{\mu,\sigma} : \mathcal{F}_2\mathcal{M}_{m_1,m_2} \to \mathcal{FM}$ is a fiber product preserving bundle functor (the details can be found in Section 3).

Thus, we have the category MVW_{m_1,m_2} of modified vertical Weil functors $V^{\mu,\sigma} : \mathcal{F}_2\mathcal{M}_{m_1,m_2} \to \mathcal{FM}$ (for all μ and σ as above) and their natural transformations, and the obvious (forgetting, inclusion) functor

$$I : \mathrm{MVW}_{m_1,m_2} \to \mathrm{FPP}_{m_1,m_2}\ ,$$

where FPP_{m_1,m_2} is the category of fiber product preserving bundle functors on $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ and their natural transformations.

In Section 5, given a fiber product preserving bundle functor F on $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ we construct canonically a natural transformation $\mu^F : A^F \to B^F$ of Weil algebra bundle functors on $\mathcal{FM}_{m_1,m_2}$ and an $\mathcal{FM}_{m_1,m_2}$-

canonical section σ^F of $T^{\mu^F}:\mathcal{FM}_{m_1,m_2}\to\mathcal{FM}$. Thus, we get the modified vertical Weil functor V^{μ^F,σ^F} on $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$. In this way, we obtain functor

$$J:\mathrm{FPP}_{m_1,m_2}\to\mathrm{MVW}_{m_1,m_2}\ .$$

The main result of the present note is the following.

Main result. There is the equivalence

$$\mathrm{FPP}_{m_1,m_2}\tilde{=}\mathrm{MVW}_{m_1,m_2}$$

of categories. More precisely, $I\circ J\tilde{=}\mathrm{Id}_{\mathrm{FPP}_{m_1,m_2}}$ and $J\circ I\tilde{=}\mathrm{Id}_{\mathrm{MVW}_{m_1,m_2}}$.

In [9], there are described all fiber product preserving bundle functors F of vertical type on $\mathcal{FM}_m$ in terms of the so-called generalized vertical Weil functors $V^A:\mathcal{FM}_m\to\mathcal{FM}$ corresponding to Weil algebra bundle functors A on $\mathcal{M}f_m$. In [12], we described also all fiber product preserving bundle functors F on $\mathcal{FM}_m$ in a similar way. There is also another pure theoretical description of all fiber product preserving bundle functors on $\mathcal{FM}_m$ by means of triples (A,H,t); see [8] (see also [6, 3]). Product preserving bundle functors on some categories over manifolds are considered in many papers, e.g., [13, 14, 15, 16]. Product preserving bundle functors on parameter dependent manifolds are studied in [1]. Natural operators to product preserving bundle functors are studied by many authors, e.g., [2].

Remark 1 *The category $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ has the same skeleton as the category of foliated fibered manifolds over foliated manifolds of dimension m_1+m_2 with leaves of dimension m_2 and their morphisms covering local leaf preserving diffeomorphisms. Thus, our main result may be easily extended on fiber product preserving bundle functors in this category.*

2. Bundle functors T^μ

We start with the following example (see [11]).

Example 1 *Let $\mu:A\to B$ be an algebra homomorphism between Weil algebras. If $p:Y\to\underline{Y}$ is an $\mathcal{FM}$-object we put*

$$T^\mu Y=T^A\underline{Y}_{\mu_{\underline{Y}}}\times_{T^Bp}T^BY=\{(u,v)\in T^A\underline{Y}\times T^BY\mid \mu_{\underline{Y}}(u)=T^Bp(v)\}$$

with the obvious projection on Y, where $\mu_{\underline{Y}}:T^A\underline{Y}\to T^B\underline{Y}$ is the natural transformation induced by μ. If $p':Y'\to\underline{Y}'$ is another $\mathcal{FM}$-object and $f:Y\to Y'$ is an $\mathcal{FM}$-map with the base map $\underline{f}:\underline{Y}\to\underline{Y}'$ we define

$$T^\mu f:=T^A\underline{f}\times T^Bf_{|T^\mu Y}:T^\mu Y\to T^\mu Y'\ .$$

The correspondence $T^\mu:\mathcal{FM}\to\mathcal{FM}$ is a product preserving bundle functor.

If $\mu':A'\to B'$ is another algebra homomorphism of Weil algebras and (φ,ψ) is a morphism $\mu\to\mu'$ (i.e. $\varphi:A\to A'$ and $\psi:B\to B'$ are algebra homomorphisms with $\mu'\circ\varphi=\psi\circ\mu$) we have the induced natural transformation $(\varphi,\psi):T^\mu\to T^{\mu'}$ defined by

$$(\varphi,\psi)_Y:=\varphi_{\underline{Y}}\times\psi_{Y|T^\mu Y}:T^\mu Y\to T^{\mu'Y}\ ,$$

where $\varphi_{\underline{Y}}:T^A\underline{Y}\to T^{A'}\underline{Y}$ and $\psi_Y:T^BY\to T^{B'}Y$ are the natural transformations induced by algebra morphisms φ and ψ of Weil algebras.

3. The generalization of T^μ

We can generalize the functors T^μ as follows.

Example 2 *Let $\mu : A \to B$ be a natural transformation between Weil algebra bundle functors A and B on $\mathcal{FM}_{m_1,m_2}$. If $Y = ((p,\underline{p}) : (q : Y \to X) \to (\underline{q} : \underline{Y} \to \underline{X}))$ is an $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-object, we put*

$$T^\mu Y := \bigcup_{y \in Y} T_y^{\mu_{p(y)}} Y$$

with the obvious projection $\pi_Y^\mu : T^\mu Y \to Y$, where Y on the right side denotes the fibered manifold $q : Y \to X$ and where $T^{\mu_{p(y)}}$ is the bundle functor on $\mathcal{FM}$ (as in the previous section) corresponding to the algebra homomorphism $\mu_{p(y)} : A_{p(y)}\underline{Y} \to B_{p(y)}\underline{Y}$ between Weil algebras (the restriction of the natural transformation $\mu_{\underline{Y}} : A\underline{Y} \to B\underline{Y}$ to the fibers as indicated), where $\underline{Y} = (\underline{q} : \underline{Y} \to \underline{X})$ is the $\mathcal{FM}_{m_1,m_2}$-object. If $Y' = ((p', \underline{p}') : (q' : Y' \to X') \to (\underline{q}' : \underline{Y}' \to \underline{X}'))$ is another $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-object and $f : Y \to Y'$ is an $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-map determining (in an obvious way) an $\mathcal{FM}_{m_1,m_2}$-map $\underline{f} : \underline{Y} \to \underline{Y}'$ (between the $\mathcal{FM}_{m_1,m_2}$-objects $\underline{Y} = (\underline{q} : \underline{Y} \to \underline{X})$ and $\underline{Y}' = (\underline{q}' : \underline{Y}' \to \underline{X}')$), we put $T^\mu f = \bigcup_{y \in Y} T_y^\mu f : T^\mu Y \to T^\mu Y'$, where $T_y^\mu f : T_y^\mu Y \to T_{f(y)}^\mu Y'$ is the composition

$$T_y^{\mu_{p(y)}} Y \to T_{f(y)}^{\mu_{p(y)}} Y' \to T_{f(y)}^{\mu_{p(f(y))}} Y'$$

of the restriction $T_y^{\mu_{p(y)}} f : T_y^{\mu_{p(y)}} Y \to T_{f(y)}^{\mu_{p(y)}} Y'$ of $T^{\mu_{p(y)}} f : T^{\mu_{p(y)}} Y \to T^{\mu_{p(y)}} Y'$ to the fibers with the restriction $(A_{p(y)}\underline{f}, B_{p(y)}\underline{f})_{f(y)} : T_{f(y)}^{\mu_{p(y)}} Y' \to T_{f(y)}^{\mu_{p(f(y))}} Y'$ of the natural transformation $(A_{p(y)}\underline{f}, B_{p(y)}\underline{f})_{Y'} : T^{\mu_{p(y)}} Y' \to T^{\mu_{p(f(y))}} Y'$ induced (as in Example 1) by morphism $(A_{p(y)}\underline{f}, B_{p(y)}\underline{f}) : \mu_{p(y)} \to \mu_{p(f(y))}$, where $Y = (q : Y \to X)$ and $Y' = (q' : Y' \to X')$ are the fibered manifolds. Every fibered-fibered chart (U, φ) on Y induces chart

$$(T^\mu U, T^\mu \varphi) \quad \text{on} \quad T^\mu Y$$

provided that we use the "translation" identification $T^\mu(\mathbf{R}^{m_1,m_2,n_1,n_2}) \tilde{=} \mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \times \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \times T_{(0,0,0,0)}^{\mu_{(0,0)}}(\mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \times \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}) (\tilde{=} \mathbf{R}^{n(m_1,m_2,n_1,n_2)})$, where $\mathbf{R}^{m_1,m_2,n_1,n_2}$ is the $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-object $(pr_{\mathbf{R}^{m_1} \times \mathbf{R}^{m_2}}, pr_{\mathbf{R}^{m_1}}) : (pr_{\mathbf{R}^{m_1} \times \mathbf{R}^{n_1}} : \mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \times \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \to \mathbf{R}^{m_1} \times \mathbf{R}^{n_1}) \to (pr_{\mathbf{R}^{m_1}} : \mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \to \mathbf{R}^{m_1})$ (the canonical projections) and where in $T_{(0,0,0,0)}^{\mu_{(0,0)}}$ we have the $\mathcal{FM}$-object $\mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \times \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} = (pr_{\mathbf{R}^{m_1} \times \mathbf{R}^{n_1}} : \mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \times \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \to \mathbf{R}^{m_1} \times \mathbf{R}^{n_1})$. Thus, the correspondence $T^\mu : \mathcal{F}_2\mathcal{M}_{m_1,m_2} \to \mathcal{FM}$ is a bundle functor.

If $\mu' : A' \to B'$ is another natural transformation between Weil algebra bundle functors on $\mathcal{FM}_{m_1,m_2}$ and (φ, ψ) is a morphism $\mu \to \mu'$ (i.e $\varphi : A \to A'$ and $\psi : B \to B'$ are natural transformations between Weil algebra bundle functors such that $\mu' \circ \varphi = \psi \circ \mu$), we have the induced natural transformation $(\varphi, \psi) : T^\mu \to T^{\mu'}$ given by

$$(\varphi, \psi)_Y = \bigcup_{y \in Y} (\varphi, \psi)_y : T^\mu Y \to T^{\mu'} Y \ ,$$

where $(\varphi, \psi)_y : T_y^{\mu_{p(y)}} Y \to T_y^{\mu'_{p(y)}} Y$ is the restriction of natural transformation $(\varphi_{p(y)}, \psi_{p(y)})_Y : T^{\mu_{p(y)}} Y \to T^{\mu'_{p(y)}} Y$ induced (as in Example 1) by the morphism $(\varphi_{p(y)}, \psi_{p(y)}) : \mu_{p(y)} \to \mu'_{p(y)}$.

Clearly, any $\mathcal{FM}_{m_1,m_2}$-object $\underline{Y}=(\underline{q}:\underline{Y}\to\underline{X})$ can be treated as the $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-object $\underline{Y}=((id_{\underline{Y}},id_{\underline{X}}):(\underline{q}:\underline{Y}\to\underline{X})\to(\underline{q}:\underline{Y}\to\underline{X}))$. Similarly, any $\mathcal{FM}_{m_1,m_2}$-map $\underline{f}=(f,\underline{f}):\underline{Y}\to\underline{Y}'$ between $\mathcal{FM}_{m_1,m_2}$-objects $\underline{Y}$ and $\underline{Y}=(q':\underline{Y}'\to\underline{X}')$ can be treated as the $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ morphism $\underline{f}=(f,\underline{f},f,\underline{f}):\underline{Y}\to\underline{Y}'$ between the $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-objects $\underline{Y}$ and $\underline{Y}'$. Thus, for any natural transformation $\mu:A\to B$ between Weil algebra bundle functors on $\mathcal{FM}_{m_1,m_2}$, we have bundle functor $T^\mu:\mathcal{FM}_{m_1,m_2}\to\mathcal{FM}$ (the "restriction" of T^μ from Example 2).

4. The bundle functors $V^{\mu,\sigma}$

We have the following general example of a fiber product preserving bundle functor on $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$.

Example 3 *Let $\mu:A\to B$ be a natural transformation between Weil algebra bundle functors on $\mathcal{FM}_{m_1,m_2}$. Suppose we have $\mathcal{FM}_{m_1,m_2}$-natural (canonical) section $\sigma:\underline{Y}\to T^\mu\underline{Y}$ with respect to the bundle functor projection $T^\mu\underline{Y}\to\underline{Y}$ for any $\mathcal{FM}_{m_1,m_2}$-object $\underline{Y}$ (the naturality means that $T^\mu\underline{f}\circ\sigma=\sigma\circ\underline{f}$ for any $\mathcal{FM}_{m_1,m_2}$-map $\underline{f}:\underline{Y}\to\underline{Y}'$). If $Y=((p,\underline{p}):(q:Y\to X)\to(\underline{q}:\underline{Y}\to\underline{X}))$ is an $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-object, we define*

$$V^{\mu,\sigma}Y:=(T^\mu p)^{-1}(im(\sigma))=\bigcup_{\underline{y}\in\underline{Y}}(T^\mu p)^{-1}(\sigma(\underline{y}))\subset T^\mu Y$$

with the obvious projection onto Y (the restriction of π^μ_Y), where $p:Y\to\underline{Y}$ is treated as the $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-morphism $(p,\underline{p},id_{\underline{Y}},id_{\underline{X}}):Y\to\underline{Y}$ between the $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-objects Y and $\underline{Y}=((id_{\underline{Y}},id_{\underline{X}}):(\underline{q}:\underline{Y}\to\underline{X})\to(\underline{q}:\underline{Y}\to\underline{X}))$. Since $T^\mu p$ is a submersion (it can be observed in fibered-fibered charts), $V^{\mu,\sigma}Y$ is a submanifold. If Y' is another $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-object and $f:Y\to Y'$ is an $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-map, then $T^\mu f(V^{\mu,\sigma}Y)\subset V^{\mu,\sigma}Y'$, and we define

$$V^{\mu,\sigma}f:=T^\mu f_{|V^{\mu,\sigma}Y}:V^{\mu,\sigma}Y\to V^{\mu,\sigma}Y'\ .$$

One can see that $V^{\mu,\sigma}:\mathcal{F}_2\mathcal{M}_{m_1,m_2}\to\mathcal{FM}$ is a bundle functor. It will be called the modified vertical Weil functor (corresponding to (μ,σ)).

If $\mu':A'\to B'$ is another natural transformation between Weil algebra bundle functors on $\mathcal{FM}_{m_1,m_2}$, $\sigma':\underline{Y}\to T^{\mu'}\underline{Y}$ (for all $\mathcal{FM}_{m_1,m_2}$-objects $\underline{Y}$) is another canonical section and $(\varphi,\psi):\mu\to\mu'$ is a morphism such that

$$\sigma'=(\varphi,\psi)_{\underline{Y}}\circ\sigma$$

for any $\mathcal{FM}_{m_1,m_2}$-object $\underline{Y}$, then $(\varphi,\psi)_Y(V^{\mu,\sigma}Y)\subset V^{\mu',\sigma'}Y$, and then we have the induced $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-natural transformation $[\varphi,\psi]:V^{\mu,\sigma}\to V^{\mu',\sigma'}$ defined by

$$[\varphi,\psi]_Y:=(\varphi,\psi)_{Y|V^{\mu,\sigma}Y}$$

for any $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-object Y, where $(\varphi,\psi)_Y:T^\mu Y\to T^{\mu'}Y$ is the natural transformation induced by (φ,ψ) (as in Example 2).

Lemma 1 *The bundle functor $V^{\mu,\sigma}:\mathcal{F}_2\mathcal{M}_{m_1,m_2}\to\mathcal{FM}$ is fiber product preserving.*

Proof Let $Z = (p : Z \to \underline{Z})$ be an $\mathcal{FM}_{m_1,m_2}$-object and $z \in Z$. Given an $\mathcal{FM}$-object $N = (q : N \to \underline{N})$ we put

$$GN := V_z^{\mu,\sigma}(Z \times N)$$

with the obvious projection onto N, where $Z \times N = ((pr_Z, pr_{\underline{Z}}) : (p \times q : Z \times N \to \underline{Z} \times \underline{N}) \to (p : Z \to \underline{Z}))$ is the $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-object. If $N' = (q' : N' \to \underline{N}')$ is another $\mathcal{FM}$-object and $f = (f, \underline{f}) : N \to N'$ is an $\mathcal{FM}$-map we define

$$Gf := V^{\mu,\sigma}(id_Z \times f)_{|GN} : GN \to GN' ,$$

where $id_Z \times f = (id_Z \times f, id_{\underline{Z}} \times \underline{f}, id_Z, id_{\underline{Z}}) : Z \times N \to Z \times N'$ is the $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-map. The correspondence $G : \mathcal{FM} \to \mathcal{FM}$ is a bundle functor.

Clearly, it is sufficient to show that G is product preserving. We have $GN \subset T^{\mu_z}(Z \times N)$, where $Z \times N = (q \times p : Z \times N \to \underline{Z} \times \underline{N})$ is the $\mathcal{FM}$-object. Moreover, we have

$$GN = \{\sigma(z)\} \times T^{\mu_z} N$$

modulo the identification $T^{\mu_z}(Z \times N) = T^{\mu_z} Z \times T^{\mu_z} N$ ($T^{\mu_z} : \mathcal{FM} \to \mathcal{FM}$ is product preserving). We see that

$$Gf = T^{\mu_z}(id_Z \times f)_{|GN}$$

(as $A_z(id_Z) = id$ and $B_z(id_Z) = id$). Thus, G is $\mathcal{FM}$-natural isomorphic with T^{μ_Z}, and so G is product preserving, as well. □

5. The induced bundle functors V^{μ^F,σ^F}

Now we are going to show that (conversely) any fiber product preserving bundle functor $F : \mathcal{F}_2\mathcal{M}_{m_1,m_2} \to \mathcal{FM}$ determines a natural transformation $\mu^F : A^F \to B^F$ of Weil algebra bundle functors on $\mathcal{FM}_{m_1,m_2}$ and a canonical section $\sigma^F : Y \to T^{\mu^F} Y$ for any $\mathcal{FM}_{m_1,m_2}$-object Y. We will use the following notation.

Given an $\mathcal{FM}_{m_1,m_2}$-object $Y = (q : Y \to X)$ and a manifold N we have the $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-objects

$$[Y \times N] := ((pr_Y, pr_X) : (q \times id_N : Y \times N \to X \times N) \to (q : Y \to X))$$

and

$$< Y \times N >:= ((pr_Y, id_X) : (q \circ pr_Y : Y \times N \to X) \to (q : Y \to X)) ,$$

where $pr_Y : Y \times N \to Y$ and $pr_X : X \times N \to X$ are the canonical projections. We have the $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-map

$$[id_{Y \times N}] := (id_{Y \times N}, pr_X, id_Y, id_X) : [Y \times N] \to < Y \times N > \ .$$

If N' is another manifold and $f : N \to N'$ is a map we have $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-maps

$$[id_Y \times f] := (id_Y \times f, id_X \times f, id_Y, id_X) : [Y \times N] \to [Y \times N']$$

and

$$< id_Y \times f >:= (id_Y \times f, id_X, id_Y, id_X) :< Y \times N > \to < Y \times N' > .$$

If $Y' = (q' : Y' \to X')$ is another $\mathcal{FM}_{m_1,m_2}$-object and $\varphi = (\varphi, \underline{\varphi}) : Y \to Y'$ is an $\mathcal{FM}_{m_1,m_2}$-map we have $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-maps

$$[\varphi \times id_N] := (\varphi \times id_N, \underline{\varphi} \times id_N, \varphi, \underline{\varphi}) : [Y \times N] \to [Y' \times N]$$

and

$$< \varphi \times id_N >:= (\varphi \times id_N, \underline{\varphi}, \varphi, \underline{\varphi}) :< Y \times N > \to < Y' \times N > \ .$$

We have the following example.

Example 4 *Let* $F : \mathcal{F}_2\mathcal{M}_{m_1,m_2} \to \mathcal{FM}$ *be a bundle functor. If* Y *is an* $\mathcal{FM}_{m_1,m_2}$*-object we have fibered manifolds*

$$A^F Y := F([Y \times \mathbf{R}]) \quad \textit{and} \quad B^F Y := F(< Y \times \mathbf{R} >)$$

with the projections $A^F Y \to Y$ *and* $B^F Y \to Y$ *being the composition of the bundle functor projections with the canonical projection* $Y \times \mathbf{R} \to Y$*. We have the map*

$$\mu^F_Y := F([id_{Y\times\mathbf{R}}]) : A^F Y \to B^F Y \ .$$

If Y' *is another fibered manifold and* $\varphi : Y \to Y'$ *is an* $\mathcal{FM}_{m_1,m_2}$*-map we have the induced maps*

$$A^F\varphi := F([\varphi \times id_{\mathbf{R}}]) : A^F Y \to A^F Y' \ , \quad B^F\varphi := F(< \varphi \times id_{\mathbf{R}} >) : B^F Y \to B^F Y' \ .$$

Since F *is a functor,*

$$B^F\varphi \circ \mu^F_Y = \mu^F_{Y'} \circ A^F\varphi \ .$$

Thus, we have the bundle functors A^F *and* B^F *on* $\mathcal{FM}_{m_1,m_2}$ *and the natural transformation* $\mu^F : A^F \to B^F$*.*

If $F' : \mathcal{F}_2\mathcal{M}_{m_1,m_2} \to \mathcal{FM}$ *is another bundle functor and* $\eta : F \to F'$ *is a natural transformation we have the natural transformations* $\varphi^\eta : A^F \to A^{F'}$ *and* $\psi^\eta : B^F \to B^{F'}$ *given by*

$$\varphi^\eta_Y := \eta_{[Y\times\mathbf{R}]} \quad \textit{and} \quad \psi^\eta_Y := \eta_{<Y\times\mathbf{R}>}$$

for any $\mathcal{FM}_{m_1,m_2}$*-object* Y*. Since* η *is a natural transformation,*

$$\mu^{F'} \circ \varphi^\eta = \psi^\eta \circ \mu^F \ .$$

Thus, $(\varphi^\eta, \psi^\eta) : \mu^F \to \mu^{F'}$ *is a morphism of natural transformations.*

Lemma 2 *If* F *is fiber product preserving, then* A^F *and* B^F *are Weil algebra bundle functors and* $\mu^F : A^F \to B^F$ *is a natural transformation of Weil algebra bundle functors.*

If additionally F' *is fiber product preserving, then* $(\varphi^\eta, \psi^\eta) : \mu^F \to \mu^{F'}$ *is a morphism between natural transformations of Weil algebra bundle functors.*

Proof Let Y be an $\mathcal{FM}_{m_1,m_2}$-object and $y \in Y$ be a point. For any manifold N we define

$$F^{[y]}N := F_y([Y \times N]) \ \text{ and } \ F^{<y>}N := F_y(< Y \times N >)$$

(the fibers over y of $F([Y \times N) \to Y \times N \to Y$ and of $F(< Y \times N >) \to Y \times N \to Y$) with the obvious projections onto N. If N' is another manifold and $f : N \to N'$ is a map we have

$$F^{[y]} f := F([id_Y \times f])_{|F^{[y]}N} : F^{[y]}N \to F^{[y]}N'$$

and similarly

$$F^{<y>} f := F(< id_Y \times f >)_{|F^{<y>}N} : F^{<y>}N \to F^{<y>}N' \ .$$

Thus, we have the bundle functors $F^{[y]} : \mathcal{M}f \to \mathcal{FM}$ and $F^{<y>} : \mathcal{M}f \to \mathcal{FM}$. Since F is fiber product preserving, $F^{[y]}$ and $F^{<y>}$ are product preserving. Then $F^{[y]}\mathbf{R}$ and $F^{<y>}\mathbf{R}$ are Weil algebras because of the well-known result concerning Weil functors. (If $m : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $+ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are the multiplication and sum maps for $\mathbf{R}$, then $F^{[y]}m : F^{[y]}\mathbf{R} \times F^{[y]}\mathbf{R} = F^{[y]}(\mathbf{R} \times \mathbf{R}) \to F^{[y]}\mathbf{R}$ and $F^{[y]}(+) : F^{[y]}\mathbf{R} \times F^{[y]}\mathbf{R} = F^{[y]}(\mathbf{R} \times \mathbf{R}) \to F^{[y]}\mathbf{R}$ are the multiplication and sum maps in $F^{[y]}\mathbf{R}$ (and similarly for $F^{<y>}$ instead of $F^{[y]}$), where $F^{[y]}(\mathbf{R} \times \mathbf{R}) = F^{[y]}\mathbf{R} \times F^{[y]}\mathbf{R}$ modulo the identification given by $(F^{[y]}pr_1, F^{[y]}pr_2) : F^{[y]}(\mathbf{R} \times \mathbf{R}) \to F^{[y]}\mathbf{R} \times F^{[y]}\mathbf{R}$, where $pr_i : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are the canonical projections.) Since $A^F_y Y = F^{[y]}\mathbf{R}$ and $B^F_y Y = F^{<y>}\mathbf{R}$, $A^F Y$ and $B^F Y$ are Weil algebra bundles.

If Y' is another $\mathcal{FM}_{m_1,m_2}$-object and $\varphi : Y \to Y'$ is an $\mathcal{FM}_{m_1,m_2}$-map, we define

$$\tilde{\varphi}^{[y]}_N := F([\varphi \times id_N])_{|F^{[y]}N} : F^{[y]}N \to F^{[\varphi(y)]}N$$

and

$$\tilde{\varphi}^{<y>}_N := F(< \varphi \times id_N >)_{|F^{<y>}N} : F^{<y>}N \to F^{<\varphi(y)>}N \ .$$

Then $\tilde{\varphi}^{[y]} : F^{[y]} \to F^{[\varphi(y)]}$ and $\tilde{\varphi}^{<y>} : F^{<y>} \to F^{<\varphi(y)>}$ are natural transformations between product preserving bundle functors on $\mathcal{M}f$. Then $\tilde{\varphi}^{[y]}_{\mathbf{R}} : F^{[y]}\mathbf{R} \to F^{[\varphi(y)]}\mathbf{R}$ and $\tilde{\varphi}^{<y>}_{\mathbf{R}} : F^{<y>}\mathbf{R} \to F^{<\varphi(y)>}\mathbf{R}$ are algebra homomorphisms, but $\tilde{\varphi}^{[y]}_{\mathbf{R}} = A^F_y(\varphi) : A^F_y Y \to A^F_{\varphi(y)} Y'$ and $\tilde{\varphi}^{<y>}_{\mathbf{R}} = B^F_y(\varphi) : B^F_y Y \to B^F_{\varphi(y)} Y'$.

Define

$$\tilde{\mu}^y_N := F([id_{Y \times N}])_{|F^{[y]}N} : F^{[y]}N \to F^{<y>}N \ .$$

Then $\tilde{\mu}^y : F^{[y]} \to F^{<y>}$ is a natural transformation. Then $\tilde{\mu}^y_{\mathbf{R}} : F^{[y]}\mathbf{R} \to F^{<y>}\mathbf{R}$ is an algebra homomorphism, but $\tilde{\mu}^y_{\mathbf{R}} = \mu^F_y : A^F_y Y \to B^F_y Y$ (the restriction of $\mu^F_Y : A^F Y \to B^F Y$ to the fibers over y).

We have proved that $\mu^F : A^F \to B^F$ is a natural transformation between Weil algebra bundle functors on $\mathcal{FM}_{m_1 m_2}$.

Define

$$\Phi^{[y]}_N := \eta_{[Y \times N]|F^{[y]}N} : F^{[y]}N \to F'^{[y]}N$$

and

$$\Psi^{<y>}_N := \eta_{<Y \times N>|F^{<y>}N} : F^{<y>}N \to F'^{<y>}N.$$

Then $\Phi^{[y]} : F^{[y]} \to F'^{[y]}$ and $\Psi^{<y>} : F^{<y>} \to F'^{<y>}$ are natural transformations of product preserving bundle functors on $\mathcal{M}f$. Then $\Phi^{[y]}_{\mathbf{R}} : F^{[y]}\mathbf{R} \to F'^{[y]}\mathbf{R}$ and $\Psi^{<y>}_{\mathbf{R}} : F^{<y>}\mathbf{R} \to F'^{<y>}\mathbf{R}$ are algebra homomorphisms, but $\Phi^{[y]}_{\mathbf{R}} = \varphi^\eta_y : A^F_y Y \to A^{F'}_y Y$ and $\Psi^{<y>}_{\mathbf{R}} = \psi^\eta_y : B^F_y Y \to B^{F'}_y Y$ (the restrictions of $\varphi^\eta_Y : A^F Y \to A^{F'} Y$ and $\psi^\eta_Y : B^F Y \to B^{F'} Y$ to the fibers over y).

We have proved that $(\varphi^\eta, \psi^\eta) : \mu^F \to \mu^{F'}$ is a morphism between natural transformations between Weil algebra bundle functors on $\mathcal{FM}_{m_1m_2}$. □

Let $F : \mathcal{F}_2\mathcal{M}_{m_1,m_2} \to \mathcal{FM}$ be a fiber product preserving bundle functor. We use the above notation.

Example 5 *Let $Y = (q : Y \to X)$ be an $\mathcal{FM}_{m_1,m_2}$-object and $y \in Y$ be a point. Similarly as in Section 2, Y is treated as the $\mathcal{F}_2\mathcal{M}_{m_2,m_2}$-object $Y = ((id_Y, id_X) : (q : Y \to X) \to (q : Y \to X)$. We define $\sigma_1^F(y) : C^\infty_{q(y)}(X) \to A^F_y Y$ from the algebra $C^\infty_{q(y)}(X)$ of germs at $q(y)$ of smooth maps $f : X \to \mathbf{R}$ by*

$$\sigma_1^F(y)(f) := F(f^{[]})(\theta_y) \in F_y([Y \times \mathbf{R}]) = A^F_y Y \ ,$$

where $f^{[]} := ((id_Y, f \circ q), (id_X, f), id_Y, id_X) : Y \to [Y \times \mathbf{R}]$ and $\theta_y \in F_yY$ is the unique point. (Here and later, for simplicity of notations we write f instead of $germ_z(f)$ if z is clear.) Since F is a functor, using the definition of the multiplication and sum in the Weil algebra $A^F_y Y$, one can standardly show that $\sigma_1^F(y)$ is an algebra homomorphism (for example, if we apply F to the equality $(fg)^{[]} = (id_Y \times m) \circ (id_Y, (f \circ q, g \circ q))$, we obtain $\sigma_1^F(y)(fg) = \sigma_1^F(y)(f) \cdot \sigma_1^F(y)(g)$). Then

$$\sigma_1^F(y) \in Hom(C^\infty_{q(y)}(X), A^F_y Y) = T^{A^F_y Y}_{q(y)} X \ .$$

Similarly, we define $\sigma_2^F(y) : C^\infty_y(Y) \to B^F_y Y$ by

$$\sigma_2^F(y)(f) := F(f^{<>})(\theta_y) \in F_y(< Y \times \mathbf{R} >) = B^F_y Y \ ,$$

where $f^{<>} = ((id_Y, f), id_X, id_Y, id_X) : Y \to < Y \times \mathbf{R} >$. Then

$$\sigma_2^F(y) \in Hom(C^\infty_y(Y), B^F_y Y) = T^{B^F_y Y}_y Y \ .$$

Since F is a functor, $\mu^F_y(\sigma_1^F(y)(f)) = \sigma_2^F(y)(f \circ q)$ for any $f : X \to \mathbf{R}$ (we apply functor F to the equality $id_{Y \times \mathbf{R}} \circ (id_Y, f \circ q) = (id_Y, f \circ q)$ with respective $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$-maps and evaluate at θ_y), i.e.

$$(\mu^F_y)_X(\sigma_1^F(y)) = T^{B^F_y Y} q(\sigma_2^F(y)) \ .$$

Then

$$\sigma^F(y) := (\sigma_1^F(y), \sigma_2^F(y)) \in T^{\mu^F_y}_y Y = T^{\mu^F}_y Y \ .$$

Thus, we have defined canonical section $\sigma^F : Y \to T^{\mu^F} Y$ for any $\mathcal{FM}_{m_1,m_2}$-object Y.

One can easily see that if $F' : \mathcal{F}_2\mathcal{M}_{m_1,m_2} \to \mathcal{FM}$ is another fiber product preserving bundle functor and $\eta : F \to F'$ is a natural transformation, then

$$(\varphi^\eta, \psi^\eta)_Y \circ \sigma^F = \sigma^{F'}$$

for any $\mathcal{FM}_{m_1,m_2}$-object Y, where $(\varphi^\eta, \psi^\eta)_Y : T^{\mu^F} Y \to T^{\mu^{F'}} Y$ is the natural transformation induced by the morphism $(\varphi^\eta, \psi^\eta) : \mu^F \to \mu^{F'}$ (as in Example 2 for $\mu = \mu^F$, $\mu' = \mu^{F'}$, and $(\varphi, \psi) = (\varphi^\eta, \psi^\eta)$).

Thus, for any fiber product preserving bundle functor $F : \mathcal{F}_2\mathcal{M}_{m_1,m_2} \to \mathcal{FM}$ *we have defined fiber product preserving bundle functor* $V^{\mu^F,\sigma^F} : \mathcal{F}_2\mathcal{M}_{m_1,m_2} \to \mathcal{FM}$ *(as in Example 3 for* $(\mu,\sigma) = (\mu^F,\sigma^F)$*). Moreover, for any natural transformation* $\eta : F \to F'$ *between fiber product preserving bundle functors on* $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ *we have defined natural transformation* $[\varphi^\eta,\psi^\eta] : V^{\mu^F,\sigma^F} \to V^{\mu^{F'},\sigma^{F'}}$ *(as in Example 3 for* $(\varphi,\psi) = (\varphi^\eta,\psi^\eta)$*).*

6. An equivalence $F\tilde{=}V^{\mu^F,\sigma^F}$

Let $F : \mathcal{F}_2\mathcal{M}_{m_1,m_2} \to \mathcal{FM}$ be a fiber product preserving bundle functor. We prove that $F\tilde{=}V^{\mu^F,\sigma^F}$. We start with the following example.

Example 6 *Let* $Y = ((p,\underline{p}) : (q : Y \to X) \to (\underline{q} : \underline{Y} \to \underline{X}))$ *be an* $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$*-object. Let* $v \in F_yY$, $y \in Y$*. Define* $\Theta^{1F}_Y(v) : C^\infty_{q(y)}(X) \to A^F_{p(y)}\underline{Y}$ *by*

$$\Theta^{1F}_Y(v)(f) := Ff^{[]}(v) \in F_{p(y)}([\underline{Y}\times\mathbf{R}]) = A^F_{p(y)}\underline{Y}\ ,$$

where $[\underline{Y}\times\mathbf{R}]$ *is the* $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$*-object defined by the* $\mathcal{FM}_{m_1,m_2}$*-object* $\underline{Y} = (\underline{q} : \underline{Y} \to \underline{X})$ *(as in the previous section) and* $f^{[]} := ((p, f\circ q),(\underline{p}, f), id_{\underline{Y}}, id_{\underline{X}}) : Y \to [\underline{Y}\times\mathbf{R}]$*. Then* $\Theta^{1F}_Y(v)$ *is an algebra homomorphism, i.e.*

$$\Theta^{1F}_Y(v) \in Hom(C^\infty_{q(y)}(X), A^F_{p(y)}\underline{Y}) = T^{A^F_{p(y)}\underline{Y}}_{q(y)}X\ .$$

Define also $\Theta^{2F}_Y : C^\infty_y(Y) \to B^F_y\underline{Y}$ *by*

$$\Theta^{2F}_Y(v)(f) = Ff^{<>}(v) \in F_{p(y)}(<\underline{Y}\times\mathbf{R}>) = B^F_{p(y)}\underline{Y}\ ,$$

where $f^{<>} := ((p,f),\underline{p}, id_{\underline{Y}}, id_{\underline{X}}) : Y \to <\underline{Y}\times\mathbf{R}>$*. Then* $\Theta^{2F}_Y(v)$ *is also an algebra homomorphism, i.e.*

$$\Theta^{2F}_Y(v) \in Hom(C^\infty_y(Y), B^F_{p(y)}\underline{Y}) = T^{B^F_{p(y)}\underline{Y}}_yY\ .$$

Since F *is a functor,* $\mu^F_{p(y)}(\Theta^{1F}_Y(v)(f)) = \Theta^{2F}_Y(v)(f\circ q)$ *for* $f : X \to \mathbf{R}$ *(we apply functor* F *to the equality* $id_{\underline{Y}\times\mathbf{R}}\circ(p, f\circ q) = (p, f\circ q)$ *with respective* $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$*-morphisms and next evaluate at* v*). Thus,* $(\mu^F_{p(y)})_X(\Theta^{1F}_Y(v)) = T^{B^F_{p(y)}\underline{Y}}q(\Theta^{2F}_Y(v))$*, i.e.*

$$\Theta^F_Y(v) := (\Theta^{1F}_Y(v), \Theta^{2F}_Y(v)) \in T^{\mu^F_{p(y)}}_yY = T^{\mu^F}_yY\ .$$

Similarly, one can verify that $T^{\mu^F}p(\Theta^F_Y(v)) = \sigma^F(p(y))$*, where* $p : Y \to \underline{Y}$ *is treated as the* $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$*-map as in Example 3, i.e.*

$$\Theta^F_Y(v) \in V^{\mu^F,\sigma^F}_yY\ .$$

Thus, we have defined the natural transformation $\Theta^F : F \to V^{\mu^F,\sigma^F}$*.*

Proposition 1 *We have the* $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$*-natural isomorphism* $\Theta^F : F \to V^{\mu^F,\sigma^F}$ *(canonical in* F*).*

Proof Since F and V^{μ^F,σ^F} are fiber product preserving, it suffices to show that

$$(\Theta^F_{[\underline{Y}\times\mathbf{R}]})_{(\underline{y},0)} : F_{(\underline{y},0)}([\underline{Y}\times\mathbf{R}]) \to V^{\mu^F,\sigma^F}_{(\underline{y},0)}([\underline{Y}\times\mathbf{R}])$$

and

$$(\Theta^F_{<\underline{Y}\times\mathbf{R}>})_{(\underline{y},0)} : F_{(\underline{y},0)}(<\underline{Y}\times\mathbf{R}>) \to V^{\mu^F,\sigma^F}_{(\underline{y},0)}(<\underline{Y}\times\mathbf{R}>)$$

are diffeomorphisms for any $\mathcal{FM}_{m_1,m_2}$-object $\underline{Y}=(\underline{q}:\underline{Y}\to\underline{X})$ and any $\underline{y}\in\underline{Y}$. One can easily see that the inverse diffeomorphisms are the restrictions of

$$(Hom(C^\infty_{(\underline{y},0)}(\underline{Y}\times\mathbf{R}), A^F_{\underline{y}}\underline{Y}), Hom(C^\infty_{(\underline{y},0)}(\underline{Y}\times\mathbf{R}), B^F_{\underline{y}}\underline{Y})) \to A^F_{\underline{y}}\underline{Y}$$

and

$$(Hom(C^\infty_{\underline{y}}(\underline{Y}), A^F_{\underline{y}}\underline{Y}), Hom(C^\infty_{(\underline{y},0)}(\underline{Y}\times\mathbf{R}), B^F_{\underline{y}}\underline{Y})) \to B^F_{\underline{y}}\underline{Y}$$

given by $(w^1,w^2)\to w^1(pr_2)$ and $(u^1,u^2)\to u^2(pr_2)$, where $pr_2:\underline{Y}\times\mathbf{R}\to\mathbf{R}$ is the projection onto the second factor. □

7. An equivalence $V^{\mu,\sigma}\tilde{=}V^{\mu^{V^{\mu,\sigma}},\sigma^{V^{\mu,\sigma}}}$

Let $\mu:A\to B$ be a natural transformation between Weil algebra bundle functors on $\mathcal{FM}_{m_1,m_2}$ and $\sigma:Y\to T^\mu Y$ be a canonical section for any $\mathcal{FM}_{m_1,m_2}$-object $Y=(q:Y\to X)$.

Example 7 *Let $Y=(q:Y\to X)$ be an $\mathcal{FM}_{m_1,m_2}$-object.*

Define $\Phi^{1(\mu,\sigma)}_Y : AY \to V^{\mu,\sigma}([Y\times\mathbf{R}]) = A^{V^{\mu,\sigma}}Y$ as follows. Let $y\in Y$. Identifying $\mathbf{R}$ with the $\mathcal{FM}$-object $id_{\mathbf{R}}:\mathbf{R}\to\mathbf{R}$ we have the "product preserving" $\mathcal{FM}$-identification $T^{\mu_y}(Y\times\mathbf{R})=T^{\mu_y}Y\times T^{\mu_y}\mathbf{R}$. Under this identification, $V^{\mu,\sigma}_y([Y\times\mathbf{R}])=\{\sigma(y)\}\times T^{\mu_y}\mathbf{R}$. On the other hand, $T^{\mu_y}\mathbf{R}=\{(b,(\mu_y)_{\mathbf{R}}(b))\in T^{A_yY}\mathbf{R}\times T^{B_yY}\mathbf{R}\mid b\in T^{A_yY}\mathbf{R}\}$ and $T^{A_yY}\mathbf{R}=A_yY$ modulo the usual identification (from the theory of Weil functors). Thus, $T^{\mu_y}\mathbf{R}=A_yY$ modulo the clear identification. Then $V^{\mu,\sigma}_yY=\{\sigma(y)\}\times A_yY$ modulo the identification, and we define

$$(\Phi^{1(\mu,\sigma)}_Y)_y := (\sigma(y), id_{A_yY})$$

(modulo the identification). Clearly, it is an algebra isomorphism. Thus, we have defined isomorphism $\Phi^{1(\mu,\sigma)}_Y : AY\to A^{V^{\mu,\sigma}}Y$ of Weil algebra bundles, and we have the natural isomorphism $\Phi^{1(\mu,\sigma)}:A\to A^{V^{\mu,\sigma}}$ of Weil algebra bundle functors on $\mathcal{FM}_{m_1,m_2}$.

Define $\Phi^{2(\mu,\sigma)}_Y : BY\to V^{\mu,\sigma}(<Y\times\mathbf{R}>)=B^{V^{\mu,\sigma}}Y$ as follows. Let $y\in Y$. We have the "product preserving" $\mathcal{FM}$-identification $T^{\mu_y}(Y\times\mathbf{R})=T^{\mu_y}Y\times T^{\mu_y}\mathbf{R}$, where $\mathbf{R}=(pt_{\mathbf{R}}:\mathbf{R}\to pt)$, where pt is the one point manifold. Under this identification, $V^{\mu,\sigma}_y(<Y\times\mathbf{R}>)=\{\sigma(y)\}\times T^{\mu_y}\mathbf{R}$. On the other hand, $T^{\mu_y}\mathbf{R}=pt\times T^{B_yY}\mathbf{R}=T^{B_yY}\mathbf{R}=B_yY$ modulo the clear identifications. Then $B^{V^{\mu,\sigma}}_yY:=\{\sigma(y)\}\times B_yY$ modulo the identifications, and we define

$$(\Phi^{2(\mu,\sigma)}_Y)_y := (\sigma(y), id_{B_yY})$$

modulo the identifications. It is an algebra isomorphism. Thus, we have defined isomorphism $\Phi^{2(\mu,\sigma)} : B \to B^{V^{\mu,\sigma}}$ *of Weil algebra bundle functors on* $\mathcal{FM}_{m_1,m_2}$.

One can standardly verify that

$$\mu_Y^{V^{\mu,\sigma}} \circ \Phi_Y^{1(\mu,\sigma)} = \Phi_Y^{2(\mu,\sigma)} \circ \mu_Y \ ,$$

i.e. that $(\Phi^{1(\mu,\sigma)}, \Phi^{(2(\mu,\sigma)}) : \mu \to \mu^{V^{\mu,\sigma}}$ *is an isomorphism of Weil algebra bundle functors on* $\mathcal{FM}_{m_1,m_2}$. *One can also standardly verify that*

$$(\Phi^{1(\mu,\sigma)}, \Phi^{2(\mu,\sigma)})_Y \circ \sigma = \sigma^{V^{\mu,\sigma}} \ ,$$

where $(\Phi^{1(\mu,\sigma)}, \Phi^{2(\mu,\sigma)})_Y : T^{\mu}Y \to T^{\mu^{V^{\mu,\sigma}}}Y$ *is the natural transformation induced by morphism* $(\Phi^{1(\mu,\sigma)}, \Phi^{2(\mu,\sigma)})$.

Let

$$\mathcal{T}^{(\mu,\sigma)} := [\Phi^{1(\mu,\sigma)}, \Phi^{2(\mu,\sigma)}] : V^{\mu,\sigma} \to V^{\mu^{V^{\mu,\sigma}}, \sigma^{V^{\mu,\sigma}}}$$

be the induced natural transformation.

Thus, we have proved.

Proposition 2 *There is the* $\mathcal{FM}_{m_1,m_2}$*-natural isomorphism* $\mathcal{T}^{(\mu,\sigma)} : V^{\mu,\sigma} \to V^{\mu^{V^{\mu,\sigma}}, \sigma^{V^{\mu,\sigma}}}$ *(canonical in* (μ,σ)*).*

8. The main result

Let FPP_{m_1,m_2} be the category of fiber product preserving bundle functors F on $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ and their natural transformations $\eta : F \to F'$. Let MVW_{m_1,m_2} be the category of modified vertical Weil functors on $\mathcal{F}_2\mathcal{M}_{m_1,m_2}$ of the form $V^{\mu,\sigma}$ and their natural transformations of the form $[\varphi,\psi]$ (see Example 3).

Theorem 1 *We have the equivalence*

$$FPP_{m_1,m_2} \tilde{=} MVW_{m_1,m_2}$$

of categories. It means that there are functors $J : FPP_{m_1,m_2} \to MVW_{m_1,m_2}$ *and* $I : MVW_{m_1,m_2} \to FPP_{m_1,m_2}$ *with* $J \circ I \tilde{=} Id_{MVW_{m_1,m_2}}$ *and* $I \circ J \tilde{=} Id_{FPP_{m_1,m_2}}$.

Proof We define J by $J(F) := V^{\mu^F,\sigma^F}$ and $J(\eta) := [\varphi^\eta, \psi^\eta]$; see Examples 4 and 5 (for (μ^F, σ^F) and $(\varphi^\eta, \psi^\eta)$) and then Example 3 (for the modified vertical Weil functors and their natural transformations). We define I to be the usual forgetting functor. The isomorphism $I \circ J \tilde{=} \mathrm{Id}$ is by Proposition 1. The isomorphism $J \circ I \tilde{=} \mathrm{Id}$ is by Proposition 2. □

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