Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2015) 39: 763 – 772
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doi:10.3906/mat-1503-79

On the second homology of the Schützenberger product of monoids

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Received: 26.03.2015 • **Accepted/Published Online:** 17.06.2015 • **Printed:** 30.09.2015

Abstract: For two finite monoids S and T, we prove that the second integral homology of the Schützenberger product $S \Diamond T$ is equal to

$$H_2(S \Diamond T) = H_2(S) \times H_2(T) \times (H_1(S) \otimes_{\mathbb{Z}} H_1(T))$$

as the second integral homology of the direct product of two monoids. Moreover, we show that $S \Diamond T$ is inefficient if there is no left or right invertible element in both S and T.

Key words: Monoid, Schützenberger product, second integral homology, efficiency

1. Introduction

It was shown by SJ Pride (unpublished) that, for a finitely presented monoid M, $\mathrm{def}_M(M) \geq \mathrm{rank}(H_2(M))$ where $H_2(M)$ is the second integral homology of the monoid and

$$\mathrm{def}_M(M) = \min\{| R | - | A | : \langle A \mid R \rangle \text{ is a finite monoid presentation for } M\}.$$

In [1] this result was extended to a finitely presented semigroup S, that is $\mathrm{def}_S(S) \geq \mathrm{rank}(H_2(S))$ where $H_2(S)$ is the second integral homology of S^1, the monoid obtained from S by adjoining an identity if necessary, and

$$\mathrm{def}_S(S) = \min\{| R | - | A | : \langle A \mid R \rangle \text{ is a finite semigroup presentation for } S\}.$$

Moreover, it was shown that the nth integral homology of a semigroup with a left or a right zero is trivial for $n \geq 1$ (see also [8, Lemma 1]), and the second integral homology of a finite rectangular band $R_{m,n}$ of order mn is $\mathbb{Z}^{(m-1)(n-1)}$. A finite semigroup S is called *efficient* as a semigroup if $\mathrm{def}_S(S) = \mathrm{rank}(H_2(S))$, and *inefficient* otherwise. The efficiency and inefficiency of a finite monoid are defined similarly. The first examples of efficient and inefficient semigroups were given in [1], which showed that finite zero semigroups and finite free semilattices are inefficient, and finite rectangular bands are efficient. More examples of efficient semigroups can be found in [2, 3, 4, 5, 6].

It was shown in [2] that the second integral homology of a finite Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$ (finite simple semigroup) is $H_2(G) \times \mathbb{Z}^{(|I|-1)(|\Lambda|-1)}$ by using the Squier resolution (see [12]). In this paper, we also use this resolution to compute the second integral homology of the Schützenberger product of two finite monoids. We show that, for two finite monoids S and T,

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2010 *AMS Mathematics Subject Classification:* 20M05, 20M50.

$$H_2(S\Diamond T) = H_2(S) \times H_2(T) \times (H_1(S) \otimes_{\mathbb{Z}} H_1(T)),$$

and it follows from [3, Equation (1)] that $H_2(S\Diamond T) = H_2(S \times T)$. Moreover, we consider the efficiency of $S\Diamond T$ and conclude that, if there is no left or right invertible element in both S and T, then $S\Diamond T$ is inefficient.

2. Preliminaries

Since the Squier resolution given in [12] is defined by using a presentation in which the set of relations is a uniquely terminating rewriting system, we give some elementary concepts about rewriting systems.

Let A be an alphabet. We denote the free semigroup on A consisting of all nonempty words over A by A^+, and the free monoid $A^+ \cup \{\varepsilon\}$ where ε denotes the empty word by A^*. A *rewriting system* R on A is a subset of $A^* \times A^*$. For $w_1, w_2 \in A^*$, if they are identical words then we write $w_1 \equiv w_2$, and if there exist $u, v \in A^*$ and $(r,s) \in R$ such that $w_1 \equiv urv$ and $w_2 \equiv usv$ then we write $w_1 \to w_2$ and we say that w_1 *rewrites to* w_2. We denote by $\stackrel{*}{\to}$ the reflexive and transitive closure of $\to$, and by $\sim$ the equivalence relation generated by $\to$. For a word $w \in A^*$ we say that w is *reducible* (R*-reducible*) if there is a word $z \in A^*$ such that $w \to z$; otherwise we say that w is *irreducible* (R*-irreducible*). If $w \stackrel{*}{\to} y$ and $y \in A^*$ is irreducible, then we say that y is an irreducible form of w. A rewriting system R is called *terminating* if there is no infinite sequence (w_n) such that $w_n \to w_{n+1}$ for all $n \geq 1$. Let $|w|$ be the length of the word $w \in A^*$. If $|r| > |s|$ for all $(r,s) \in R$ then the system R is called *length-reducing*.

It is well known that if there exists an ordering $<$ on a set S such that, for each distinct pair $s, s' \in S$, either $s < s'$ or $s' < s$, then the ordering $<$ is called *linear* (or *total*) *ordering* and the set S is called *linearly* (or *totally*) *ordered*. For $u, v \in A^*$, if $|u| > |v|$ or if $|u| = |v|$ and v precedes u in the lexicographic ordering induced by a linear ordering on A then we write $v \ll u$ and $\ll$ is called *length-lexicographic ordering*. A rewriting system R is called a *length-lexicographic rewriting system* if $s \ll r$ for all $(r,s) \in R$. It is clear that length-reducing systems and length-lexicographic rewriting systems are terminating.

A semigroup (monoid) presentation is an ordered pair $\langle A \mid R\rangle$, where $R \subseteq A^+ \times A^+$ ($R \subseteq A^* \times A^*$). Let S be a semigroup (monoid). S is called *a semigroup (monoid) defined by the semigroup (monoid) presentation* $\langle A \mid R\rangle$ if S is isomorphic to A^+/ρ (A^*/ρ), where ρ is the congruence on A^+ (A^*) generated by R. For $w_1, w_2 \in A^*$, we also write $w_1 = w_2$ if $(w_1, w_2) \in \rho$; that is, w_2 is obtained from w_1 by applying relations from R, or, equivalently, there is a finite sequence

$$w_1 \equiv \alpha_1, \alpha_2, ..., \alpha_n \equiv w_2$$

of words from A^* in which every α_i is obtained from α_{i-1} by applying a relation from R (see [9, Proposition 1.5.9]).

A rewriting system R is called *confluent* if, for any $x, y, z \in A^*$ such that $x \stackrel{*}{\to} y$, $x \stackrel{*}{\to} z$, there exists $w \in A^*$ such that $y \stackrel{*}{\to} w$, $z \stackrel{*}{\to} w$. Also, a rewriting system R is called *complete* if it is both terminating and confluent. For a given rewriting system R, let the subset $R_1 \subseteq A^*$ be the set of all $r \in A^*$ such that there exists $(r,s) \in R$ for some $s \in A^*$. The system R is called *reduced* if for each $(r,s) \in R$, $R_1 \cap A^*rA^* = \{r\}$ and s is R-irreducible. Finally, a reduced complete rewriting system R is called a *uniquely terminating rewriting system*.

Lemma 2.1 *([7, Theorem 1.1] or [12, Theorem 2.1]) Let R be a terminating rewriting system on A. Then the following are equivalent:*

(i) *R is confluent (and hence complete);*

(ii) *for any pair $(r_1r_2, s_{1,2}), (r_2r_3, s_{2,3}) \in R$, where r_2 is nonempty, there exists a word $w \in A^*$ such that $s_{1,2}r_3 \overset{*}{\rightarrow} w$ and $r_1s_{2,3} \overset{*}{\rightarrow} w$; for any pair $(r_1r_2r_3, s_{1,2,3}), (r_2, s_2) \in R$, where r_2 is nonempty, there exists a word $w \in A^*$ such that $s_{1,2,3} \overset{*}{\rightarrow} w$ and $r_1s_2r_3 \overset{*}{\rightarrow} w$;*

(iii) *any word $w \in A^*$ has exactly one irreducible form. Moreover, $w \sim w'$ if and only if w and w' have the same irreducible form.*

If there exists a pair $(r_1r_2, s_{1,2}), (r_2r_3, s_{2,3}) \in R$ or $(r_1r_2r_3, s_{1,2,3})$, $(r_2, s_2) \in R$ such that r_2 is a nonempty word, then we define the *overlaps* to be the ordered pairs $[(r_1r_2, s_{1,2}), (r_2r_3, s_{2,3})]$ and $[(r_1r_2r_3, s_{1,2,3}), (r_2, s_2)]$, respectively. Note that the overlaps of the form $[(r_1r_2r_3, s_{1,2,3}), (r_2, s_2)]$ do not exist in a reduced rewriting system.

Let $\langle A \mid R \rangle$ be a presentation for a monoid S in which R is a uniquely terminating rewriting system on A. Also, let $\mathbb{Z}S$ denote the monoid ring of S with coefficients in $\mathbb{Z}$. In [12] Squier defined the free resolution of $\mathbb{Z}$ as follows:

$$P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where P_0 is the free $\mathbb{Z}S$-module on a single formal symbol $[\ ]$ and the augmentation map $\varepsilon : P_0 \longrightarrow \mathbb{Z}$ is defined by $\varepsilon([\ ]) = 1$. P_1 is the free $\mathbb{Z}S$-module on the set of formal symbols $[a]$ for each $a \in A$ and the map $\partial_1 : P_1 \longrightarrow P_0$ is defined by

$$\partial_1([a]) = (a-1)[\ ].$$

P_2 is the free $\mathbb{Z}S$-module on the set of formal symbols $[r, s]$, for each $(r, s) \in R$. For each $a \in A$, a function $\partial/\partial_a : A^* \longrightarrow \mathbb{Z}A^*$, which is called a *derivation*, is defined by induction as follows:

$$\partial/\partial_a(1) = 0,$$

and if $w \in A^*$ and $b \in A$, then

$$\partial/\partial_a(wb) = \begin{cases} \partial/\partial_a(w) & (\text{if } b \neq a), \\ \partial/\partial_a(w) + w & (\text{if } b = a). \end{cases}$$

Then the map $\partial_2 : P_2 \longrightarrow P_1$ is defined by

$$\partial_2([r, s]) = \sum_{a \in A} \phi(\partial/\partial_a(r) - \partial/\partial_a(s))[a],$$

where $\phi : \mathbb{Z}A^* \longrightarrow \mathbb{Z}S$ is the map induced by the natural homomorphism from A^* to S. Finally, P_3 is the free $\mathbb{Z}S$-module on the set of formal symbols $[(r_1r_2, s_{1,2}),\ (r_2r_3, s_{2,3})]$, for each pair $(r_1r_2, s_{1,2})$, $(r_2r_3, s_{2,3}) \in R$ where r_2 is not an empty word. Let w be in A^* and let u be the irreducible form of w. Then we have a sequence

$$w \equiv u_1r_1v_1 \rightarrow u_1s_1v_1 \equiv u_2r_2v_2 \rightarrow \cdots \rightarrow u_qs_qv_q \equiv u$$

where $u_i, v_i \in A^*$ and $(r_i, s_i) \in R$ for each $i = 1, \ldots, q$. Then the map $\Phi : A^* \longrightarrow P_2$ is defined by

$$\Phi(w) = \sum_{i=1}^{q} \phi(u_i)[r_i, s_i],$$

and the map $\partial_3 : P_3 \longrightarrow P_2$ is defined by

$$\partial_3([(r_1r_2, s_{1,2}), (r_2r_3, s_{2,3})]) = r_1[r_2r_3, s_{2,3}] - [r_1r_2, s_{1,2}] + \Phi(r_1s_{2,3}) - \Phi(s_{1,2}r_3).$$

Squier showed that $P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$ is an exact sequence if R is a uniquely terminating rewriting system and we assume that for each word $w \in A^*$, the chosen relation chain from w to the irreducible form of w consists of reductions only; that is, if $(r,s) \in R$, then $(s,r) \notin R$.

If we apply the tensor product $\mathbb{Z} \otimes_{\mathbb{Z}S} -$ to the resolution of $\mathbb{Z}$ given above, we obtain the chain complex of abelian groups

$$\mathbb{Z} \otimes P_3 \xrightarrow{1\otimes\partial_3} \mathbb{Z} \otimes P_2 \xrightarrow{1\otimes\partial_2} \mathbb{Z} \otimes P_1 \xrightarrow{1\otimes\partial_1} \mathbb{Z} \otimes P_0 \xrightarrow{1\otimes\varepsilon} \mathbb{Z} \otimes \mathbb{Z} \longrightarrow 0,$$

or simply,

$$\bar{P}_3 \xrightarrow{\bar{\partial}_3} \bar{P}_2 \xrightarrow{\bar{\partial}_2} \bar{P}_1 \xrightarrow{\bar{\partial}_1} \mathbb{Z} \longrightarrow 0 \qquad (1)$$

where $\bar{P}_1$, $\bar{P}_2$, and $\bar{P}_3$ are the free abelian groups on the set of formal symbols $[a]$, $[r,s]$, and $[(r_1r_2, s_{1,2}),$ $(r_2r_3, s_{2,3})]$ where $a \in A$; $(r,s), (r_1r_2, s_{1,2}), (r_2r_3, s_{2,3}) \in R$ with r_2 not an empty word, respectively. Clearly the map $\bar{\partial}_1 : \bar{P}_1 \to \mathbb{Z}$ is the zero map.

For $a \in A$ and $w \in A^*$, the number of occurrences of the letter a in the word w is called *a-length of w* and denoted by $\|w\|_a$. Moreover, if $w \equiv a_1a_2\cdots a_m$, then we denote the list $[a_1, a_2, \ldots, a_m]$ by $C[w]$. (Note that in any list some of the elements can be the same; for example, $C[ab^2a^2] = [a,b,b,a,a]$.)

The maps $\bar{\partial}_2 : \bar{P}_2 \to \bar{P}_1$ and $\bar{\partial}_3 : \bar{P}_3 \to \bar{P}_2$ are defined by

$$\bar{\partial}_2([r,s]) = \sum_{a \in A} (\|r\|_a - \|s\|_a)[a]$$

and

$$\bar{\partial}_3([(r_1r_2, s_{1,2}), (r_2r_3, s_{2,3})]) = [r_2r_3, s_{2,3}] - [r_1r_2, s_{1,2}] + \bar{\Phi}(r_1s_{2,3}) - \bar{\Phi}(s_{1,2}r_3),$$

respectively, where $\bar{\Phi} : A^* \to \bar{P}_2$ is the map defined by

$$\bar{\Phi}(w) = \sum_{i=1}^{q} [r_i, s_i] \text{ if } \Phi(w) = \sum_{i=1}^{q} \phi(u_i)[r_i, s_i].$$

With this notation we have the following immediate result:

Lemma 2.2 *([3, Lemma 3.1]) If a monoid S has a presentation $\langle A \mid R \rangle$ such that R is a uniquely terminating rewriting system on A, then*

$$H_1(S) = H_1(G) = G/G' = \langle A \mid \sum_{a \in A} (\|r\|_a - \|s\|_a)[a] = 0 \quad ((r,s) \in R) \rangle,$$

where G is the group defined by $\langle A \mid R \rangle$ as a group presentation and G' is the derived subgroup of G.

Lemma 2.3 *([11, Chapter 6]) Let $\langle A \mid R \rangle$ and $\langle B \mid Q \rangle$ (A and B are distinct) be presentations for the monoids S and T, respectively. Then the tensor product of their first homologies, namely $H_1(S) \otimes_{\mathbb{Z}} H_1(T)$,* *can be given by the abelian group presentation*

$$\langle [A,B] \mid \quad \textstyle\sum_{a\in A} \quad (\|r\|_a - \|s\|_a)[ab,ba] = 0 \quad (b\in B, (r,s)\in R)$$
$$\textstyle\sum_{b\in B} \quad (\|u\|_b - \|v\|_b)[ab,ba] = 0 \quad (a\in A, (u,v)\in Q)\rangle,$$

where $[A,B] = \{[ab,ba] \mid a\in A, b\in B\}$.

3. The second integral homology of the Schützenberger product of monoids

Let S and T be two finite monoids, and let $\mathcal{P}(S\times T)$ denote the set of all subsets of $S\times T$. Now we define the sets

$$sX = \{(sx,y) : (x,y)\in X\} \text{ and } Xt = \{(x,yt) : (x,y)\in X\},$$

where $X\in\mathcal{P}(S\times T)$, $s\in S$, and $t\in T$. Then the set $S\times\mathcal{P}(S\times T)\times T$ is a monoid, denoted by $S\Diamond T$ and called the *Schützenberger product of* S *and* T, with identity $(1_S,\emptyset,1_T)$ by the multiplication

$$(s_1,X_1,t_1)(s_2,X_2,t_2) = (s_1s_2, X_1t_2\cup s_1X_2, t_1t_2).$$

If S is a finitely presented monoid then it is clear that S is linearly ordered by considering the length-lexiographic ordering. In this section we consider that the monoids S and T are well ordered. Moreover, the direct product $S\times T$ is also linearly ordered, with the ordering $(s,t)\prec(s',t')$ if $s<s'$ or if $s=s'$ and $t<t'$.

If the monoid presentations $\langle A\mid R\rangle$ and $\langle B\mid Q\rangle$ (A and B are distinct) define the monoids S and T, respectively, then the presentation $\langle A\cup B\cup C\mid R\cup Q\cup Z\rangle$ where $C=\{c_{s,t}: s\in S,\ t\in T\}$ and

$$\begin{aligned} Z = \quad \{ \quad & c_{s,t}^2 = c_{s,t} \ \ (s\in S,\ t\in T),\\ & c_{s,t}c_{s',t'} = c_{s',t'}c_{s,t} \ \ ((s',t')\prec(s,t)\in S\times T),\\ & ac_{s,t} = c_{as,t}a \ \ (a\in A,\ s\in S,\ t\in T),\\ & c_{s,t}b = bc_{s,tb} \ \ (b\in B,\ s\in S,\ t\in T),\\ & ab = ba \ \ (a\in A,\ b\in B)\ \} \end{aligned}$$

defines $S\Diamond T$ in terms of the generating set

$$\{(a,\emptyset,1_T),(1_S,\emptyset,b),(1_S,\{(s,t)\},1_T) : a\in A,\ b\in B,\ (s,t)\in S\times T\}.$$

(For a proof, see [10, Theorem 3.2].)

Note that, for ease of notation, we write $c_{as,t}$ and $c_{s,tb}$ instead of $c_{\pi_S(a)s,t}$ and $c_{s,t\pi_T(b)}$ where $\pi_S: A^*\to S$ and $\pi_T: B^*\to T$ are the natural homomorphisms, respectively. Thus, for $r,p\in A^*S$ and $u,v\in TB^*$, the words $c_{r,u}$ and $c_{p,v}$ are identical if the relations $r=p$ and $u=v$ hold in S and T, respectively.

Lemma 3.1 *Let S and T be two finite monoids, and let $\langle A\mid R\rangle$ and $\langle B\mid Q\rangle$ be their finite monoid presentations such that R and Q are uniquely terminating rewriting systems on A and B, respectively. With the above notations, the rewriting system $R\cup Q\cup Z$ is uniquely terminating on $A\cup B\cup C$.*

Proof For an arbitrary word w in $(A\cup B\cup C)^*$, it is clear that the reduced form of w has the form $w_1w_2w_3$ where w_1, w_2, and w_3 are reduced words in B, C, and A, respectively. It is also clear that $R\cup Q\cup Z$ is terminating and reduced. The overlaps are:

$$\begin{aligned}
V_1 &= [(r_1r_2, p_{1,2}),\ (r_2r_3, p_{2,3})],\\
V_2 &= [(ra, p),\ (ac_{s,t}, c_{as,t}a)],\\
V_3 &= [(ra, p),\ (ab, ba)],\\
V_4 &= [(u_1u_2, v_{1,2}),\ (u_2u_3, v_{2,3})],\\
V_5 &= [(c_{s,t}c_{s,t}, c_{s,t}),\ (c_{s,t}c_{s,t}, c_{s,t})],\\
V_6 &= [(c_{s,t}c_{s,t}, c_{s,t}),\ (c_{s,t}c_{s',t'}, c_{s',t'}c_{s,t})]((s',t') \prec (s,t)),\\
V_7 &= [(c_{s,t}c_{s,t}, c_{s,t}),\ (c_{s,t}b, bc_{s,tb})],\\
V_8 &= [(c_{s,t}c_{s',t'}, c_{s',t'}c_{s,t}),\ (c_{s',t'}c_{s',t'}, c_{s',t'})]((s',t') \prec (s,t)),\\
V_9 &= [(c_{s,t}c_{s',t'}, c_{s',t'}c_{s,t}),\ (c_{s',t'}c_{s'',t''}, c_{s'',t''}c_{s',t'})]((s'',t'') \prec (s',t') \prec (s,t)),\\
V_{10} &= [(c_{s,t}c_{s',t'}, c_{s',t'}c_{s,t}),\ (c_{s',t'}b, bc_{s',t'b})]((s',t') \prec (s,t)),\\
V_{11} &= [(ac_{s,t}, c_{as,t}a),\ (c_{s,t}c_{s,t}, c_{s,t})],\\
V_{12} &= [(ac_{s,t}, c_{as,t}a),\ (c_{s,t}c_{s',t'}, c_{s',t'}c_{s,t})]((s',t') \prec (s,t)),\\
V_{13} &= [(ac_{s,t}, c_{as,t}a),\ (c_{s,t}b, bc_{s,tb})],\\
V_{14} &= [(c_{s,t}b, bc_{s,tb}),\ (bu, v)],\\
V_{15} &= [(ab, ba),\ (bu, v)],
\end{aligned}$$

where $a \in A$; $b \in B$; $(ra = p)$, $(r_1r_2 = p_{1,2})$, $(r_2r_3 = p_{2,3}) \in R$; $(bu = v)$, $(u_1u_2 = v_{1,2})$, $(u_2u_3 = v_{2,3}) \in Q$; (s,t), (s',t'), $(s'',t'') \in S \times T$. Now it follows from Lemma 2.1 that $R \cup Q \cup Z$ is confluent and so a uniquely terminating rewriting system. □

Theorem 3.2 *If S and T are two finite monoids, then*

$$H_2(S\diamond T) = H_2(S) \times H_2(T) \times (H_1(S) \otimes_{\mathbb{Z}} H_1(T)).$$

Proof We consider the uniquely terminating rewriting system $R \cup Q \cup Z$ on $A \cup B \cup C$ given in Lemma 3.1 and the chain complex (1) arising from it.

Before we compute the second integral homology of $S\diamond T$, that is $H_2(S\diamond T) = \ker \bar{\partial}_2 / \operatorname{im} \bar{\partial}_3$, we assume that $H_2(S) = \ker \bar{\partial}_{2_{|S}} / \operatorname{im} \bar{\partial}_{3_{|S}}$ and $H_2(T) = \ker \bar{\partial}_{2_{|T}} / \operatorname{im} \bar{\partial}_{3_{|T}}$ where $\ker \bar{\partial}_{2_{|S}}$, $\operatorname{im} \bar{\partial}_{3_{|S}}$, $\ker \bar{\partial}_{2_{|T}}$, and $\operatorname{im} \bar{\partial}_{3_{|T}}$ are the free abelian groups on $\{X_i : i \in I\}$, $\{Y_j : j \in J\}$, $\{U_k : k \in K\}$, and $\{W_l : l \in L\}$ (which are found by using the Squier resolution), respectively.

Now we find a generating set for the free abelian group $\operatorname{im}\bar{\partial}_3$ by using the overlaps in the proof of Lemma 3.1. We compute the following.

$$\begin{aligned}
\bar{\partial}_3(V_1) &= \operatorname{im} \bar{\partial}_{3_{|S}}\\
\bar{\partial}_3(V_2) &= [ac_{s,t}, c_{as,t}a] - [ra, p] + \bar{\Phi}(rc_{as,t}a) - \bar{\Phi}(pc_{s,t})\\
\bar{\partial}_3(V_3) &= \sum_{a \in C[ra]} [ab, ba] - \sum_{a \in C[p]} [ab, ba]
\end{aligned}$$

$$
\begin{aligned}
\bar{\partial}_3(V_4) &= \operatorname{im}\bar{\partial}_{3_{|T}} \\
\bar{\partial}_3(V_5) &= 0 \\
\bar{\partial}_3(V_6) &= [c_{s,t}c_{s',t'}, c_{s',t'}c_{s,t}] \\
\bar{\partial}_3(V_7) &= [c_{s,t}b, bc_{s,tb}] - [c_{s,t}^2, c_{s,t}] + [c_{s,tb}^2, c_{s,tb}] \\
\bar{\partial}_3(V_8) &= -[c_{s,t}c_{s',t'}, c_{s',t'}c_{s,t}] \\
\bar{\partial}_3(V_9) &= 0 \\
\bar{\partial}_3(V_{10}) &= -[c_{s,t}c_{s',t'}, c_{s',t'}c_{s,t}] + [c_{s,tb}c_{s',t'b}, c_{s',t'b}c_{s,tb}] \\
\bar{\partial}_3(V_{11}) &= [c_{s,t}^2, c_{s,t}] - [ac_{s,t}, c_{as,t}a] - [c_{as,t}^2, c_{as,t}] \\
\bar{\partial}_3(V_{12}) &= [c_{s,t}c_{s',t'}, c_{s',t'}c_{s,t}] - [c_{as,t}c_{as',t'}, c_{as',t'}c_{as,t}] \\
\bar{\partial}_3(V_{13}) &= [c_{s,t}b, bc_{s,tb}] - [ac_{s,t}, c_{as,t}a] + [ac_{s,tb}, c_{as,tb}a] - [c_{as,t}b, bc_{as,tb}] \\
\bar{\partial}_3(V_{14}) &= -[c_{s,t}b, bc_{s,tb}] + \bar{\Phi}(c_{s,t}v) - \bar{\Phi}(c_{s,tb}u) \\
\bar{\partial}_3(V_{15}) &= \sum_{b\in C[v]}[ab, ba] - \sum_{b\in C[bu]}[ab, ba]
\end{aligned}
$$

Now let

$$
\begin{aligned}
W(ra,p) &= \sum_{a\in C[ra]}[ab, ba] - \sum_{a\in C[p]}[ab, ba], \\
W(bu,v) &= \sum_{b\in C[v]}[ab, ba] - \sum_{b\in C[bu]}[ab, ba], \\
W(a,s,t) &= [c_{s,t}^2, c_{s,t}] - [ac_{s,t}, c_{as,t}a] - [c_{as,t}^2, c_{as,t}], \\
W(b,s,t) &= [c_{s,t}b, bc_{s,tb}] - [c_{s,t}^2, c_{s,t}] + [c_{s,tb}^2, c_{s,tb}], \\
W(s',t',s,t) &= [c_{s,t}c_{s',t'}, c_{s',t'}c_{s,t}] \qquad (\,(s',t') \prec (s,t)\,)
\end{aligned}
$$

where $a \in A,\ b \in B,\ s,s' \in S,\ t,t' \in T,\ (ra,p) \in R$, and $(bu,v) \in Q$. Then we show that the set

$$
\{Y_j,\ W_l,\ W(ra,p),\ W(bu,v),\ W(a,s,t),\ W(b,s,t),\ W(s',t',s,t)\,((s',t') \prec (s,t)) :
$$
$$
j \in J;\ l \in L;\ a \in A;\ b \in B;\ s,s' \in S;\ t,t' \in T;\ (ra,p) \in R;\ (bu,v) \in Q\,\}
$$

is a generating set for the free abelian group $\operatorname{im}\bar{\partial}_3$ as follows.

If $r \equiv a_1 \cdots a_m$ and $p \equiv a'_1 \cdots a'_n$ $(a_1,\ldots,a_m,a'_1,\ldots,a'_n \in A)$ then we define

$$
\begin{aligned}
W_0 &= W(a_m, as, t), \\
W_i &= W(a_{m-i}, a_{m+1-i}\cdots a_m as, t)\ \ (1 \le i \le m-1), \\
W'_0 &= W(a'_n, s, t), \\
W'_j &= W(a'_{n-j}, a'_{n+1-j}\cdots a'_n s, t)\ \ (1 \le j \le n-1).
\end{aligned}
$$

Thus, we have

$$\begin{aligned}\bar{\partial}_3(V_2) &= [ac_{s,t}, c_{as,t}a] + \bar{\Phi}(rc_{as,t}) - \bar{\Phi}(pc_{s,t}) = [ac_{s,t}, c_{as,t}a] \\ &\quad +[a_m c_{as,t}, c_{a_m as,t}a_m] + \sum_{i=1}^{m-1}[a_{m-i}c_{a_{m+1-i}\cdots a_m as,t}, c_{a_{m-i}\cdots a_m as,t}a_{m-i}] \\ &\quad -[a'_n c_{s,t}, c_{a'_n s,t}a'_n] - \sum_{j=1}^{n-1}[a'_{n-j}c_{a'_{n+1-j}\cdots a'_n s,t}, c_{a'_{n-j}\cdots a'_n s,t}a'_{n-j}] \\ &= -W(a,s,t) + \sum_{j=0}^{n-1} W'_j - \sum_{i=0}^{m-1} W_i,\end{aligned}$$

and so $\bar{\partial}_3(V_2)$ is a linear combination of $W(a,s,t)$s. Similarly, it can be shown that $\bar{\partial}_3(V_{14})$ is a linear combination of $W(b,s,t)$s. Moreover, it is clear that all of $\bar{\partial}_3(V_6)$, $\bar{\partial}_3(V_8)$, $\bar{\partial}_3(V_{10})$, and $\bar{\partial}_3(V_{12})$ are linear combinations of $W(s',t',s,t)$s, and that

$$\bar{\partial}_3(V_{13}) = W(b,s,t) + W(a,s,t) - W(a,s,tb) - W(b,as,t).$$

Next we find a generating set for $\ker \bar{\partial}_2$. Since any $\alpha \in \bar{P}_2$ has the form

$$\begin{aligned}\alpha &= \sum_{(r=s)\in R}\alpha_{(r,s)}[r,s] + \sum_{(u=v)\in Q}\alpha_{(u,v)}[u,v] + \sum_{a\in A,\, b\in B}\alpha_{(a,b)}[ab,ba] \\ &\quad + \sum_{s\in S,\, t\in T}\alpha_{(s,t)}[c^2_{s,t}, c_{s,t}] + \sum_{(s',t')\prec(s,t)\in S\times T}\alpha_{(s',t',s,t)}[c_{s,t}c_{s',t'}, c_{s',t'}c_{s,t}] \\ &\quad + \sum_{a\in A,\, s\in S,\, t\in T}\alpha_{(a,s,t)}[ac_{s,t}, c_{as,t}a] + \sum_{b\in B,\, s\in S,\, t\in T}\alpha_{(b,s,t)}[c_{s,t}b, bc_{s,tb}]\end{aligned}$$

where all the coefficients are integers, then $\alpha \in \ker \bar{\partial}_2$ if and only if

$$\bar{\partial}_2(\sum_{(r=s)\in R}\alpha_{(r,s)}[r,s]) = 0, \quad \bar{\partial}_2(\sum_{(u=v)\in Q}\alpha_{(u,v)}[u,v]) = 0 \ \text{ and}$$

$$\sum_{s\in S,\, t\in T}\alpha_{(s,t)}[c_{s,t}] + \sum_{a\in A}\alpha_{(a,s,t)}([c_{s,t}] - [c_{as,t}]) + \sum_{b\in B}\alpha_{(b,s,t)}([c_{s,t}] - [c_{s,tb}]) = 0.$$

From the first two equations given above we obtain the generators $\{X_i : i \in I\}$ and $\{U_k : k \in K\}$ for $\ker \bar{\partial}_{2_{|S}}$ and $\ker \bar{\partial}_{2_{|T}}$, respectively. Now we concentrate on the last equation. By rearranging it, we have

$$\alpha_{(s,t)} = -\sum_{a\in A}\alpha_{(a,s,t)} - \sum_{b\in B}\alpha_{(b,s,t)} + \sum_{\substack{a'\in A,\, s'\in S\\ a's'=s}}\alpha_{(a',s',t)} + \sum_{\substack{b'\in B,\, t'\in T\\ t'b'=t}}\alpha_{(b',s,t')} \tag{2}$$

for each $(s,t) \in S\times T$. For fixed $\alpha_{(a,s,t)}$, we assume that $\alpha_{(a,s,t)} = 1$ and all the other variables on the right-hand side of Equation (2) are zero, and so we obtain $\alpha_{(s,t)} = -1$ and $\alpha_{(as,t)} = 1$. Thus, we have the following generators:

$$W_1(a,s,t) = [ac_{s,t}, c_{as,t}a] - [c^2_{s,t}, c_{s,t}] + [c^2_{as,t}, c_{as,t}].$$

Similarly, we have

$$W_2(b,s,t) = [c_{s,t}b, bc_{s,tb}] - [c^2_{s,t}, c_{s,t}] + [c^2_{s,tb}, c_{s,tb}].$$

Therefore,

$$\{X_i,\ U_k,\ [ba,ab],\ W_1(a,s,t),\ W_2(b,s,t),\ [c_{s,t}c_{s',t'}, c_{s',t'}c_{s,t}] : i \in I;\ k \in K;\ a \in A;$$
$$b \in B;\ s,s' \in S;\ t,t' \in T\ ((s',t')) \prec (s,t))\}$$

is a generating set for $\ker \bar{\partial}_2$.

Notice that $W_1(a,s,t)$, $W_2(b,s,t)$ and $[c_{s,t}c_{s',t'}, c_{s',t'}c_{s,t}]$ are also in the generating set for $\operatorname{im} \bar{\partial}_3$ given above, and so

$$\begin{aligned} H_2(S\Diamond T) &= \langle X_i,\ U_k,\ [ab,ba]\,(i \in I,\ k \in K,\ a \in A,\ b \in B) \mid \\ &\qquad Y_j = 0,\ W_l = 0,\ W(ra,p) = 0,\ W(bu,v) = 0 \\ &\qquad (j \in J,\ l \in L,\ (ra,p) \in R,\ (bu,v) \in Q)\rangle \\ &= H_2(S) \times H_2(T) \times \langle [ab,ba]\,(a \in A,\ b \in B) \mid W(ra,p) = 0, \\ &\qquad W(bu,v) = 0\,((ra,p) \in R,\ (bu,v) \in Q)\rangle. \end{aligned}$$

Since $\langle [ab,ba]\,(a \in A,\ b \in B) \mid W(ra,p) = 0, W(bu,v) = 0, ((ra,p) \in R,\ (bu,v) \in Q)\rangle$ is equal to $H_1(S) \otimes_{\mathbb{Z}} H_1(T)$, from Lemma 2.3, the proof is complete. □

Notice that one may consider the Schützenberger product $S\Diamond T$ as "a kind of direct product" of the monoids $S \times T$ and the free semilattice over $S \times T$ (the monoid considered as the set of all subsets of $S \times T$ with set-theoretical union as a multiplication). Therefore, from [1, Proposition 3.1] and [3, Equation (1), p. 282], the result in the last theorem is perhaps not surprising.

4. Remark

In [1, Theorem 3.3] it was shown that if A is a finite nonempty set of size n, then

$$def_S(SL_A) = n(n-1)/2, \tag{3}$$

and for $n \geq 2$ SL_A is inefficient, where SL_A is the set of all nonempty subsets of A with set-theoretic union as multiplication.

For convenience, first we state a probably well-known lemma that can be proved easily.

Lemma 4.1 *Let S be a monoid, $P = \langle A \mid R\rangle$ be a presentation of S, T be a subsemigroup of S, and $S\backslash T$ be an ideal of S. Then T has a presentation $\langle B \mid Q\rangle$ such that $B \subset A$ and $Q \subset R$.*

Corollary 4.2 *If S and T are two finite monoids without any left or right invertible element, then $S\Diamond T$ is inefficient.*

Proof Consider the sets

$$\begin{aligned} U &= \{(1_S, X, 1_T) \mid X \subset S \times T\} \text{ and} \\ V &= (S\Diamond T)\backslash U = \{(s,X,t) \in S\Diamond T \mid (s,t) \neq (1_S, 1_T)\}. \end{aligned}$$

It is clear that U is a subsemigroup of $S\Diamond T$ and isomorphic to the free semilattice $SL_{S\times T}$. Moreover, V is an ideal of $S\Diamond T$. It follows from Lemma 4.1, Equation (3), and Theorem 3.2 that $S\Diamond T$ is inefficient. □

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