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On pseudohyperbolic space motions

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Abstract: In the present paper, the geometrical instantaneous invariants of the motion H_m/H_f in dual Lorentzian 3-space are determined. Depending on this, the dual Lorentzian instantaneous screw axis of the motion of K_m with respect to the dual pseudohyperbolic space K_m is constructed. On the other hand, we show that, in each position of H_m, the fixed and moving axodes have the instantaneous screw axis of this position in common. We also give relations between the geodetic curvature and the curvature of the polodes.

Key words: Pseudohyperbolic space, instantaneous screw axis, Lorentzian 3-space, axodes

1. Introduction

This paper deals with the motion of a rigid body relative to a reference system in dual Lorentzian 3-space $\mathbb{D}_1^3$. It is well known that the aggregations of instantaneous screw axes (ISAs) of all instants form a pair of ruled surfaces with the ISA as their straight line generatrix in the stationary reference space and in the moving body, respectively. These surfaces are named the fixed and moving axodes [4, 5, 6].

There are various recent works in the literature dealing with ISA and the invariants of the axodes [4, 5, 9, 12, 13, 14]. In particular, Chen [5] derived (by a special approach) the formulas for computing surface normal and surface curvatures of axodes in Euclidean 3-space. A new geometric and kinematic approach to one parameter spatial motion for the calculation of instantaneous invariants based on information specifying the motion of axodes in dual 3-space $\mathbb{D}^3$ was provided by Abdel-Baky and Al-Solamy [1].

The rolling space problem was extended to rolling pseudo-Riemannian manifolds by Jurdjevic and Zimmerman [7]. In that work, they extended this problem to situations in which an oriented sphere S_ρ^n of radius ρ rolls on stationary sphere S_σ^n of radius σ as well as to its hyperbolic analogue.

Korolko and Leite [8] obtained the equations of motion for the n-dimensional Lorentzian sphere rolling, without slipping and twisting, over the affine tangent space at a point. Along the same lines, Marques and Leite deduced the kinematic equations for rolling, without slipping or twisting, a pseudohyperbolic space over its affine tangent space at a point in an earlier work [10] .

So far, there seems to be no discussion about the geometrical instantaneous invariants of the dual unit pseudohyperbolic space motion. In this current work, we derive the geometrical instantaneous invariants of

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the dual unit pseudohyperbolic space motion in dual Lorentzian 3-space $\mathbb{D}_1^3$ by using a dual semiorthogonal matrix. Moreover, we show that the moving Π_m and the fixed Π_f axodes maintain the contact with each other along the ISA. It is also proved that the real part $\overrightarrow{U}$ and the dual part $\overrightarrow{U}_0$ of a dual angular velocity vector $\overrightarrow{\hat{U}}$ correspond to the rolling and sliding motions of the instantaneous helicoidal motion around the ISA at the instant t respectively. Furthermore, we verify that the pair of axodes during the one parameter pseudohyperbolic space contact each other along the ISA. Finally, we specify the relations between the geodetic curvatures and the curvatures of the polodes.

2. Setting and notations

Let $\mathbb{D}_1^3$ be the dual Lorentzian space with the inner product

$$<\overrightarrow{\hat{X}},\overrightarrow{\hat{Y}}>=<\overrightarrow{X},\overrightarrow{Y}>+\varepsilon(<\overrightarrow{X},\overrightarrow{Y}_0>+<\overrightarrow{X}_0,\overrightarrow{Y}>),$$

for which the inner product of the vectors $\overrightarrow{X}$ and $\overrightarrow{Y}$ is defined to be

$$<\overrightarrow{X},\overrightarrow{Y}>=-x_1y_1+x_2y_2+x_3y_3,$$

together with the vector product

$$\overrightarrow{\hat{X}}\times\overrightarrow{\hat{Y}}=\overrightarrow{X}\times\overrightarrow{Y}+\varepsilon(\overrightarrow{X}\times\overrightarrow{Y}_0+\overrightarrow{X}_0\times\overrightarrow{Y}),$$

in which the vector product of the vectors $\overrightarrow{X}=(x_1,x_2,x_3)$ and $\overrightarrow{Y}=(y_1,y_2,y_3)$ is given by

$$\overrightarrow{X}\times\overrightarrow{Y}=\left(-\begin{vmatrix} x_2 & x_3\\ y_2 & y_3\end{vmatrix},\begin{vmatrix} x_3 & x_1\\ y_3 & y_1\end{vmatrix},\begin{vmatrix} x_1 & x_2\\ y_1 & y_2\end{vmatrix}\right).$$

A dual vector $\overrightarrow{\hat{X}}$ of $\mathbb{D}_1^3$ is said to be spacelike if $<\overrightarrow{\hat{X}},\overrightarrow{\hat{X}}>>0$ or $\overrightarrow{\hat{X}}=0$, timelike if $<\overrightarrow{\hat{X}},\overrightarrow{\hat{X}}><0$, and lightlike or null if $<\overrightarrow{\hat{X}},\overrightarrow{\hat{X}}>=0$ and $\overrightarrow{\hat{X}}\neq 0$ [2, 3, 11, 15, 17]. The norm of a dual vector $\overrightarrow{\hat{X}}$ in $\mathbb{D}_1^3$ is defined to be $\left\|\overrightarrow{\hat{X}}\right\|=\sqrt{\left|<\overrightarrow{\hat{X}},\overrightarrow{\hat{X}}>\right|}$. On the other hand, a 3×3 matrix $\hat{A}$ is called a dual semiorthogonal matrix if it satisfies the equation $\hat{A}^t=\hat{S}\hat{A}^{-1}\hat{S}$, where the matrix $\hat{S}$ is the signature matrix given as [11]:

$$\begin{bmatrix} -1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

The space given by

$$\hat{H}_0^2=\{\overrightarrow{\hat{X}}=\overrightarrow{X}+\varepsilon\overrightarrow{X}_0\epsilon\mathbb{D}_1^3\left|<\overrightarrow{\hat{X}},\overrightarrow{\hat{X}}>=-1 \text{ and } \overrightarrow{X},\overrightarrow{X}_0\epsilon\mathbb{R}_1^3\right.\}$$

is the dual unit pseudohyperbolic space. To ease the notation, we abbreviate it to K, and in a similar vein, the Lorentzian 3-space $\mathbb{R}_1^3$ is shortly denoted by H.

Throughout our work, the fixed and the moving Lorentzian 3-spaces are denoted by H_f and H_m respectively. We consider the Cartesian frames of references $\{O_f; E_i\}$ and $\{O_m; R_i\}$ in H_f and H_m having the same orientation under the semiorthogonal matrix transformations. Moreover, if O is the common center, then these frames of references correspond to the dual Cartesian frames of references $\hat{E} = \{O_f; \overrightarrow{\hat{E}}_i = \overrightarrow{E}_i + \varepsilon \overrightarrow{E}_{oi}\}$ and $\hat{R} = \{O_m; \overrightarrow{\hat{R}}_i = \overrightarrow{R}_i + \varepsilon \overrightarrow{R}_{oi}\}$, for which $\overrightarrow{E}_{0i} = \overrightarrow{OO}_f \times \overrightarrow{E}_i$ and $\overrightarrow{R}_{0i} = \overrightarrow{OO}_m \times \overrightarrow{R}_i$. Therefore, two orthonormal dual frames are linked rigidly to the dual unit fixed and moving pseudohyperbolic spaces K_f and K_m in $\mathbb{D}^3_1$ respectively. The coordinates of a point $\hat{X}$ on K_m with respect to the basis $\hat{R}$ will be denoted by $\overrightarrow{\hat{X}}_i = \overrightarrow{X}_i + \varepsilon \overrightarrow{X}_{oi}$ $(i = 1, 2, 3)$. This point coincides with a point of K_f having the coordinates $\overrightarrow{{}_f\hat{X}}_i = \overrightarrow{{}_fX}_i + \varepsilon \overrightarrow{{}_fX}_{oi}$ with respect to $\hat{E}$. The matrix forms of the points ${}_f\hat{X}$ and $\hat{X}$ and the base vectors $\hat{E}$ and $\hat{R}$ can be given as follows:

$$
{}_f\hat{X} = \begin{bmatrix} \overrightarrow{{}_f\hat{X}}_1 \\ \overrightarrow{{}_f\hat{X}}_2 \\ \overrightarrow{{}_f\hat{X}}_3 \end{bmatrix}, \ \hat{X} = \begin{bmatrix} \overrightarrow{\hat{X}}_1 \\ \overrightarrow{\hat{X}}_2 \\ \overrightarrow{\hat{X}}_3 \end{bmatrix}, \ \hat{E} = \begin{bmatrix} \overrightarrow{\hat{E}}_1 \\ \overrightarrow{\hat{E}}_2 \\ \overrightarrow{\hat{E}}_3 \end{bmatrix} \text{ and } \hat{R} = \begin{bmatrix} \overrightarrow{\hat{R}}_1 \\ \overrightarrow{\hat{R}}_2 \\ \overrightarrow{\hat{R}}_3 \end{bmatrix}.
$$

We can therefore write the position vector of a point $\hat{X}$ on the dual unit pseudohyperbolic space K_m in terms of $\hat{R}$ as

$$
\overrightarrow{\hat{X}} = \hat{X}^t \hat{R},
$$

which in turn implies

$$
< \overrightarrow{\hat{X}}, \overrightarrow{\hat{X}} >= -\hat{X}_1^2 + \hat{X}_2^2 + \hat{X}_3^2 = -1.
$$

If the real and the dual parts of this equation are separated, we have the following equations:

$$
\begin{aligned}
-x_1^2 + x_2^2 + x_3^2 &= -1, \\
-x_1 x_{01} + x_2 x_{02} + x_3 x_{03} &= 0.
\end{aligned}
$$

Observe that in the last equations, there are six unknown terms, which are called the Pluckerian coordinates of the oriented line $\overrightarrow{\hat{X}}$ in H_m.

If the dual unit pseudohyperbolic space K_m moves with respect to the fixed dual unit pseudohyperbolic space K_f, we call such a motion the dual pseudohyperbolic space, denoted by K_m/K_f. It is obvious that if the dual unit pseudohyperbolic spaces K_m and K_f correspond to the line spaces H_m and H_f respectively, the motion K_m/K_f corresponds to H_m/H_f. In such a case, H_m is called the moving space with respect to the fixed space H_f. Therefore, we may describe the motion K_m/K_f by

$$
{}_f\hat{X} = \hat{X}^t \hat{A} \hat{E}, \tag{2.1}
$$

where

$$
\hat{A} = \begin{bmatrix} \alpha_1 + \varepsilon\alpha_{01} & \alpha_2 + \varepsilon\alpha_{02} & \alpha_3 + \varepsilon\alpha_{03} \\ \beta_1 + \varepsilon\beta_{01} & \beta_2 + \varepsilon\beta_{02} & \beta_3 + \varepsilon\beta_{03} \\ \gamma_1 + \varepsilon\gamma_1 & \gamma_2 + \varepsilon\gamma_2 & \gamma_3 + \varepsilon\gamma_3 \end{bmatrix} = A + \varepsilon A_0
$$

and the equality

$$\hat{R} = \hat{A}\hat{E}$$

holds. Note that the elements of dual semiorthogonal matrix $\hat{A}$ are real functions of t, and these functions are assumed to be differentiable of any order with respect to t. These all together yield the following:

Theorem 1 *The Lorentzian motions in $\mathbb{R}^3_1$ are represented by the dual semiorthogonal matrix $\hat{A}$ in the dual Lorentzian space $\mathbb{D}^3_1$, where $\hat{A}\hat{S}\hat{A}^t\hat{S} = \hat{I}$ and $\hat{S}$ is the 3×3 signature matrix.*

3. The dual instantaneous screw axis of the motion K_m/K_f

In this section we construct the dual instantaneous screw axis of the motion K_m/K_f. Since the dual Lorentzian matrix $\hat{A}$ is semiorthogonal, the identity

$$\hat{A}\hat{S}\hat{A}^t\hat{S} = \hat{I}$$

holds. We may therefore conclude by differentiating the above equation with respect to the motion parameter t that

$$\dot{\hat{A}}\hat{A}^{-1} + \hat{S}(\dot{\hat{A}}\hat{A}^{-1})^t\hat{S} = 0.$$

Furthermore, if we define $\hat{\Omega} = \dot{\hat{A}}\hat{A}^{-1}$, then the matrix $\hat{\Omega}$ is a skew-symmetric dual Lorentzian matrix ($\hat{\Omega}^t = -\hat{S}\hat{\Omega}\hat{S}$) so that the matrix $\hat{\Omega}$ can be written in the following form:

$$\hat{\Omega} = W + \varepsilon W_0 = \begin{bmatrix} 0 & \hat{\Omega}_3 & -\hat{\Omega}_2 \\ \hat{\Omega}_3 & 0 & \hat{\Omega}_1 \\ -\hat{\Omega}_2 & -\hat{\Omega}_1 & 0 \end{bmatrix},$$

where

$$W = \dot{A}A^{-1}, \; W_0 = \dot{A}_0 A^{-1} + \dot{A}A_0^{-1}$$

and

$$\hat{\Omega}_i = w_i + \varepsilon w_{0i}, \quad 1 \le i \le 3.$$

By differentiating equation (2.1) and considering the equalities $\hat{E} = \hat{S}\hat{A}^t\hat{S}\hat{R}$ and $\hat{\Omega} = \dot{\hat{A}}\hat{A}^{-1}$, we obtain the velocity of a point $\hat{X}$ on K_m during the motion K_m/K_f as

$$_f\dot{\hat{X}} = \dot{\hat{X}}^t\hat{R} + \hat{X}^t\hat{\Omega}\hat{R}. \tag{3.2}$$

Here $\dot{\hat{X}}^t\hat{R}$ is the relative velocity of a point $\hat{X}$ on K_m, and $\hat{X}^t\hat{\Omega}\hat{R}$ is the velocity of the motion K_m/K_f. Henceforth, we take $\hat{X}$ as the fixed point on K_m by which equation (3.2) can be written as

$$_f\dot{\hat{X}} = \hat{X}^t\hat{\Omega}\hat{R}. \tag{3.3}$$

Now we look for points having zero velocity at any instant of the moving dual pseudohyperbolic space K_m. For this purpose, we define the following dual vector with respect to the basis $\hat{R}$:

$$\vec{\hat{U}} = \sum_{i=1}^{3}[w_i r_i + \varepsilon(w_i r_{0i} + w_{0i} r_i)] = \vec{U} + \varepsilon\vec{U}_0.$$

Then equation (3.3) can be rewritten as

$$\overrightarrow{{}_f\dot{\hat{X}}} = \overrightarrow{\hat{U}} \times \overrightarrow{\hat{X}}, \tag{3.4}$$

and hence the module of $\overrightarrow{\hat{U}}$ can be given by

$$\left\| \overrightarrow{\hat{U}} \right\| = w + \varepsilon w_0 = \hat{W},$$

where

$$w = \left\| \overrightarrow{U} \right\| \text{ and } w_0 = w^{-2}(< \overrightarrow{U}, \overrightarrow{U}_0 >).$$

Therefore, we obtain the dual unit vector of $\overrightarrow{\hat{U}}$ as

$$\begin{aligned} \overrightarrow{\hat{U}}_1 &= \frac{\overrightarrow{\hat{U}}}{\left\|\overrightarrow{\hat{U}}\right\|} \\ &= w^{-1}\overrightarrow{\hat{U}} + \varepsilon w^{-2}(w\overrightarrow{\hat{U}}_0 - w_0 \overrightarrow{\hat{U}}) \\ &= \overrightarrow{U}_1 + \varepsilon \overrightarrow{U}_{01}. \end{aligned}$$

Now equation (3.4) can be brought to the following form:

$$\overrightarrow{{}_f\dot{\hat{X}}} = \hat{W}(\overrightarrow{\hat{U}}_1 \times \overrightarrow{\hat{X}}).$$

As a consequence of this equation, it is easily seen that $\overrightarrow{{}_f\dot{\hat{X}}} = 0$ if and only if $\overrightarrow{\hat{U}}_1 = \overrightarrow{\hat{X}}$.

By reconsidering equation (3.3), we conclude that a point $\hat{X}$ on K_m has zero velocity at instant t if and only if

$$\begin{aligned} \hat{X}^t\hat{\Omega} &= 0, \\ -\hat{X}_1^2 + \hat{X}_2^2 + \hat{X}_3^2 &= -1 \end{aligned} \tag{3.5}$$

under the dual unit pseudohyperbolic space motion (2.1), since $< \overrightarrow{\hat{X}}, \overrightarrow{\hat{X}} >= -1$. Note that the first equation in (3.5) is equivalent to the partial differential equations $\dfrac{\partial}{\partial x_i} < \overrightarrow{{}_f\dot{\hat{X}}}, \overrightarrow{{}_f\dot{\hat{X}}} >= 0$ $(i \in \{1,2,3\})$. When we separate equation (3.5) into the real and the dual parts, we can obtain the following system of equations in six unknown terms with coefficients w_i and w_{0i}.

$$\begin{aligned} w_3x_2 - w_2x_3 &= 0, \\ w_3x_1 - w_1x_3 &= 0, \\ -x_1^2 + x_2^2 + x_3^2 &= -1, \end{aligned} \quad \text{and} \quad \begin{aligned} w_3x_{02} - w_2x_{03} &= w_{02}x_3 - w_{03}x_2, \\ w_3x_{01} - w_1x_{03} &= w_{01}x_3 - w_{03}x_2, \\ -x_1x_{01} + x_2x_{02} + x_3x_{03} &= 0. \end{aligned}$$

The solutions of this system of equations are

$$x_i = \pm w^{-1}w_i \ (i \in \{1,2,3\})$$

and

$$\begin{aligned} x_{01} &= w^{-3}[w_{01}(w_2^2+w_3^2)-w_1(w_{02}w_2+w_{03}w_3)], \\ x_{02} &= w^{-3}[w_{02}(-w_1^2+w_3^2)+w_2(w_{01}w_1-w_{03}w_3)], \\ x_{03} &= w^{-3}[w_{03}(-w_1^2+w_2^2)+w_3(w_{01}w_1-w_{02}w_2)]. \end{aligned}$$

These results indicate that the unit timelike dual vector $\vec{\hat{X}}$ is nothing but $\vec{\hat{U}}_1$; that is, they are consistent with Veldkamp's results [16] on ISAs, since the timelike dual unit vector $\vec{\hat{U}}_1$ determines the oriented lines in Lorentzian 3-spaces H_m and H_f. Therefore, we call the timelike dual unit vector

$$\hat{U}_1 = \hat{U}_1^t \hat{R} \tag{3.6}$$

the ISA of the position, while the timelike dual unit vector

$${}_f\hat{U}_1 = \hat{U}_1^t \hat{A}\hat{E} \tag{3.7}$$

is called the ISA of the position on K_f. On the other hand, the vector $\vec{\hat{U}}$ and its dual module $\hat{W}$ are called the dual angular velocity and speed of the motion H_m/H_f.

As a result of these, the following corollary can be stated.

Corollary 2 *The real part $\vec{U}$ and the dual part $\vec{U}_0$ of $\vec{\hat{U}}$ correspond to the rolling and sliding motions of the instantaneous helicoidal motion around the ISA at the instant t. The instantaneous screw pitch of this motion at instant t is given by*

$$k = \frac{\hat{w}_0}{\hat{w}}.$$

4. The polodes and dual angular velocity along ISA

We observe that $\vec{\hat{U}}_1$ is the function of t during the pseudohyperbolic space motions. The timelike dual unit vector $\vec{\hat{U}}_1$ represents the locus of the ISA on K_m, and this locus is a dual curve π_m on K_m known as the moving polode. Note that this curve corresponds to a timelike ruled surface Π_m in Lorentzian 3-space H_m, the moving axode. The locus of the ISA on K_f is also a dual curve π_f, the fixed polode. This polode likewise corresponds to a ruled surface Π_f in H_f and it is called the fixed axode. By differentiating the equations (3.6) and (3.7) with respect to t, we get that

$$\dot{\hat{U}}_1 = \dot{\hat{U}}_1^t \hat{R} + \hat{U}_1^t \hat{\Omega}\hat{R}$$

and

$${}_f\dot{\hat{U}}_1 = \dot{\hat{U}}_1^t \hat{R} + \hat{U}_1^t \hat{\Omega}\hat{R}.$$

As a result of this fact, the equality $\dot{\hat{U}}_1 = {}_f\dot{\hat{U}}_1$ holds; that is, the polodes roll without slipping on each other.

Thus, in analogy with the dual spherical motion, we can state the following theorem.

Theorem 3 *The pair of axodes during the one-parameter pseudohyperbolic space K_m/K_f contact each other along the ISA.*

At the instant t, the ISA is $\vec{\hat{U}}_1$, while when the instant is $t+\Delta t$, we can represent the timelike ISA by

$$\vec{\hat{U}}_1(t+\Delta t)=\vec{\hat{U}}_1+\Delta\vec{\hat{U}}_1.$$

Suppose now that Π_m is a not a cylinder. Then we may take Δt such that $\vec{\hat{U}}_1$ and $\vec{\hat{U}}_1(t+\Delta t)$ are not parallel; hence, $\Delta\vec{\hat{U}}_1$ is not a pure dual vector. As a consequence, the common normal of $\vec{\hat{U}}_1$ and $\vec{\hat{U}}_1(t+\Delta t)$ is well defined.

We next define the ISA dual angular velocity as the vector

$$\vec{\hat{\Psi}}=\vec{\hat{U}}_1(t)\times\dot{\vec{\hat{U}}}_1(t),$$

which can be rewritten as

$$\dot{\vec{\hat{U}}}_1=\vec{\hat{\Psi}}\times\vec{\hat{U}}_1.$$

Once we assume that

$$\hat{P}=\left\|\dot{\vec{\hat{U}}}_1\right\|=p+\varepsilon p_0,$$

we define the following spacelike dual unit vectors

$$\vec{\hat{U}}_2=\frac{\dot{\vec{\hat{U}}}_1}{\hat{P}}=\vec{U}_2+\varepsilon\vec{U}_{02} \text{ and } \vec{\hat{U}}_3=\vec{\hat{U}}_1\times\vec{\hat{U}}_2=\vec{U}_3+\varepsilon\vec{U}_{03},$$

from which we conclude that

$$\vec{\hat{\Psi}}=\hat{P}\vec{\hat{U}}_3=p\vec{U}_3+\varepsilon(p_0\vec{U}_3+p\vec{U}_{03}). \tag{4.8}$$

We remark that the spacelike dual unit vector $\vec{\hat{U}}_3$ is the axis of $\vec{\hat{\Psi}}$. Moreover, the dual unit vectors $\vec{\hat{U}}_1$, $\vec{\hat{U}}_2$, and $\vec{\hat{U}}_3$ represent mutually orthogonal spears in H_m and H_f. The intersection point of these spears is the point of striction $S(t)$ on the ruling $\vec{\hat{U}}_1(t)$ of Π_m and Π_f. On the other hand, the location of points of striction $S(t)$ are the curves of striction on Π_m and Π_f so that the position vector of $S(t)$ can be given by

$$\vec{S}(t)=\vec{U}_1\times\vec{U}_{01}+<\vec{U}_3,\vec{U}_{02}>\vec{U}_1.$$

This in turn implies that the spears $\vec{\hat{U}}_1$, $\vec{\hat{U}}_2$, and $\vec{\hat{U}}_3$ are referred to as the ISA trihedron at the point of striction. Note also that at the same time, this is a trihedron of polodes at the pole.

The dual number $\hat{P}$ in equation (4.8) is the ISA angular speed from which the vector $\vec{\hat{\Psi}}$ may be expressed with respect to basis $\hat{E}$ as

$$\hat{\Psi}=\hat{P}\hat{U}_3^t\hat{A}\hat{E}.$$

For the sake of simplicity, let us denote the coordinates of the ISA in the basis $\hat{R}$ as $(\hat{a}_1, \hat{a}_2, \hat{a}_3)$. Then the coordinate representation of $\vec{\hat{U}}_2$ is given by

$$\vec{\hat{U}}_2 = \frac{\dot{\vec{\hat{U}}}_1}{\left\|\dot{\vec{\hat{U}}}_1\right\|} = \left(\frac{\dot{\hat{a}}_1}{\hat{P}}, \frac{\dot{\hat{a}}_2}{\hat{P}}, \frac{\dot{\hat{a}}_3}{\hat{P}}\right),$$

while that of the spacelike dual unit vector $\vec{\hat{U}}_3$ is

$$\vec{\hat{U}}_3 = \vec{\hat{U}}_1 \times \vec{\hat{U}}_2 = \left(\frac{\hat{\Delta}_1}{\hat{P}}, \frac{\hat{\Delta}_2}{\hat{P}}, \frac{\hat{\Delta}_3}{\hat{P}}\right),$$

where

$$\hat{\Delta}_1 = \hat{a}_2\dot{\hat{a}}_3 - \hat{a}_3\dot{\hat{a}}_2,\ \hat{\Delta}_2 = \hat{a}_1\dot{\hat{a}}_3 - \hat{a}_3\dot{\hat{a}}_1 \text{ and } \hat{\Delta}_3 = \hat{a}_2\dot{\hat{a}}_1 - \hat{a}_1\dot{\hat{a}}_2.$$

We therefore conclude that

$$\begin{aligned}\begin{bmatrix} \vec{\hat{U}}_1 \\ \vec{\hat{U}}_2 \\ \vec{\hat{U}}_3 \end{bmatrix} &= \begin{bmatrix} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 \\ \frac{\dot{\hat{a}}_1}{\hat{P}} & \frac{\dot{\hat{a}}_2}{\hat{P}} & \frac{\dot{\hat{a}}_3}{\hat{P}} \\ \frac{\hat{\Delta}_1}{\hat{P}} & \frac{\hat{\Delta}_2}{\hat{P}} & \frac{\hat{\Delta}_3}{\hat{P}} \end{bmatrix} \begin{bmatrix} \vec{\hat{R}}_1 \\ \vec{\hat{R}}_2 \\ \vec{\hat{R}}_3 \end{bmatrix} \\ &= \hat{M}\hat{R},\end{aligned}$$

in which $\hat{M}$ denotes the dual semiorthogonal matrix.

Overall we are ready to state the following theorem.

Theorem 4 *The displacements of the ISA trihedron along the moving and the fixed polodes are*

$$\begin{bmatrix} \dot{\vec{\hat{U}}}_1 \\ \dot{\vec{\hat{U}}}_2 \\ \dot{\vec{\hat{U}}}_3 \end{bmatrix} = \begin{bmatrix} 0 & \hat{P} & 0 \\ \hat{P} & 0 & \hat{Q} \\ 0 & -\hat{Q} & 0 \end{bmatrix} \begin{bmatrix} \vec{\hat{U}}_1 \\ \vec{\hat{U}}_2 \\ \vec{\hat{U}}_3 \end{bmatrix} \tag{4.9}$$

and

$$\frac{d_f}{dt}\begin{bmatrix} \vec{\hat{U}}_1 \\ \vec{\hat{U}}_2 \\ \vec{\hat{U}}_3 \end{bmatrix} = \begin{bmatrix} 0 & \hat{P} & 0 \\ \hat{P} & 0 & -\hat{W}+\hat{Q} \\ 0 & \hat{W}-\hat{Q} & 0 \end{bmatrix} \begin{bmatrix} \vec{\hat{U}}_1 \\ \vec{\hat{U}}_2 \\ \vec{\hat{U}}_3 \end{bmatrix}, \tag{4.10}$$

respectively, where $\hat{P} = p + \varepsilon p_0$, $\hat{Q} = \frac{(\vec{\hat{U}}_1, \dot{\vec{\hat{U}}}_1, \ddot{\vec{\hat{U}}}_1)}{\hat{P}^2} = q + \epsilon q_0$ *and* $\hat{Q} - \hat{W}$ *are the invariants of the motion* K_m/K_f.

As a result of Theorem (4.2), we obtain the arc element of moving dual curve $\overrightarrow{\hat{U}}_3(t)$ that can be written as

$$\hat{Q} = \frac{(\overrightarrow{\dot{\hat{U}}}_1, \overrightarrow{\ddot{\hat{U}}}_1, \overrightarrow{\dddot{\hat{U}}}_1)}{\hat{P}^2} = q + \epsilon q_0,$$

and the arc element of the dual fixed curve $\overrightarrow{\hat{U}}_3(t)$ is $\hat{Q} - \hat{W}$ in which $\hat{P}$ is the arc element of the polodes. Thus, the integral $\int \hat{P}dt = \hat{S}_1$ is the dual arc length of the moving and the fixed polodes, whereas the integrals $\int \hat{Q}dt = \hat{S}_3$ and $\int(\hat{Q} - \hat{W})dt = \hat{S}_3 - \int \hat{W}dt$ are dual arc lengths of the curves $\overrightarrow{\hat{U}}_3(t)$ on dual unit pseudohyperbolic spaces K_f and K_m respectively.

5. Angular acceleration and geodetic curvature

This section is devoted to the investigation of the geodetic curvatures of the moving and the fixed polodes, and the existence of the relation between the geodetic curvature and the curvature of these polodes. For these purposes, we first construct the Frenet trihedrons of the moving and the fixed polodes at the pole.

By differentiating equation (4.8) with respect to t and considering equations (4.9) and (4.10), we obtain the angular accelerations of the pole with respect to K_m and K_f as

$$\overrightarrow{\dot{\hat{\Psi}}} = \dot{\hat{P}}\overrightarrow{\hat{U}}_3 - \hat{P}\hat{Q}\overrightarrow{\hat{U}}_2$$

and

$$_f\overrightarrow{\dot{\hat{\Psi}}} = \dot{\hat{P}}\overrightarrow{\hat{U}}_3 + \hat{P}(\hat{W} - \hat{Q})\overrightarrow{\hat{U}}_2.$$

If we denote by $\hat{\Theta}_m$ a dual angle between the spacelike vectors $\overrightarrow{\dot{\hat{\Psi}}}$ and $\overrightarrow{\hat{U}}_2$, we then have

$$\hat{Q} = \frac{\dot{\hat{P}}}{\hat{P}} \cot \hat{\Theta}_m. \tag{5.11}$$

In a similar vein, if $\hat{\Theta}_f$ is a dual angle between the spacelike vectors $_f\overrightarrow{\dot{\hat{\Psi}}}$ and $\overrightarrow{\hat{U}}_2$, we then obtain

$$\hat{W} - \hat{Q} = \frac{\dot{\hat{P}}}{\hat{P}} \cot \hat{\Theta}_f. \tag{5.12}$$

From equations (5.11) and (5.12), we conclude the equality:

$$\cot \hat{\Theta}_f + \cot \hat{\Theta}_m = \frac{\hat{P}\hat{W}}{\dot{\hat{P}}}.$$

As a next step, we now construct a trihedron on the moving polode at the pole in order to get the geodetic curvature of the moving polode. The spacelike binormal unit vector of the moving polode at the pole is

$$\overrightarrow{\hat{B}}_m = \frac{\overrightarrow{\dot{\hat{U}}}_1 \times \overrightarrow{\ddot{\hat{U}}}_1}{\left\| \overrightarrow{\dot{\hat{U}}}_1 \times \overrightarrow{\ddot{\hat{U}}}_1 \right\|} = \frac{\hat{P}\overrightarrow{\hat{U}}_3 + \hat{Q}\overrightarrow{\hat{U}}_1}{(\hat{P}^2 + \hat{Q}^2)^{1/2}}, \tag{5.13}$$

which is the axis of dual Darboux vector $\vec{\hat{B}} = \hat{Q}\vec{\hat{U}}_1 + \hat{P}\vec{\hat{U}}_3$.

Definition 5 *The point C_m on the moving dual pseudohyperbolic space K_m indicated by $\vec{\hat{B}}_m$ that coincides with a point C_f on the fixed dual pseudohyperbolic space K_f at a given instant t is called the dual spherical center of curvature π_m so that we call the spacelike dual unit vector $\vec{\hat{B}}_m$ the axis of curvature of Π_m.*

Since $\vec{\hat{U}}_2$ is the common tangent of the polodes, the timelike principle dual unit vector of the moving polode can be described as

$$\vec{\hat{N}}_m = \vec{\hat{B}}_m \times \vec{\hat{U}}_2 = \pm\frac{\hat{P}\vec{\hat{U}}_1 + \hat{Q}\vec{\hat{U}}_3}{(\hat{P}^2 - \hat{Q}^2)^{1/2}}. \tag{5.14}$$

Similarly, we obtain the following spacelike binormal and timelike principle dual unit vectors for the fixed polode at the pole:

$$\vec{\hat{B}}_f = \frac{\dot{\vec{\hat{U}}}_1 \times \ddot{\vec{\hat{U}}}_1}{\left\| \dot{\vec{\hat{U}}}_1 \times \ddot{\vec{\hat{U}}}_1 \right\|} = \frac{\hat{P}\vec{\hat{U}}_3 + (\hat{Q} - \hat{W})\vec{\hat{U}}_1}{(\hat{P}^2 - (\hat{Q} - \hat{W})^2)^{1/2}}$$

and

$$\vec{\hat{N}}_f = \vec{\hat{B}}_f \times \vec{\hat{U}}_2 = \pm\frac{\hat{P}\vec{\hat{U}}_1 + (\hat{Q} - \hat{W})\vec{\hat{U}}_3}{(\hat{P}^2 - (\hat{Q} - \hat{W})^2)^{1/2}}.$$

At the same time $\vec{\hat{B}}_f$ is the axis of curvature Π_f .Therefore, the trihedrons $\{\vec{\hat{U}}_2, \vec{\hat{N}}_m, \vec{\hat{B}}_m\}$ and $\{\vec{\hat{U}}_2, \vec{\hat{N}}_f, \vec{\hat{B}}_f\}$ are the Frenet trihedrons of the polodes at the pole.

Let ρ_m be a dual Lorentzian timelike angle between the ISA and $\vec{\hat{N}}_m$. If we let ρ_f be another dual Lorentzian timelike angle between the ISA and $\vec{\hat{N}}_f$, we have that

$$\begin{bmatrix} \vec{\hat{U}}_2 \\ \vec{\hat{N}}_m \\ \vec{\hat{B}}_m \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \cosh\rho_m & 0 & \sinh\rho_m \\ \sinh\rho_m & 0 & \cosh\rho_m \end{bmatrix} \begin{bmatrix} \vec{\hat{U}}_1 \\ \vec{\hat{U}}_2 \\ \vec{\hat{U}}_3 \end{bmatrix} \tag{5.15}$$

and

$$\begin{bmatrix} \vec{\hat{U}}_2 \\ \vec{\hat{N}}_f \\ \vec{\hat{B}}_f \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \cosh\rho_f & 0 & \sinh\rho_f \\ \sinh\rho_f & 0 & \cosh\rho_f \end{bmatrix} \begin{bmatrix} \vec{\hat{U}}_1 \\ \vec{\hat{U}}_2 \\ \vec{\hat{U}}_3 \end{bmatrix}.$$

The following two pairs of identities,

$$\cosh\rho_m = \frac{\hat{P}}{(\hat{P}^2 - \hat{Q}^2)^{1/2}} \text{ and } \sinh\rho_m = \frac{\hat{Q}}{(\hat{P}^2 - \hat{Q}^2)^{1/2}} \tag{5.16}$$

and

$$\cosh\rho_f = \frac{\hat{P}}{(\hat{P}^2-(\hat{Q}-\hat{W})^2)^{1/2}} \text{ and } \sinh\rho_f = \frac{\hat{Q}-\hat{W}}{(\hat{P}^2-(\hat{Q}-\hat{W})^2)^{1/2}},$$

are the consequences of equations (5.13) and (5.14), respectively. Note also that the dual Lorentzian timelike angles ρ_m and ρ_f are the dual Lorentzian spherical radii of π_m and π_f. These all together justify the following result.

Theorem 6 *The geodetic curvatures of the moving and the fixed polodes are*

$$\mathcal{K}_m = \coth\rho_m = \frac{\hat{P}}{\hat{Q}} \text{ and } \mathcal{K}_f = \coth\rho_f = \frac{\hat{P}}{\hat{Q}-\hat{W}}, \tag{5.17}$$

respectively, where $\hat{Q}$ and $\hat{Q}-\hat{W}$ are the elements of the geodetic curvatures of the polodes.

One of the immediate results of Theorem (5.2) is the following:

$$\mathcal{K}_f + \mathcal{K}_m = \frac{2\hat{P}\hat{Q}-\hat{P}\hat{W}}{\hat{Q}(\hat{Q}-\hat{W})} \text{ and } \mathcal{K}_f - \mathcal{K}_m = \frac{\hat{P}\hat{W}}{\hat{Q}(\hat{Q}-\hat{W})}. \tag{5.18}$$

As a result, in analogy with the dual spherical motion [1, 13], the invariants $\hat{P}$, $\hat{Q}$, and $\hat{W}$ can be considered asinstantaneous geometrical invariants of the motion H_m/H_f. In particular, the second one in equation (5.18) is the dual Lorentzian counterpart of Euler–Savary formula for spherical kinematics.

If we differentiate equation (5.15) and consider equations (4.9), (5.15), and (5.16), we conclude that

$$\frac{d_m}{dt}\begin{bmatrix} \vec{\hat{U}}_2 \\ \vec{\hat{N}}_m \\ \vec{\hat{B}}_m \end{bmatrix} = \begin{bmatrix} 0 & (\hat{P}^2-\hat{Q}^2)^{1/2} & 0 \\ (\hat{P}-\hat{Q}^2)^{1/2} & 0 & \dot{\rho}_m \\ 0 & \dot{\rho}_m & 0 \end{bmatrix} \begin{bmatrix} \vec{\hat{U}}_2 \\ \vec{\hat{N}}_m \\ \vec{\hat{B}}_m \end{bmatrix} \tag{5.19}$$

and

$$\frac{d_f}{dt}\begin{bmatrix} \vec{\hat{U}}_2 \\ \vec{\hat{N}}_f \\ \vec{\hat{B}}_f \end{bmatrix} = \begin{bmatrix} 0 & (\hat{P}^2-(\hat{Q}-\hat{W})^2)^{1/2} & 0 \\ (\hat{P}^2-(\hat{Q}-\hat{W})^2)^{1/2} & 0 & \dot{\rho}_f \\ 0 & \dot{\rho}_f & 0 \end{bmatrix} \begin{bmatrix} \vec{\hat{U}}_2 \\ \vec{\hat{N}}_f \\ \vec{\hat{B}}_m \end{bmatrix}$$

for the variation of the trihedrons $\{\vec{\hat{U}}_2, \vec{\hat{N}}_m, \vec{\hat{B}}_m\}$ and $\{\vec{\hat{U}}_2, \vec{\hat{N}}_f, \vec{\hat{B}}_f\}$. Thus, we have the curvature and the torsion of the moving and the fixed polodes

$$\kappa_m = -\frac{(\hat{P}-\hat{Q}^2)^{1/2}}{\hat{P}} \text{ and } \tau_m = -\frac{\dot{\rho}_m}{P}, \tag{5.20}$$

$$\kappa_f = -\frac{(\hat{P}^2-(\hat{Q}-\hat{W})^2)^{1/2}}{\hat{P}} \text{ and } \tau_f = -\frac{\dot{\rho}_f}{P},$$

respectively. We may therefore rewrite equations (5.18) and (5.19) as

$$\frac{d_m}{d\hat{s}}\begin{bmatrix}\vec{\hat{U}}_2\\ \vec{\hat{N}}_m\\ \vec{\hat{B}}_m\end{bmatrix}=\begin{bmatrix}0&\kappa_m&0\\ \kappa_m&0&\tau_m\\ 0&\tau_m&0\end{bmatrix}\begin{bmatrix}\vec{\hat{U}}_2\\ \vec{\hat{N}}_m\\ \vec{\hat{B}}_m\end{bmatrix}$$

and

$$\frac{d_f}{d\hat{s}}\begin{bmatrix}\vec{\hat{U}}_2\\ \vec{\hat{N}}_f\\ \vec{\hat{B}}_f\end{bmatrix}=\begin{bmatrix}0&\kappa_f&0\\ \kappa_f&0&\tau_f\\ 0&\tau_f&0\end{bmatrix}\begin{bmatrix}\vec{\hat{U}}_2\\ \vec{\hat{N}}_f\\ \vec{\hat{B}}_f\end{bmatrix}.$$

On the other hand, by using the first equations in (5.20), (5.16), and (5.17), we reach the following corollary.

Corollary 7 *The relations*

$$\mathcal{K}_m^2=\frac{1}{1-\kappa_m^2}\ \textit{and}\ \mathcal{K}_f^2=\frac{1}{1-\kappa_f^2}$$

hold among the geodetic curvatures and the curvature of the polodes.

6. Conclusions

We derive the geometrical instantaneous invariants of the dual pseudohyperbolic space motion in dual Lorentzian 3-space. We also show that the moving and fixed axodes maintain contact with each other along the ISA. We verify that the rolling and sliding motions of the instantaneous helicoidal motion around the ISA at the instant t correspond to the real part $\vec{U}$ and the dual part $\vec{U}_0$ of dual angular velocity vector $\vec{\hat{U}}$, respectively.

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