Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2015) 39: 618 – 644
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doi:10.3906/mat-1410-18

Moduli spaces of arrangements of 11 projective lines with a quintuple point

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Received: 08.10.2014 • **Accepted/Published Online:** 04.03.2015 • **Printed:** 30.09.2015

Abstract: In this paper, we try to classify moduli spaces of arrangements of 11 lines with quintuple points. We show that moduli spaces of arrangements of 11 lines with quintuple points can consist of more than 2 connected components. We also present defining equations of the arrangements whose moduli spaces are not irreducible after taking quotients by the complex conjugation by Maple and supply some "potential Zariski pairs".

Key words: Line arrangements, moduli spaces, irreducibility

1. Introduction

Let $\mathcal{A} = \{H_1, H_2, \dots, H_n\}$ be a line arrangement in the complex projective plane $\mathbb{CP}^2$, and denote by $M(\mathcal{A})$ the corresponding complement of the arrangement.

An essential topic in hyperplane arrangement theory is to study the intersection between topology of complements and combinatorics of intersection lattices. It is important to study how closely topology and combinatorics of a given arrangement are related. For line arrangements, Jiang and Yau [8] showed that homeomorphism of the complement always implies lattice isomorphism. However, the converse is not true in general for line arrangements. In [7] and [12], the authors found a large class of line arrangements whose intersection lattices determine topology of the complements, called nice arrangements and simple arrangements respectively. The notion of nice line arrangements has been generalized to arrangements of hyperplanes in higher dimensional projective spaces (see [13, 14, 15]).

We call a pair of line arrangements a *Zariski pair* if they are lattice isomorphic, but the fundamental groups of their complements are different. The first Zariski pair of line arrangements was constructed by Rybnikov [11]. On the other hand, combining the results of Fan [4], Garber et al. [5] proved that there is no Zariski pair of arrangements of up to 8 real lines. This result was recently generalized to arrangements of 8 complex lines by Nazir and Yoshinaga [9]. In the same paper, Nazir and Yoshinaga also claimed that there is no Zariski pair of arrangements of 9 complex lines. A complete proof of their claim was presented in [16]. Recently, Amram et al. classified arrangements of 10 complex lines in [2, 1] and found some "potential Zariski pairs".

Let $\mathcal{A}$ be a complex line arrangement. We define the *moduli space* of line arrangements with the fixed lattice $L(\mathcal{A})$ (or simply, the moduli space of $\mathcal{A}$) as

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2010 *AMS Mathematics Subject Classification:* 14N20, 32S22, 52C35.

$$\mathcal{M}_{\mathcal{A}} = \{\mathcal{B} \in ((\mathbb{CP}^2)^*)^n | \mathcal{B} \sim \mathcal{A})\}/PGL_{\mathbb{C}}(2),$$

where $\mathcal{B} \sim \mathcal{A}$ means $\mathcal{B}$ and $\mathcal{A}$ are lattice isomorphic. We denote by $\mathcal{M}^c_{\mathcal{A}}$ the quotient of $\mathcal{M}_{\mathcal{A}}$ under the complex conjugation. By Randell's lattice-isotopy theorem in [10] and Cohen and Suciu's theorem [3, Theorem 3.9], we know that arrangements in the same connected component of the moduli space, or in two complex conjugate components, can not form Zariski pairs. Therefore, to investigate the existence of Zariski pairs of arrangements of 11 lines, it is very important to know the geometry of moduli spaces of arrangements. In this paper, we try to classify the moduli spaces of arrangements of 11 lines with quintuple points, and in particular we completely classify the arrangements of 11 lines with a quintuple point and at least one quadruple point. On this basis, we give forty new "potential Zariski pairs" of arrangements of 11 lines.

The classification of moduli spaces consists of three steps. First, we will roughly classify intersection lattices according the number of multiple intersection points. Second, we divide our classification into some different cases according to different positions between quintuple point and the other multiple intersection points. Third, we will write down defining equations involving parameters for a given intersection lattice.

This paper is structured as follows. Section 2 provides preliminaries and ideas on classifying moduli spaces of arrangements of 11 lines. Section 3 shows that moduli spaces of arrangements with multiple points of high multiplicity are most likely irreducible. In Section 4 and Section 5, we completely classify the arrangements of 11 lines with a quintuple point and at least one quadruple point. In Section 6 we deal with the arrangements of 11 lines with a quintuple point and no quadruple point. Sections 4, 5, and 6 are the main parts of this work and in total forty "potential Zariski pairs" can be found there. In the Appendix (on the journal's website), we give an example to show how to compute the defining equations of the arrangements by Maple.

2. Preliminaries

Let $\mathcal{A} = \{L_1, L_2, \cdots, L_n\}$ be a line arrangement in $\mathbb{CP}^2$. We say a singularity of $L_1 \cup L_2 \cup \cdots \cup L_n$ is a *multiple point* of $\mathcal{A}$ if it has multiplicity of at least 3. We call the set $L(\mathcal{A}) = \{\bigcap_{i \in S} L_i | S \subseteq \{1, 2, \ldots, n\}\}$ partially ordered by reverse inclusion in the *intersection lattice* of $\mathcal{A}$.

Definition 2.1 *Two line arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ are lattice isomorphic, denoted as $\mathcal{A}_1 \sim \mathcal{A}_2$, if their intersection lattices $L(\mathcal{A}_1)$ and $L(\mathcal{A}_2)$ are isomorphic, i.e. there is a permutation ϕ of $\{1, 2, \ldots, n\}$ such that*

$$\dim\Big(\bigcap_{\substack{i \in S \\ L_i \in \mathcal{A}_1}} L_i\Big) = \dim\Big(\bigcap_{\substack{j \in \phi(S) \\ H_j \in \mathcal{A}_2}} H_j\Big)$$

for any nonempty subset $S \subseteq \{1, 2, \ldots, n\}$.

Definition 2.2 *([9, Definition 3.10]) Let $k \in \mathbb{N}$. We say that a line arrangement $\mathcal{A}$ is of type C_k if k is the minimum number of lines in $\mathcal{A}$ containing all points of multiplicity of at least three.*

Definition 2.3 *([9, Definition 3.13]) Let $\mathcal{A}$ be an line arrangement of type C_3. Then $\mathcal{A}$ is a* simple C_3 *arrangement if there are three lines $L_1, L_2, L_3 \in \mathcal{A}$ such that all points of multiplicity of at least three are contained in $L_1 \bigcup L_2 \bigcup L_3$ and one of the following holds:*

1. *$L_1 \cap L_2 \cap L_3 \neq \emptyset$, or*

2. $L_1 \cap L_2 \cap L_3 = \emptyset$ *and one of* L_1, L_2, *and* L_3 *contains only one multiple point apart from the other two lines.*

Theorem 2.4 *([9, Theorem 3.15]) Let* $\mathcal{A}$ *be an arrangement of* C_3 *of simple type. Then the moduli space* $\mathcal{M}_{\mathcal{A}}$ *is irreducible.*

Theorem 2.5 *([9, Lemma 3.2]) Let* $\mathcal{A} = \{L_1, L_2, \cdots, L_n\}$ *be a line arrangement. Assume that* L_n *passes through at most* 2 *multiple points. Set* $\mathcal{A}' = \{L_1, L_2, \cdots, L_{n-1}\}$, *and then* $\mathcal{M}_{\mathcal{A}}$ *is irreducible if* $\mathcal{M}_{\mathcal{A}'}$ *is irreducible.*

We say that a line arrangement is *nonreductive* if each line of the arrangement passes through at least 3 multiple points. Otherwise, we say the arrangement is *reductive*.

Denote by n_r the number of intersection points of multiplicity r. We recall the following useful results.

Lemma 2.6 *(See for instance [6].) Let* $\mathcal{A}$ *be an arrangement of* k *lines in* $\mathbb{CP}^2$. *Then*

$$\frac{k(k-1)}{2} = \sum_{r \geq 2} \frac{r(r-1)n_r}{2}.$$

Theorem 2.7 *(See [6].) Let* $\mathcal{A}$ *be an arrangement of* k *lines in* $\mathbb{CP}^2$. *Assume that* $n_k = n_{k-1} = n_{k-2} = 0$. *Then*

$$n_2 + \frac{3}{4}n_3 \geq k + \sum_{r \geq 5}(2r-9)n_r.$$

The following lemma is well known and is used to facilitate the calculation in our paper.

Lemma 2.8 *Let* $\{L_1, L_2, L_3, L_4\}$ *and* $\{L_5, L_6, L_7\}$ *be two pencils of lines who intersect at one point and intersect transversally in* 12 *points. Then there is an automorphism of the dual projective plane such that the 7 lines under the automorphism are defined by* $Y = Z$, $Y = t_3 Z$, $Y = t_2 Z$, $Y = 0$, $X = 0$, $X = t_1 Z$, $X = Z$.

Remark 2.9 *All the computations in Sections 4 and 5 are based on Lemma 2.8 above. First, we let* $L_1, \cdots, L_7$ *be as in Lemma 2.8 and let* L_{11} *be the line at infinity. Second, by the intersection points we obtain the defining equations of* L_8, L_9, L_{10}, *and by the conditions of slope, parallel, and intersection points, we get the equations on the coefficients* t_1, t_2, t_3. *Third, using Maple, it is easy to get the solutions of* t_1, t_2, t_3, *and the defining equations of the arrangements or the arrangements cannot be realized. In Section 6, similarly as in Lemma 2.8, we can establish similar vertical nets and the methods of computing the defining equations of the arrangements is the same as in the above three steps.*

3. Arrangements of 11 lines with multiple points of multiplicity at least 6

Theorem 3.1 *Let* $\mathcal{A}$ *be an arrangement of* n $(n \geq 9)$ *lines. If there is a multiple point of multiplicity* $\geq n-4$, *then the moduli space* $\mathcal{M}_{\mathcal{A}}$ *is irreducible.*

Proof For $n = 9, 10$, it was proved in [16, Prop 3.3] and [2, Theorem 3.1]. Now we consider $n \geq 11$. Assume that $L_1 \cap L_2 \cap \cdots \cap L_{n-4} \neq \emptyset$. It is easy to see that at least one of the $n-4$ lines contains at most 2 multiple points. By Theorem 2.5 and [2, Theorem 3.1], we see that $\mathcal{M}_{\mathcal{A}}$ is irreducible. □

In particular, if $n = 11$ and $n_7 \geq 1$, then $\mathcal{M}_{\mathcal{A}}$ is irreducible.

Theorem 3.2 *Let $\mathcal{A}$ be an arrangement of* 11 *lines with a multiple point of multiplicity* 6 *and no multiple points of higher multiplicities; then $\mathcal{A}$ is reductive.*

Proof Assume that $\mathcal{A}$ is nonreductive, and then by Lemma 2.6 and Theorem 2.7 we have

$$44 - 18n_6 \geq \frac{9}{4}(n_3 + n_4 + n_5). \tag{1}$$

On the other hand, it is easy to see that there must be at least $13 - n_6$ multiple points of multiplicity ≤ 5. Thus, $13 - n_6 \leq n_3 + n_4 + n_5$. Together with (1), we get $n_6 \leq \frac{59}{63} < 1$, a contradiction. □

4. Arrangements of 11 lines with a quintuple point and 2 quadruple points

In this section, we investigate arrangements of 11 lines with a quintuple point and no multiple points of higher multiplicities.

First, we show the possible values of the numerical invariants n_4, n_5 such that the arrangement is nonreductive.

Lemma 4.1 *Let $\mathcal{A}$ be a nonreductive arrangement of* 11 *lines in $\mathbb{CP}^2$ with a quintuple point and $n_r = 0$ for $r \geq 6$. Then $n_5 = 1$ and $n_4 \leq 2$.*

Proof By Lemma 2.6 and Theorem 2.7, we have $n_3 + n_4 \geq \frac{4}{9}(44 - 11n_5)$. On the other hand, it is easy to see that there must be at least $11 - n_5$ multiple points of multiplicity ≤ 4. Thus, $11 - n_5 \leq \frac{4}{9}(44 - 11n_5)$. It follows that $n_5 \leq 2$. If $n_5 = 2$ and these 2 quintuple points are not collinear, then it is easy to see that there is a line with at most 2 multiple points. If $n_5 = 2$ and these 2 quintuple points are collinear, let $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_{11}$ and $L_5 \cap L_6 \cap L_7 \cap L_8 \cap L_{11}$ be 2 quintuple points, and let L_{11} be the line at infinity. Each of L_9 and L_{10} must pass through 4 points of $L_i \cap L_j, i = 1, 2, 3, 4;\ j = 5, 6, 7, 8$. Assume that L_9 passes through $L_1 \cap L_8, L_2 \cap L_7, L_3 \cap L_6, L_4 \cap L_5$, and then to make the arrangement nonreductive, L_{10} should pass through $L_1 \cap L_6, L_2 \cap L_5, L_3 \cap L_8, L_4 \cap L_7$ and $L_9 \cap L_{10}$ is on L_{11}. After an easy computation, such an arrangement can not be realized. Therefore, $n_5 = 1$.

Also by Lemma 2.6 and Theorem 2.7, we obtain $\frac{9}{4}n_3 + 6n_4 \leq 33$. Since each line contains at least 3 multiple points, then there must be at least $11 - n_5 = 10$ multiple points. It follows that $n_4 \leq \frac{42}{15}$, and thus $n_4 \leq 2$. □

Theorem 4.2 *Let $\mathcal{A}$ be a nonreductive arrangement of* 11 *lines in $\mathbb{CP}^2$ with a qunituple point such that $n_4 = 2$ and $n_r = 0$ for $r \geq 6$. Then the moduli space $\mathcal{M}_{\mathcal{A}}$ is irreducible.*

Proof First, we assume that the quintuple point and a quadruple point are not collinear in $\mathcal{A}$. We show that there is a line containing only 2 multiple points. Let $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5$ be the quintuple point and let $L_6 \cap L_7 \cap L_8 \cap L_9$ be the quadruple point. Then the other quadruple point must be $L_i \cap L_j \cap L_{10} \cap L_{11}$ for some $i \in 1, 2, 3, 4, 5,\ j \in 6, 7, 8, 9$. Then L_i passes through at most 2 multiple points. If there are two noncollinear quadruple points, each one being collinear with the quintuple point, it is easy to see that the arrangement is *reductive.*

Second, we consider that any 2 of the quintuple points and 2 quadruple points are collinear, but all of them are not collinear. Let $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_{11}$, $L_5 \cap L_6 \cap L_7 \cap L_{11}$ and $L_2 \cap L_6 \cap L_9 \cap L_{10}$ be the quintuple point and 2 quadruple points. There must be another triple point on L_{11} so that it contains 3 multiple points. We may assume $L_8 \cap L_{10} \cap L_{11}$ is the triple point.

Case 1. $L_8 \cap L_9$ is not a triple point. It is easy to see that $L_9 \cup L_{10}$ pass through at most 4 points of $\Delta := \{L_i \cap L_j,\ i = 1,2,3,4;\ j = 5,6,7\}$ except $L_2 \cap L_6$. Thus, L_8 passes through 3 points of Δ and $L_9 \cup L_{10}$ pass through 5 points of Δ to make the arrangement nonreductive. Up to a lattice isomorphism, we may assume that L_8 passes through $\{L_2 \cap L_5,\ L_3 \cap L_6,\ L_4 \cap L_7\}$. Then $L_1 \cap L_5$ and $L_3 \cap L_7$ are on L_9 or L_{10}. Up to a permutation, we can assume they are on L_9, and then $L_1 \cap L_7$ and $L_4 \cap L_5$ are on L_{10} (see Figure 1).

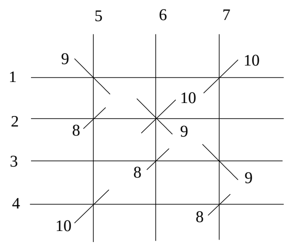

Figure 1.

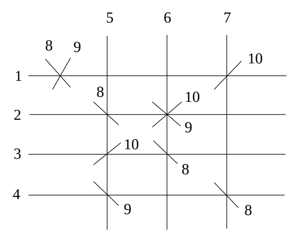

Figure 2.

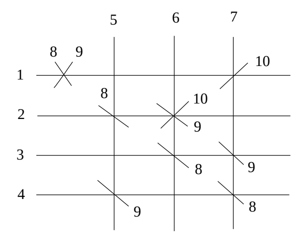

Figure 3.

After an easy computation, we see that Figure 1 cannot be realized.

Case 2. $L_8 \cap L_9$ is a triple point. We assume that $L_8 \cap L_9$ is on L_1, and up to a lattice isomorphism, we assume $L_1 \cap L_7$ is on L_{10} so that L_1 contains 3 multiple points. Note that $L_2 \cap L_5$ or $L_2 \cap L_7$ is on L_8 so that L_2 contains 3 multiple points and $L_3 \cap L_6$ or $L_4 \cap L_6$ is on L_8 so that L_6 contains 3 multiple points.

Subcase 1. $L_2 \cap L_5$ and $L_3 \cap L_6$ are on L_8.

(I). $L_4 \cap L_7$ is on L_8, so then $L_4 \cap L_5$ is on L_9 or L_{10} so that L_4 passes through 3 multiple points.

①**.** $L_4 \cap L_5$ is on L_9, so then $L_3 \cap L_5$ is on L_{10} (Figure 2) or $L_3 \cap L_7$ is on L_9 (Figure 3).

②**.** $L_4 \cap L_5$ is on L_{10}, so then $L_3 \cap L_5$ (Figure 4) or $L_3 \cap L_7$ is on L_9 (Figure 5).

(II). $L_4 \cap L_7$ is on L_9, so then $L_4 \cap L_5$ is on L_{10} and $L_3 \cap L_5$ is on L_9 so that L_3, L_4 pass through 3 multiple points (Figure 6).

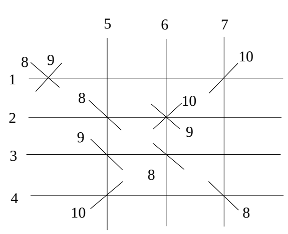

Figure 4.

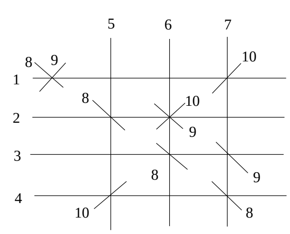

Figure 5.

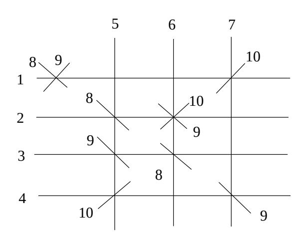

Figure 6.

After an easy computation, we see that Figures 2, 3, 4, 5, and 6 cannot be realized.

Subcase 2. $L_2 \cap L_7$ and $L_3 \cap L_6$ are on L_8. If $L_4 \cap L_5$ is on L_9, then L_4 contains only 2 multiple points, and thus $L_4 \cap L_5$ is on L_8 or L_{10}.

(I). $L_4 \cap L_5$ is on L_8, so then to make the arrangement nonreductive, $L_4 \cap L_7$ is on L_9 and $L_3 \cap L_5$ is on L_9 or L_{10} (Figure 7).

(II). $L_4 \cap L_5$ is on L_{10}, so then to make the arrangement nonreductive, $L_4 \cap L_7$ and $L_3 \cap L_5$ are on L_9 (Figure 8).

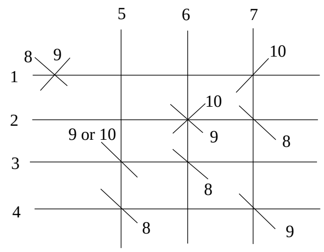

Figure 7.

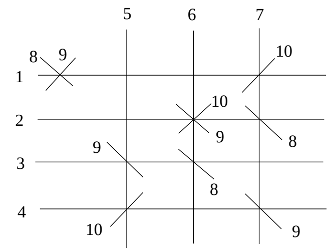

Figure 8.

After an easy computation, we see that Figures 7 and 8 cannot be realized.

Assume that the quintuple point and 2 quadruple points are collinear in $\mathcal{A}$. We assume that $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_{11}$, $L_5 \cap L_6 \cap L_7 \cap L_{11}$, and $L_8 \cap L_9 \cap L_{10} \cap L_{11}$ are the quintuple point and 2 quadruple points. It is easy to see that $L_8,\ L_9,\ L_{10}$ pass through 8 or 9 points of Δ so that $L_1,\ L_2,\ L_3,\ L_4$ contain at least 3 multiple points each.

If $L_8,\ L_9,\ L_{10}$ pass through 9 points of Δ, then each of $L_5,\ L_6,\ L_7,\ L_8,\ L_9,\ L_{10}$ contains 4 multiple points. Then $\mathcal{A}' = \mathcal{A} \backslash \{L_i\},\ i \in \{1,2,3,4\}$ is a line arrangement of 10 lines with 3 quadruple points, which are collinear in $\mathcal{A}'$. By the last paragraph in the proof [2, Theorem 4.2], $\mathcal{M}_{\mathcal{A}'}$ is irreducible. Hence, $\mathcal{M}_{\mathcal{A}}$ is irreducible.

If $L_8,\ L_9,\ L_{10}$ pass through 8 points of Δ, then each of $L_1,\ L_2,\ L_3,\ L_4$ passes through 2 triple points. We assume that 8 points of Δ are $L_2 \cap L_5,\ L_3 \cap L_5,\ L_4 \cap L_5,\ L_1 \cap L_6,\ L_4 \cap L_6,\ L_1 \cap L_7,\ L_2 \cap L_7,\ L_3 \cap L_7$. Furthermore, we assume that $L_1 \cap L_6$ is on L_8 and $L_4 \cap L_6$ is on L_9.

(I). $L_2 \cap L_5$ is on L_8, so then we see that $L_3 \cap L_7$ is on L_8, $L_3 \cap L_5$ is on L_9, and $L_4 \cap L_5$ is on L_{10}.

① $L_1 \cap L_7$ is on L_9 and $L_2 \cap L_7$ is on L_{10} (Figure 9).

② $L_1 \cap L_7$ is on L_{10} and $L_2 \cap L_7$ is on L_9 (Figure 10).

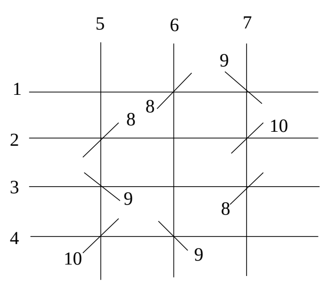

Figure 9.

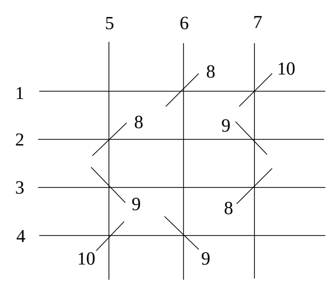

Figure 10.

After an easy computation, we conclude that Figures 9 and 10 cannot be realized.

(II). $L_3 \cap L_5$ is on L_8, so then we see that $L_2 \cap L_7$ is on L_8, $L_2 \cap L_5$ is on L_9, and $L_4 \cap L_5$ is on L_{10}. Exchanging $L_2,\ L_3$, we see it is lattice isomorphic to **(1)**.

(III). $L_4 \cap L_5$ is on L_8, and up to a lattice isomorphism, we assume that $L_3 \cap L_5$ is on L_9, so then $L_2 \cap L_5$ is on L_{10} (Figures 11 and 12).

① $L_1 \cap L_7$ is on L_9, so then $L_2 \cap L_7$ is on L_8, $L_3 \cap L_7$ is on L_{10}.

② $L_2 \cap L_7$ is on L_9, so then $L_3 \cap L_7$ is on L_8 , $L_1 \cap L_7$ is on L_{10}.

After an easy computation, we conclude that Figures 11 and 12 cannot be realized.

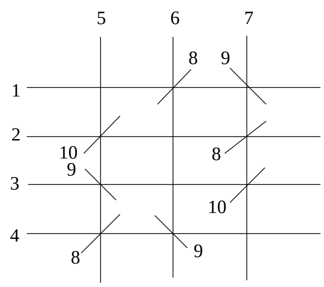

Figure 11.

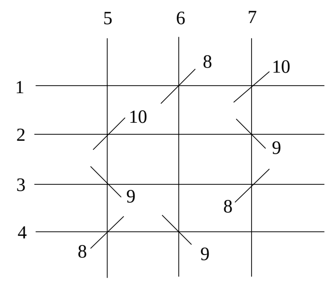

Figure 12.

Thus, $\mathcal{M}_{\mathcal{A}}$ is irreducible.

□

5. Arrangements of 11 lines with a quintuple point and exactly 1 quadruple point

In this section, we investigate an arrangement of 11 lines with a quintuple point and exactly 1 quadruple point.

Lemma 5.1 *Let $\mathcal{A}$ be a nonreductive arrangement of 11 lines in $\mathbb{CP}^2$ with $n_5 = n_4 = 1$ and $n_r = 0$ for $r \geq 6$. If the quintuple point and the quadruple point are not collinear, then $\mathcal{M}_{\mathcal{A}}$ is empty.*

Proof Assume that $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5$ is the quintuple point and $L_6 \cap L_7 \cap L_8 \cap L_9$ is the quadruple point. Since L_{10} and L_{11} pass through at most 8 triple points of $\{L_6 \cup L_7 \cup L_8 \cup L_9\}$, then one of $\{L_1, L_2, L_3, L_4, L_5\}$ contains at most 2 multiple points, and then $\mathcal{A}$ is reductive, contradiction. Then $\mathcal{M}_{\mathcal{A}}$ is empty. □

In the following theorem, we assume that the quintuple point and the quadruple point are collinear.

Theorem 5.2 *Let $\mathcal{A}$ be a nonreductive arrangement of 11 lines in $\mathbb{CP}^2$ with $n_5 = n_4 = 1$ and $n_r = 0$ for $r \geq 6$. If the quintuple point and the quadruple point are collinear, then $\mathcal{M}_{\mathcal{A}}$ or $\mathcal{M}^c_{\mathcal{A}}$ is irreducible except in the cases of Figures 14, 15, 17, 26, 27, 30, 31, 32, 33, 35, 37, 46, 55, 56, 57, 58, and 60 and the corresponding arrangements of these figures are "potential Zariski pairs".*

Proof Assume that $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_{11}$ is the quintuple point and $L_5 \cap L_6 \cap L_7 \cap L_{11}$ is the quadruple point. Then one of $\{L_8 \cap L_9, L_8 \cap L_{10}, L_9 \cap L_{10}\}$ is on L_{11} so that it contains at least 3 multiple points. We may assume $L_8 \cap L_9$ is on L_{11}.

Case 1. Neither of $L_8 \cap L_{10}$ and $L_9 \cap L_{10}$ is a triple point. Then L_{10} must pass through at 3 points of Δ. Then L_8, L_9 must pass through at least 5 points of Δ so that L_1, L_2, L_3, L_4 contains at least 3 multiple points.

Subcase 1. Both L_8 and L_9 pass through 3 points of Δ. Let L_4 be the line such that $L_4 \cap L_{10}$ is not a triple point and let $\mathcal{A}' = \mathcal{A} \backslash \{L_4\}$. Then $\mathcal{A}'$ is an arrangement of 10 lines with 2 quadruple points on the same line and none of $L_{10} \cap (L_8 \cup L_9)$ is a triple point, it is just [2, Theorem 4.4, Case 1]. Then $\mathcal{M}_{\mathcal{A}'}$ is either empty or irreducible, and then $\mathcal{M}_{\mathcal{A}}$ is either empty or irreducible.

Subcase 2. One of L_8, L_9 passes through 2 points of Δ. We assume that L_8 passes through 2 points of Δ. Up to a lattice isomorphism, we assume that $\{L_1 \cap L_7, L_2 \cap L_6, L_3 \cap L_5\}$ are on L_{10}. To make L_4 pass through at least 3 multiple points, L_8, L_9 must pass through 2 points of $\{L_4 \cap L_5, L_4 \cap L_6, L_4 \cap L_7\}$. Up to a permutation, let L_8 contain $L_4 \cap L_6$ and let L_9 contain $L_4 \cap L_5$.

(I). $L_1 \cap L_5$ is on L_8. It is easy to see that $L_2 \cap L_7$ and $L_3 \cap L_6$ are on L_9 so that L_2, L_3 pass through 3 multiple points (Figure 13).

(II). $L_2 \cap L_5$ is on L_8. To make L_1, L_3 pass through 3 multiple points, $L_1 \cap L_6$ and $L_3 \cap L_7$ must be on L_9 (Figure 14).

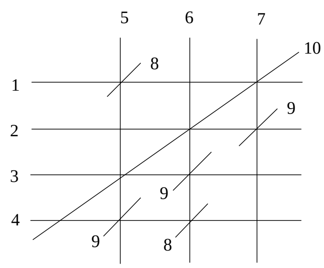

Figure 13.

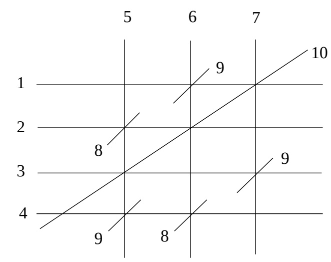

Figure 14.

(III). $L_2 \cap L_7$ is on L_8. Obviously, L_9 must pass through $L_1 \cap L_6$ and $L_3 \cap L_7$ so that L_1, L_3 pass through 3 multiple points (see Figure 15).

(IV). $L_3 \cap L_7$ is on L_8. Note that $L_1 \cap L_6$ and $L_2 \cap L_7$ should be on L_9, and then L_1, L_2 contains 3 multiple points (see Figure 16).

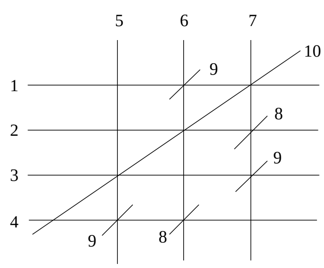

Figure 15.

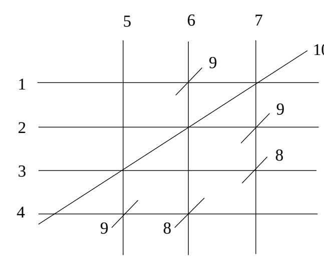

Figure 16.

An easy computation shows that Figures 13, 14, 15, and 16 cannot be realized.

Case 2. One of $(L_8 \cup L_9) \cap L_{10}$ is a triple point in $\mathcal{A}$. We assume that $L_8 \cap L_{10}$ is a triple point, and then L_{10} passes through 2 or 3 points of Δ.

Subcase 1. L_{10} passes through 3 points of Δ. We assume that $L_1 \cap L_7, L_2 \cap L_6, L_3 \cap L_5$ are on L_{10} and $L_8 \cap L_{10}$ is on L_4. Note that $(L_8 \cup L_9)$ contain at least 4 points of Δ so that L_1, L_2, L_3, L_4 pass through at least 3 multiple points.

(I). Both L_8 and L_9 contain 2 points of Δ. Note that L_9 must pass through one of $\{L_4 \cap L_5, L_4 \cap L_6, L_4 \cap L_7\}$. Up to a lattice isomorphism, we assume $L_4 \cap L_5$ is on L_9.

① $L_1 \cap L_6$ is on L_9, and then $L_2 \cap L_5, L_3 \cap L_7$ or $L_2 \cap L_7, L_3 \cap L_6$ is on L_8 so that L_1, L_2, L_3, L_4 pass through at least 3 multiple points (Figures 17 and 18).

Figure 17 can be defined by the following equation:

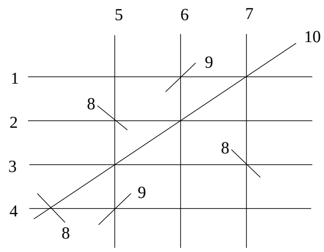

Figure 17.

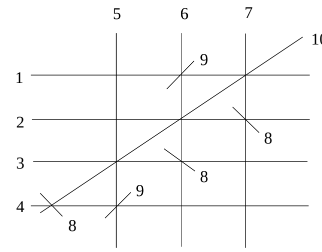

Figure 18.

$$XYZ(X-Z)(X+tZ)(Y-Z)(Y-tZ)(Y-t^2Z)(Y-(t-t^2)X-t^2Z)(tY+X)(Y+(t-1)X-tZ)=0,$$

where t satisfies $t^3-t^2-1=0$.

Figure 18 can be defined by the following equation:

$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(t_1Y-X)((1-t_1)Y-(t_3-t_2)(X-1)-t_3Z)(Y+(t_2-1)X-t_2Z)=0$, where $t_1=t, t_2=2t^2+5t-3, t_3=t^2+3t-1$ and t satisfies $t^3+2t^2-3t+1=0$.

② $L_3\cap L_6$ is on L_9, and then $L_2\cap L_7, L_1\cap L_5$ or $L_2\cap L_7, L_1\cap L_6$ is on L_8 so that L_1, L_2, L_7 pass through at least 3 multiple points (Figures 19 and 20).

Figure 19 cannot be realized. Figure 20 can be defined by the following equation:

$$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(t_1Y-t_2Z)(Y-(t_3-1)X-Z)(Y+(t_2-1)X-t_2Z)=0,$$

where $t_1=t^2+t+1, t_2=t, t_3=t^2+t$, and t satisfies $t^3+t^2-1=0$.

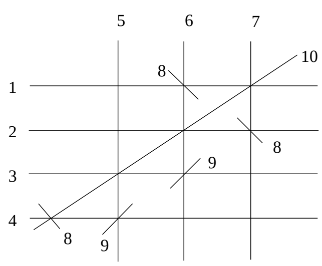

Figure 19.

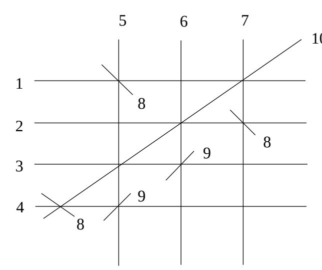

Figure 20.

③ $L_2\cap L_7$ is on L_9; it is lattice isomorphic to ①.

④ $L_3\cap L_7$ is on L_9; it is lattice isomorphic to ②.

(II). L_8 contains 2 points of Δ and L_9 contains 3 points of Δ. Then L_9 passes through one of $\{(L_1\cap L_6, L_2\cap L_7), (L_1\cap L_6, L_3\cap L_7)\}, (L_3\cap L_6, L_2\cap L_7)$.

① $L_1\cap L_6, L_2\cap L_7$ is on L_9, and then $L_3\cap L_6$ or $L_3\cap L_7$ is on L_8; up to a permutation, we assume that $L_3\cap L_6$ is on L_8. Then $L_1\cap L_5$ $L_2\cap L_5$ is on L_8 (Figures 21 and 22).

Figure 21 cannot be realized.

Figure 22 can be defined by the following equation:

$$XYZ(X+2Z)(X-Z)(Y-Z)(2Y-Z)(2Y+Z)(X+2Y)(X+2Y+Z)(X-2Y+Z)=0.$$

② $L_1\cap L_6, L_3\cap L_7$ is on L_9. Then $L_2\cap L_5$ or $L_2\cap L_7$ is on L_8 so that L_2 contains 3 multiple points. If $L_2\cap L_5$ is on L_8, then $L_3\cap L_6$ is on L_8 (Figure 23). If $L_2\cap L_7$ is on L_8, then $L_1\cap L_5$ or $L_3\cap L_6$ is on L_8 (Figures 24 and 25).

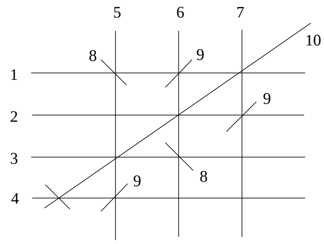

Figure 21.

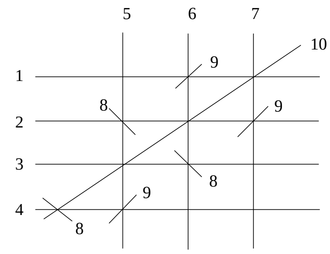

Figure 22.

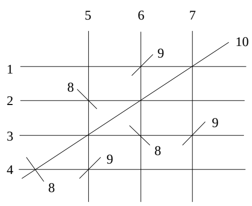

Figure 23.

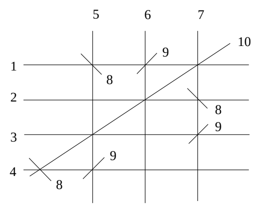

Figure 24.

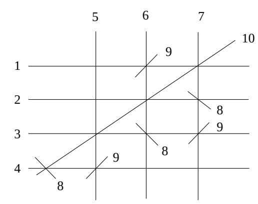

Figure 25.

Figures 23, 24, and 25 cannot be realized.

③ $L_3 \cap L_6, L_2 \cap L_7$ is on L_9. It is lattice isomorphic to ② (under permutation (6,7)(1,2)).

(III). L_8 contains 3 points of Δ and L_9 contains 2 points of Δ. Since $L_4 \cap L_5$ is on L_9, then one of $\{L_1 \cap L_6, L_3 \cap L_6, L_2 \cap L_7, L_3 \cap L_7\}$ is on L_9.

① $L_1 \cap L_6$ is on L_9. To make L_2, L_3 contain at least 3 multiple points, $\{L_1 \cap L_5, L_2 \cap L_7, L_3 \cap L_6\}$ are on L_8 (Figure 26).

② $L_3 \cap L_6$ is on L_9. To make L_1, L_2 contain at least 3 multiple points, $\{L_1 \cap L_6, L_2 \cap L_5, L_3 \cap L_7\}$ are on L_8 (Figure 27).

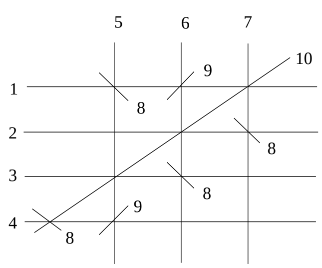

Figure 26.

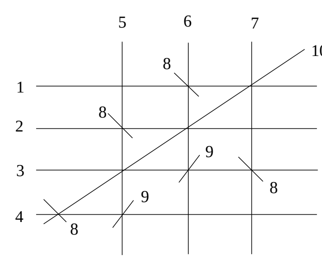

Figure 27.

Figures 26 and 27 cannot be realized.

③ $L_2 \cap L_7$ is on L_9. After a permutation $(6,7)(1,2)$, it is isomorphic to ①.

④ $L_3 \cap L_7$ is on L_9. After a permutation $(6,7)(1,2)$, it is isomorphic to ②.

(IV). L_8 contains 3 points of Δ and L_9 contains 3 points of Δ. Since **(III)** cannot be realized, case **(IV)** cannot be realized.

(V). L_8 contains 1 point of Δ and L_9 contains 3 points of Δ. From **(III)**, we need to remove one L_8 intersecting with Δ, and it is easy to see that there are 3 cases (see Figures 28, 29, and 30).

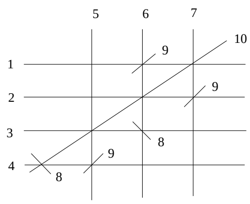

Figure 28.

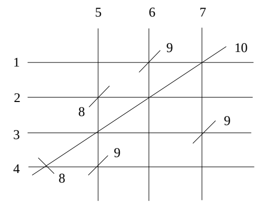

Figure 29.

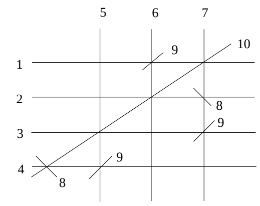

Figure 30.

Figure 28 cannot be realized.

Figure 29 can be defined by the following equation:

$$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(t_1Y-X)(t_1Y-X-t_1t_3Z)(Y+(t_2-1)X-t_2Z)=0,$$

where $t_1=2t^2-2t+2, t_2=t, t_3=2t^2-t+1$, and t satisfies $2t^3-2t^2+2t-1=0$.

Figure 30 can be defined by the following equation:

$$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(t_1Y-X)(t_1Y-X-(t_1t_3-1)Z)(Y+(t_2-1)X-t_2Z)=0,$$

where $t_1=t^2-t+2, t_2=t, t_3=t^2+1$, and t satisfies $t^3-t^2+2t-1=0$.

Subcase 2. L_{10} passes through 2 points of Δ. We assume that $L_1\cap L_7, L_2\cap L_6$ are on L_{10} and $L_8\cap L_{10}$ is on L_4. Note that $(L_8\cup L_9)$ contain at least 5 points of Δ so that L_1,L_2,L_3,L_4 pass through at least 3 multiple points. To make L_3,L_4 contain at least 3 multiple points, $L_9\cap(L_3\cup L_4)$ is a triple point.

(I). L_8 contains 3 points of Δ and L_9 contains 2 points of Δ.

① $L_4\cap L_5$ is on L_9.

(a) $L_3\cap L_6$ is on L_9. Then $L_1\cap L_6$ is on L_8, and $(L_2\cap L_5, L_3\cap L_7)$ or $(L_2\cap L_7, L_3\cap L_5)$ is on L_8 (see Figures 28 and 29).

(b) $L_3\cap L_7$ is on L_9. By a permutation, it is lattice isomorphic to the previous case.

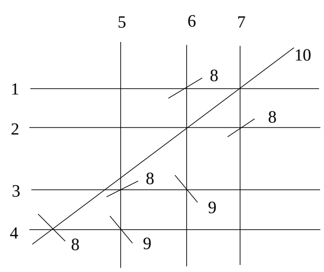

Figure 31.

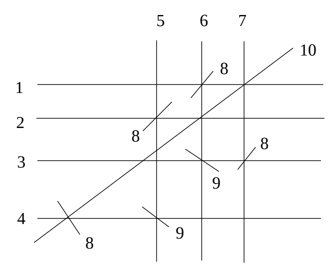

Figure 32.

Figure 31 can be defined by the following equation:

$$XYZ(X-Z)(3X+Z)(Y-Z)(2Y-Z)(Y+Z)(2Y+3X-Z)(2Y+3X)(2Y-3X+Z)=0.$$

Figure 32 can be defined by the following equation:

$$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y+(t_3-t_2)X-t_3Z)(t_1Y-t_2Z)(Y-\tfrac{1-t_3}{1-t_1}X-\tfrac{t_3-t_1}{1-t_1}Z)=0,$$

where $t_1=\frac{1}{3}+\frac{1}{3}t, t_2=1-t, t_3=t$, and t satisfies $t^2-t+1=0$.

② $L_4\cap L_6$ is on L_9. Then $L_3\cap L_5$ is on L_9 and $L_1\cap L_5$ or $L_2\cap L_5$ is on L_8 so that L_5 passes through 3 multiple points.

(a) $L_1\cap L_5$ is on L_8. Then $L_2\cap L_7$ or $L_3\cap L_6$ is on L_8 so that $\mathcal{A}$ is nonreductive (Figure 33).

(b) $L_2 \cap L_5$ is on L_8 Then $L_1 \cap L_6$ or $L_3 \cap L_7$ is on L_8 so that $\mathcal{A}$ is nonreductive (Figure 34).

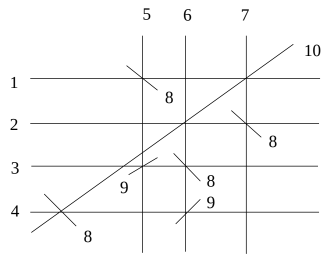

Figure 33.

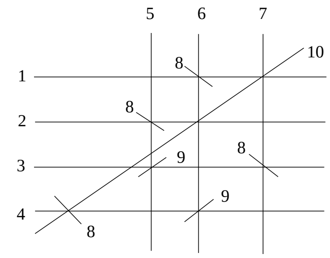

Figure 34.

Figure 33 can be defined by the following equation:

$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-(t_3-1)X-Z)(Y+\frac{t_2}{t_1}X-t_2Z)(Y-\frac{1-t_3}{1-t_1}X-\frac{t_3-t_1}{1-t_1}Z)=0$, where $t_1=\pm t, t_2=\frac{1}{2}, t_3=\pm t-1$, and t satisfies $2t^2-4t+1=0$.

Figure 34 can be defined by the following equation:

$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-(t_2-t_3)X-t_3Z)(Y+\frac{t_2}{t_1}X-t_2Z)(Y-\frac{1-t_3}{1-t_1}X-\frac{t_3-t_1}{1-t_1}Z)=0$, where $t_1=\pm t-1, t_2=\pm t-1, t_3=\pm t$, and t satisfies $t^2-t-1=0$.

(II). L_8 contains 2 points of Δ and L_9 contains 3 points of Δ.

① $L_4 \cap L_5$ is on L_9. Then $(L_3 \cap L_6, L_2 \cap L_7)$ or $(L_1 \cap L_6, L_3 \cap L_7)$ are on L_9. Up to a permutation $(6,7)(1,2)$, we may assume that $(L_3 \cap L_6, L_2 \cap L_7)$ are on L_9. Then $(L_1 \cap L_5, L_3 \cap L_7)$ or $(L_1 \cap L_6, L_3 \cap L_5)$ are on L_8 (Figures 35 and 36).

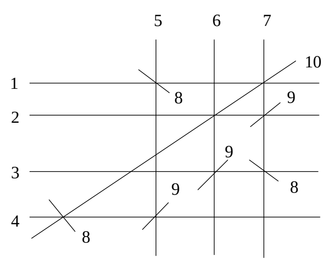

Figure 35.

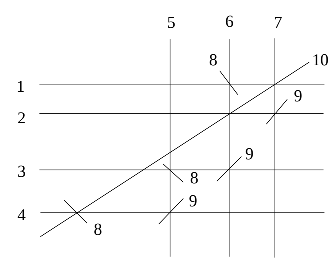

Figure 36.

Figure 35 can be defined by the following equation:

$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_3X)(Y-\frac{1-t_2}{t_1}X-t_2Z)(Y-\frac{1-t_3}{1-t_1}X-\frac{t_3-t_1}{1-t_1}Z)=0$, where $t_1=\frac{1}{2}t+\frac{1}{2}, t_2=\frac{1}{2}, t_3=t$, and t satisfies $t^2-2t-1=0$.

Figure 36 can be defined by the following equation:

$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_3X)(Y-(t_2-1)X-Z)(Y-\frac{1-t_3}{1-t_1}X-\frac{t_3-t_1}{1-t_1}Z)=0$, where $t_1=\frac{1}{4}+\frac{1}{2}t, t_2=t, t_3=t-1$, and t satisfies $2t^2+t-2=0$.

② $L_4 \cap L_6$ is on L_9. Then $L_3 \cap L_5$ or $L_3 \cap L_7$ is on L_9.

If $L_3 \cap L_5$ is on L_9, then $(L_1 \cap L_5, L_3 \cap L_6)$ or $(L_1 \cap L_5, L_3 \cap L_7)$ are on L_8 so that $\mathcal{A}$ is nonreductive (Figures 37 and 38).

Figure 37 can be defined by the following equation:

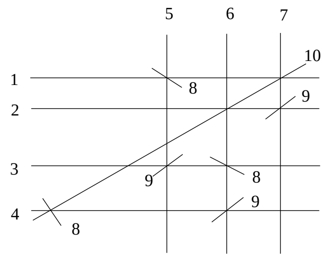

Figure 37.

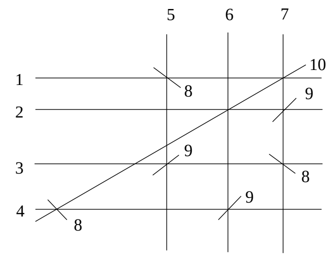

Figure 38.

$XYZ(X-Z)(3X-Z)(Y-Z)(2Y-Z)(Y+Z)(2Y+3X-2Z)(2Y+3X-Z)(Y-3X+2Z)=0$.

Figure 38 can be defined by the following equation:

$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-(t_3-t_2)X-t_2Z)(Y-(t_2-1)X-Z)(Y-\frac{1-t_3}{1-t_1}X-\frac{t_3-t_1}{1-t_1}Z)=0$, where $t_1=2t+1, t_2=t, t_3=2t-1$, and t satisfies $2t^2-1=0$.

If $L_3 \cap L_7$ is on L_9, then $L_1 \cap L_5$ or $L_2 \cap L_5$ is on L_9 so that L_5 contains at least 3 multiple points.

(1). $L_1 \cap L_5$ is on L_9, so then $(L_2 \cap L_5, L_3 \cap L_6)$ or $(L_2 \cap L_7, L_3 \cap L_5)$ is on L_8 so that $\mathcal{A}$ is nonreductive (Figures 39 and 40).

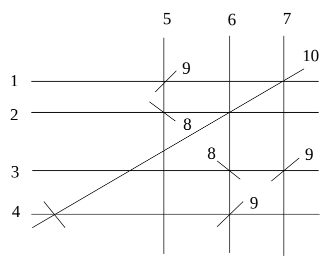

Figure 39.

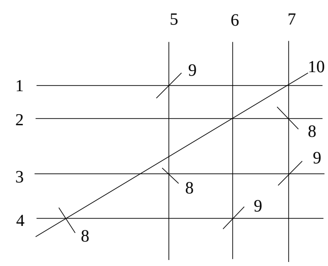

Figure 40.

Figure 39 can be defined by the following equation:

$XYZ(X-Z)(3X-2Z)(Y-Z)(2Y+Z)(Y-2Z)(2Y+3X-2Z)(2Y+3X-Z)(2Y-3X-Z)=0$.

Figure 40 can be defined by the following equation:

$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-(t_3-t_2)X-t_2Z)(Y-(t_2-1)X-Z)(Y-\frac{1-t_3}{1-t_1}X-\frac{t_3-t_1}{1-t_1}Z)=0$, where $t_1=t, t_2=\frac{1}{4}t+\frac{1}{4}, t_3=-\frac{1}{2}+\frac{1}{2}t$, and t satisfies $t^2-3t+4=0$.

(2). $L_2 \cap L_5$ is on L_9, so then $(L_1 \cap L_5, L_3 \cap L_6)$ or $(L_1 \cap L_6, L_3 \cap L_5)$ are on L_8 so that $\mathcal{A}$ is nonreductive (Figures 41 and 42).

Figure 41 cannot be realized.

Figure 42 can be defined by the following equation:

$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-(t_2-t_3)X-t_3Z)(Y-\frac{1-t_2}{t_1}X-t_2Z)(Y-\frac{1-t_3}{1-t_1}X-\frac{t_3-t_1}{1-t_1}Z)=0$, where $t_1=-t, t_2=1+t, t_3=t$, and t satisfies $t^2+2t-1=0$.

(3). $L_4 \cap L_7$ is on L_9. Up to a permutation $(6,7)(1,2)$, it is lattice isomorphic to (2).

(III). Both L_8 and L_9 contain 3 points of Δ. We only need to add 1 point of Δ to L_8 for **(I)** or to L_9 for **(II)**, so we obtain 5 cases (Figures 43, 44, 45, 46, and 47).

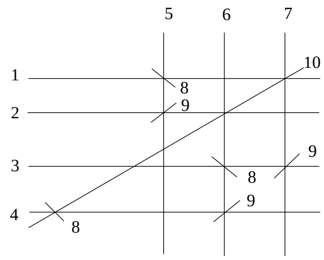

Figure 41.

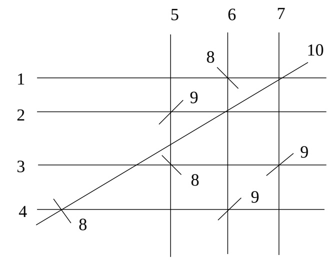

Figure 42.

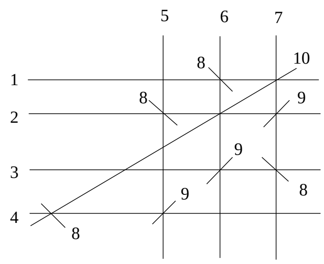

Figure 43.

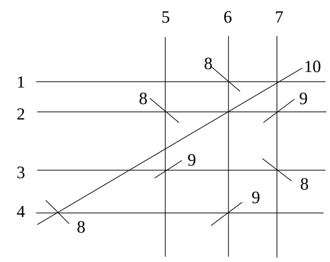

Figure 44.

Figures 43, 44, 45, 46, and 47 cannot be realized.

Case 3. $(L_8 \cup L_9) \cap L_{10}$ are triple points in $\mathcal{A}$. Then L_{10} passes through at least 1 point of Δ so that it contains at least 3 multiple points.

Subcase 1. L_{10} passes through 2 points of Δ. We assume that L_{10} passes through $(L_1 \cap L_7, L_2 \cap L_6)$, $L_8 \cap L_{10}$ is on L_4, and $L_9 \cap L_{10}$ is on L_3. Note that $L_8 \cup L_9$ pass through at least 4 points of Δ so that $\mathcal{A}$ is nonreductive.

(I). L_8 contains 3 points of Δ and L_9 contains 1 point of Δ. To make L_4, L_5 contain at least 3 multiple points, $L_4 \cap L_5$ is on L_9.

① $L_3 \cap L_5$ is on L_8. Then $L_1 \cap L_6, L_2 \cap L_7$ are on L_8 (Figure 48).

② $L_3 \cap L_6$ is on L_8. Then $L_1 \cap L_5, L_2 \cap L_7$ are on L_8 (Figure 49).

③ $L_3 \cap L_7$ is on L_8. Then $L_1 \cap L_6, L_2 \cap L_5$ are on L_8. After a permutation $(6,7)(1,2)$, it is lattice isomorphic to ②.

Figure 48 cannot be realized.

Figure 49 can be defined by the following equation:

$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-(t_3-1)X)(Y-(t_3-1)X-Z)(Y-\frac{1-t_3}{1-t_1}X-\frac{t_3-t_1}{1-t_1}Z) = 0$, where $t_1 = t, t_2 = 1+t^2-2t, t_3 = t-1$, and t satisfies $t^3 - 4t^2 + 5t - 3 = 0$.

(II). L_8 contains 3 points of Δ and L_9 contains 2 points of Δ.

① $L_3 \cap L_5$ is on L_8. Then $L_1 \cap L_6, L_2 \cap L_7$ are on L_8. To make L_4, L_5 contain at least 3 multiple points, up to a permutation $(6,7)(1,2)$, $L_1 \cap L_5, L_4 \cap L_6$, or $L_2 \cap L_5, L_4 \cap L_6$ must be on L_9 (Figures 50 and 51).

② $L_3 \cap L_6$ is on L_8. Then $L_1 \cap L_5, L_2 \cap L_7$ are on L_8. To make L_4, L_5 contain at least 3 multiple points, L_9 passes through one of $\{(L_4 \cap L_5, L_1 \cap L_6), (L_4 \cap L_6, L_2 \cap L_5), (L_4 \cap L_7, L_2 \cap L_5)\}$ (Figures 52, 53, and 54).

Figures 50, 51, 52, 53, and 54 cannot be realized.

③ $L_3 \cap L_7$ is on L_8. Up to a permutation $(6,7)(1,2)$, it is lattice isomorphic to ②.

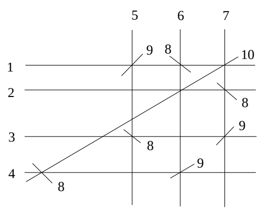

Figure 45.

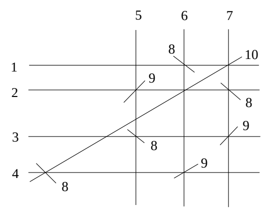

Figure 46.

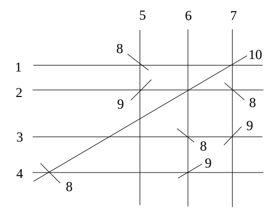

Figure 47.

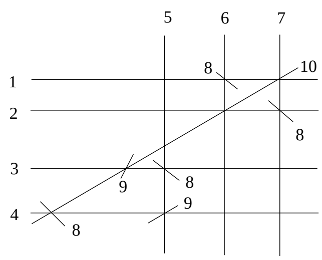

Figure 48.

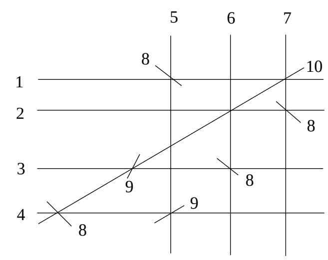

Figure 49.

(III). Both L_8 and L_9 contain 3 points of Δ.

① $L_3 \cap L_5$ is on L_8, and it is easy to see L_9 passes through at most 2 points of Δ.

② $L_3 \cap L_6$ is on L_8, so then $L_1 \cap L_5, L_2 \cap L_7$ are on L_8 and L_9 passes through $L_1 \cap L_6, L_2 \cap L_5, L_4 \cap L_7$ (Figure 55).

③ $L_3 \cap L_7$ is on L_8. Up to a permutation $(6,7)(1,2)$, it is lattice isomorphic to ②.

Figure 55 cannot be realized.

(IV). L_8 contains 1 or 2 points of Δ and L_9 contains 3 points of Δ. Up to a permutation $(8,9)(3,4)$, it is lattice isomorphic to **(I)** or **(II)**.

(V). Both L_8 and L_9 contain 2 points of Δ.

If $L_4 \cap L_5$ is on L_9, then $L_1 \cap L_6$ or $L_2 \cap L_7$ is on L_9.

① $L_1 \cap L_6$ is on L_9. Then $(L_2 \cap L_5, L_3 \cap L_7)$ or $(L_2 \cap L_7, L_3 \cap L_5)$ are on L_8 so that $\mathcal{A}$ is nonreductive (Figures 56 and 57).

② $L_2 \cap L_7$ is on L_9. Up to a permutation $(6,7)(1,2)$, it is lattice isomorphic to ①.

Figures 56 and 57 cannot be realized.

If $L_4 \cap L_6$ is on L_9, then $L_1 \cap L_5$ or $L_2 \cap L_5$ is on L_9.

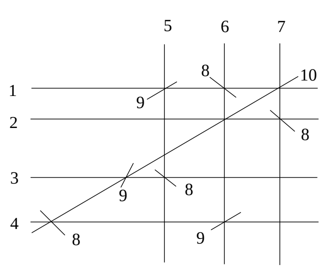

Figure 50.

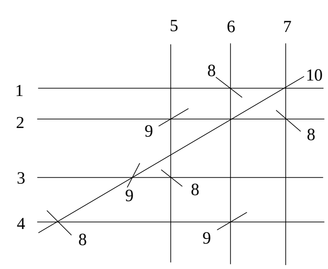

Figure 51.

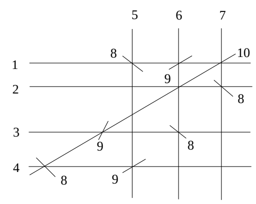

Figure 52.

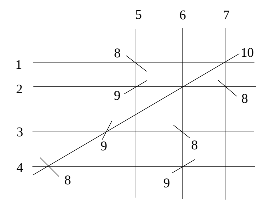

Figure 53.

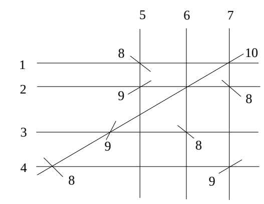

Figure 54.

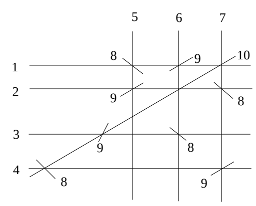

Figure 55.

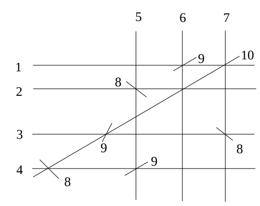

Figure 56.

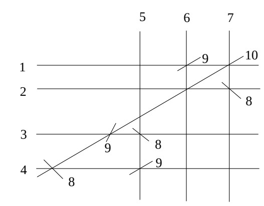

Figure 57.

① $L_1 \cap L_5$ is on L_9. Then $(L_2 \cap L_5, L_3 \cap L_7)$ or $(L_2 \cap L_7, L_3 \cap L_5)$ are on L_8 so that $\mathcal{A}$ is nonreductive (Figures 58 and 59).

② $L_2 \cap L_5$ is on L_9. Then $(L_1 \cap L_5, L_3 \cap L_7)$ are on L_8 so that $\mathcal{A}$ is nonreductive (Figure 60).

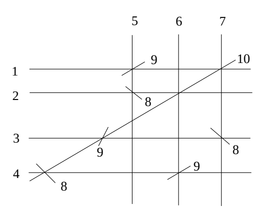

Figure 58.

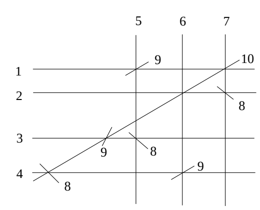

Figure 59.

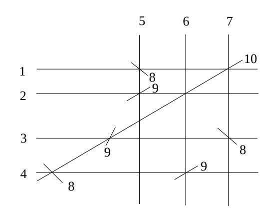

Figure 60.

Figure 58 can be defined by the following equation:

$$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-(t_2-t_3)X)(Y-(t_2-t_3)X-t_3Z)(Y-\tfrac{1-t_3}{1-t_1}X-\tfrac{t_3-t_1}{1-t_1}Z)=0,$$ where $t_1=-1, t_2=1\pm i, t_3=\pm i$.

Figure 59 can be defined by the following equation:

$$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-(t_3-t_2)X-Z)(Y-(t_3-t_2)X-t_2Z)(Y-\tfrac{1-t_3}{1-t_1}X-\tfrac{t_3-t_1}{1-t_1}Z)=0,$$ where $t_1=2t-t^2, t_2=1+t^2, t_3=t$, and t satisfies $t^3-2t^2+t-1=0$.

Figure 60 can be defined by the following equation:

$$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-(t_2-1)X-Z)(Y-(t_2-1)X-t_3Z)(Y-\tfrac{1-t_3}{1-t_1}X-\tfrac{t_3-t_1}{1-t_1}Z)=0,$$ where $t_1=-1, t_2=\pm\sqrt{2}, t_3=\sqrt{2}-1$.

Subcase 2. L_{10} passes through 1 point of Δ. Then $L_8 \cup L_9$ passes through at least 5 points of Δ. We assume that $L_1 \cap L_7$ is on L_{10}, $L_8 \cap L_{10}$ is on L_4, and $L_9 \cap L_{10}$ is on L_3.

(I). $L_8 \cup L_9$ passes through 5 points of Δ. We assume that L_8 contains 2 points of Δ and L_9 contains 3 points of Δ.

① $L_4 \cap L_5$ is on L_9. Then $L_1 \cap L_6, L_2 \cap L_7$ are on L_9. To make L_2, L_3, L_5, L_6 pass through at least 3 multiple points, L_8 passes through $(L_2 \cap L_5, L_3 \cap L_6)$ or $(L_2 \cap L_6, L_3 \cap L_5)$ (Figures 61 and 62).

② $L_4 \cap L_6$ is on L_9. Up to a permutation $(5,6)$, it is lattice isomorphic to ①.

③ $L_4 \cap L_7$ is on L_9. L_9 passes through $(L_1 \cap L_5, L_2 \cap L_6)$ or $(L_1 \cap L_6, L_2 \cap L_5)$. Up to a permutation $(5,6)$, we assume $(L_1 \cap L_5, L_2 \cap L_6)$ are on L_9 (Figure 63).

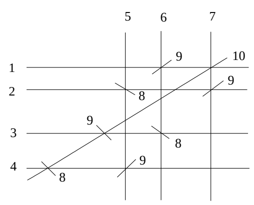

Figure 61.

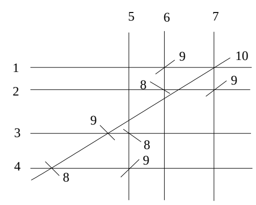

Figure 62.

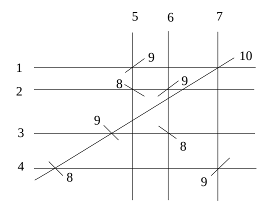

Figure 63.

Figure 61 can be defined by the following equation:

$$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_3X)(Y-t_3X-t_3Z)(Y-\tfrac{1-t_3}{1-t_1}X-\tfrac{t_3-t_1}{1-t_1}Z)=0,$$

where $t_1 = \pm\sqrt{2}, t_2 = 1 \pm \frac{\sqrt{2}}{2}, t_3 = \pm\frac{\sqrt{2}}{2}$.

Figure 62 can be defined by the following equation:

$$XYZ(X-Z)(3X-2Z)(Y-Z)(2Y-Z)(2Y-3Z)(2Y-3X)(2Y-3X-Z)(2Y+3X+5Z)=0.$$

Figure 63 can be defined by the following equation:

$$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y+X-Z)(Y+X-t_3Z)(Y-\tfrac{1-t_3}{1-t_1}X-\tfrac{t_3-t_1}{1-t_1}Z)=0,$$

where $t_1 = \frac{-1\pm\sqrt{3}}{2}, t_2 = 2 \pm \sqrt{3}, t_3 = \frac{3\pm\sqrt{3}}{2}$.

(II). $L_8 \cup L_9$ passes through 6 points of Δ. It is easy to see from **(I)** that this case is impossible. □

6. Arrangements of 11 lines with a quintuple point and no quadruple point

Let $\mathcal{A}$ be a nonreductive arrangement of 11 with a quintuple point and no quadruple point. By Lemma 2.6 and Theorem 2.7, we know that there are at most 14 triple points.

We say that 2 multiple points of $\mathcal{A}$ are disjoint if they are not on the same line of $\mathcal{A}$. We say that 2 multiple points of $\mathcal{A}$ are adjoint if they are on the same line of $\mathcal{A}$.

We claim that there is at most 1 disjoint triple point apart from the quintuple point. Assume that $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5$ is the quintuple point and $L_6 \cap L_7 \cap L_8$ is the triple point apart from the quintuple point. Suppose there is another triple point apart from the quintuple point. It is easy to see that $L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5$ pass through at most 9 triple points, but $L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5$ pass through at least 10 triple points, a contradiction.

6.1. One disjoint triple point apart from the pencil of the quintuple point

First, we show that there are at most 13 triple points in $\mathcal{A}$.

Lemma 6.1 *Let $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5$ be the quintuple point and let $L_6 \cap L_7 \cap L_8$ be the triple point apart from the quintuple point. Then there are at most 13 triple points in $\mathcal{A}$.*

Proof By Bézout's theorem, the intersection number of $(L_1 \cup L_2 \cup \cdots \cup L_8)$ and $(L_9 \cup L_{10} \cup L_{11})$ is 24. Since the intersection multiplicity of a triple point is 2, there will be at most 12 triple points in $(L_1 \cup L_2 \cup \cdots \cup L_8) \cap (L_9 \cup L_{10} \cup L_{11})$. In addition with $L_6 \cap L_7 \cap L_8$, we will have at most 13 triple points. □

Theorem 6.2 *Let $\mathcal{A}$ be a nonreductive line arrangement of* 11 *lines in* $\mathbb{CP}^2$ *with a quintuple point* $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5$. *Assume that* $L_6 \cap L_7 \cap L_8$ *is the triple point apart from the quintuple point; then there are exactly* 11 *triple points in* $\mathcal{A}$. *Then there are* 28 *cases that can be realized,* 7 *of whose moduli spaces are irreducible and* 21 *of them are "potential Zariski pairs".*

Proof Note that we have at least 11 triple points in $\mathcal{A}$, because it is nonreductive.

Case 1. There are 13 triple points in $\mathcal{A}$. Then $(L_9 \cap L_{10}, L_9 \cap L_{11}, L_{10} \cap L_{11})$ are triple points, and each of L_9, L_{10}, L_{11} passes through 3 triple points on $(L_6 \cup L_7 \cap L_8)$. Up to a lattice isomorphism, we assume that $(L_1 \cap L_8, L_2 \cap L_7, L_3 \cap L_6)$ are on L_{11}, and $L_9 \cap L_{10}, L_9 \cap L_{11}, L_{10} \cap L_{11}$ are on L_1, L_4, L_5, respectively. Note that L_9 must pass through one of $L_5 \cap L_6, L_5 \cap L_7, L_5 \cap L_8$ so that L_5 contains at least 3 multiple points. Up to lattice isomorphism, we assume that $L_5 \cap L_6$ is on L_9. Then $L_3 \cap L_7, L_2 \cap L_8$ are on L_9 and $(L_2 \cap L_6, L_3 \cap L_8, L_4 \cap L_7)$ are on L_{10} so that $\mathcal{A}$ is nonreductive (Figure 64). Figure 64 cannot be realized.

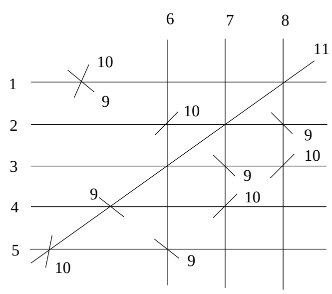

Figure 64.

Case 2. There are 12 triple points in $\mathcal{A}$. Then at least 2 points of $(L_9 \cap L_{10}, L_9 \cap L_{11}, L_{10} \cap L_{11})$ are triple points.

Subcase 1. All of $(L_9 \cap L_{10}, L_9 \cap L_{11}, L_{10} \cap L_{11})$ are triple points. Note that L_9, L_{10}, L_{11} pass through 8 triple points on $(L_6 \cup L_7 \cap L_8)$, and we assume L_{10} passes through 2 triple points on $(L_6 \cup L_7 \cap L_8)$. Similarly as in **Case 1**, we assume that $(L_1 \cap L_8, L_2 \cap L_7, L_3 \cap L_6)$ are on L_{11}, $(L_9 \cap L_{10},\ L_9 \cap L_{11},\ L_{10} \cap L_{11})$ are on L_1, L_4, L_5, respectively, and $L_5 \cap L_6$ is on L_9. Then $L_3 \cap L_7, L_2 \cap L_8$ are on L_9. Hence, L_{10} contains one of $\{(L_4 \cap L_6, L_3 \cap L_8), (L_4 \cap L_7, L_2 \cap L_6), (L_4 \cap L_7, L_3 \cap L_8), (L_4 \cap L_8, L_2 \cap L_6)\}$.

Subcase 1 cannot be realized.

Subcase 2. Two of $(L_9 \cap L_{10}, L_9 \cap L_{11}, L_{10} \cap L_{11})$ are triple points. Let $L_9 \cap L_{11}, L_{10} \cap L_{11}$ be the triple points. Then L_9, L_{10}, and L_{11} pass through 3 triple points on $(L_6 \cup L_7 \cap L_8)$. We assume that $(L_1 \cap L_8, L_2 \cap L_7, L_3 \cap L_6)$ are on L_{11}, $(L_9 \cap L_{10}, L_{10} \cap L_{11})$ are on L_4, L_5 respectively, and $L_5 \cap L_6$ is on L_9. Then L_9 must pass through one of $\{(L_1 \cap L_7, L_2 \cap L_8), (L_3 \cap L_7, L_2 \cap L_8), (L_1 \cap L_7, L_3 \cap L_8)\}$.

(1). L_9 passes through $(L_1 \cap L_7, L_2 \cap L_8)$. Up to a permutation $(7,8)(1,2)$, L_{10} contains $(L_1 \cap L_6, L_3 \cap L_8, L_4 \cap L_7)$ or $(L_2 \cap L_6, L_3 \cap L_8, L_4 \cap L_7)$.

(2). L_9 passes through $(L_3\cap L_7, L_2\cap L_8)$. To make $\mathcal{A}$ nonreductive, L_{10} contains one of $\{(L_4\cap L_6, L_3\cap L_8, L_1\cap L_7), (L_4\cap L_7, L_3\cap L_8, L_1\cap L_6), (L_4\cap L_8, L_2\cap L_6, L_1\cap L_7)\}$.

(3). L_9 passes through $(L_1\cap L_7, L_3\cap L_8)$. After a permutation $(7,8)(1,2)$, it is lattice isomorphic to **(2)**.

Subcase 2 cannot be realized.

Case 3. There are 11 triple points in $\mathcal{A}$.

Subcase 1. One of $(L_9\cap L_{10}, L_9\cap L_{11}, L_{10}\cap L_{11})$ is a triple point. Let $L_{10}\cap L_{11}$ be the triple point. Then each of L_9, L_{10}, and L_{11} passes through 3 triple points in $(L_6\cup L_7\cap L_8)$. We assume that $(L_1\cap L_8, L_2\cap L_7, L_3\cap L_6)$ are on L_{11} and $L_{10}\cap L_{11}$ is on L_4. Note that L_9 must pass through one of $(L_5\cap L_6, L_5\cap L_7, L_5\cap L_8)$. Up to a lattice isomorphism, let L_9 pass through $(L_5\cap L_6, L_4\cap L_7, L_2\cap L_8)$ or $(L_5\cap L_6, L_4\cap L_7, L_3\cap L_8)$.

(I). L_9 passes through $L_5\cap L_6, L_4\cap L_7, L_2\cap L_8$. Then L_{10} passes through $(L_4\cap L_6, L_1\cap L_7, L_3\cap L_8)$ or $(L_4\cap L_8, L_3\cap L_7, L_1\cap L_6)$.

The first equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-t_4X)(Y-(t_3-t_2)X-t_2Z)(Y-(1-t_3)X-t_3Z)=0$, where $t_1=t^3, t_2=t^2, t_3=t, t_4=t^3-1$, and t satisfies $t^4-t-1=0$.

The second equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-t_4X)(Y-(t_2-1)X-Z)(Y-(1-t_3)X-t_3Z)=0$, where $t_1=t, t_2=t^3, t_3=-t, t_4=t^2$, and t satisfies $t^4+1=0$.

(II). L_9 passes through $(L_5\cap L_6, L_4\cap L_7, L_3\cap L_8)$. Then L_{10} passes through $(L_4\cap L_6, L_1\cap L_7, L_2\cap L_8)$ or $(L_4\cap L_8, L_2\cap L_6, L_1\cap L_7)$ (the first case cannot be realized).

The equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-t_3X)(Y-(t_2-t_4)X-t_4Z)(Y-(1-t_3)X-t_3Z)=0$, where $t_1=t^3+t^2-2t-2, t_2=t^3-t-1, t_3=t, t_4=t^2-1$, and t satisfies $t^4-2t^2-t+1=0$.

Subcase 2. Two of $(L_9\cap L_{10}, L_9\cap L_{11}, L_{10}\cap L_{11})$ are triple points. Let $L_9\cap L_{11}$ and $L_{10}\cap L_{11}$ be triple points on L_4 and L_5, respectively. Then L_9, L_{10}, and L_{11} pass 8 triple points in $(L_6\cup L_7\cap L_8)$.

(I). L_{11} passes through 2 triple points in $(L_6\cup L_7\cap L_8)$. Assume that $L_1\cap L_8$ and $L_2\cap L_7$ are on L_{11}. Note that L_9 must pass through one of $(L_5\cap L_6, L_5\cap L_7, L_5\cap L_8)$, so that L_5 contains 3 multiple points.

① $L_5\cap L_6$ is on L_9, so then $(L_1\cap L_7, L_3\cap L_8)$ or $(L_3\cap L_7, L_2\cap L_8)$ is on L_9.

If $(L_1\cap L_7, L_3\cap L_8)$ are on L_9, then L_{10} must pass through one of $\{(L_4\cap L_6, L_3\cap L_7, L_2\cap L_8), (L_4\cap L_7, L_3\cap L_6, L_2\cap L_8), (L_4\cap L_8, L_3\cap L_7, L_2\cap L_6)\}$ (only the first case can be realized).

The equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-t_3X)(Y-(t_4-t_2)X-t_2Z)(Y-\frac{1-t_4}{1-t_1}X-\frac{t_1-t_4}{1-t_1}Z)=0$, where $t_1=t, t_2=-2t+6, t_3=-t+3, t_4=-2+t$, and t satisfies $t^2-3t+1=0$.

If $(L_3\cap L_7, L_2\cap L_8)$ are on L_9, then by a permutation $(7,8)(1,2)$, it is lattice isomorphic to the case that $(L_1\cap L_7, L_3\cap L_8)$ are on L_9.

② $L_5\cap L_7$ is on L_9, so then L_9 must pass through one of $\{(L_1\cap L_6, L_3\cap L_8), (L_2\cap L_6, L_3\cap L_8), (L_3\cap L_6, L_2\cap L_8)\}$.

If $L_1 \cap L_6, L_3 \cap L_8$ are on L_9, then L_{10} must pass through one of $\{(L_4 \cap L_6, L_3 \cap L_7, L_2 \cap L_8), (L_4 \cap L_7, L_3 \cap L_6, L_2 \cap L_8), (L_4 \cap L_8, L_3 \cap L_7, L_2 \cap L_6)\}$.

The first equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-(t_4-t_3)X-t_3Z)(Y-(t_3-t_2)X-t_2Z)(Y-\frac{1-t_4}{1-t_1}X-\frac{t_1-t_4}{1-t_1}Z)=0$, where $t_1 = 1 \pm (t-t^2+t^3), t_2 = 4 \mp (t^2+2t^3), t_3 = 2 \mp (t^2+t^3), t_4 = \pm t$, and t satisfies $t^4 - t^3 + 2t - 1 = 0$.

The second equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-(t_4-t_3)X-t_3Z)(Y-(t_3-1)X-Z)(Y-\frac{1-t_4}{1-t_1}X-\frac{t_1-t_4}{1-t_1}Z)=0$, where $t_1 = 1 \mp t, t_2 = -\frac{1}{2} \pm t, t_3 = -1 \pm t, t_4 = \pm t$, and t satisfies $2t^2 - 4t + 1 = 0$.

The third equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-(t_4-t_3)X-t_3Z)(Y-(t_2-1)X-Z)(Y-\frac{1-t_4}{1-t_1}X-\frac{t_1-t_4}{1-t_1}Z)=0$, where $t_1 = \pm(t+t^2), t_2 = -2 \pm t^2, t_3 = -1 \pm t^2, t_4 = \pm t$, and t satisfies $t^3 - t^2 - 2t + 1 = 0$.

If $L_2 \cap L_6, L_3 \cap L_8$ are on L_9, then L_{10} must pass through $\{(L_4 \cap L_8, L_3 \cap L_6, L_1 \cap L_7)\}$, $\{(L_4 \cap L_8, L_3 \cap L_7, L_1 \cap L_6)\}$ (the second case cannot be realized).

The first equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-(t_3-t_4)X-t_4Z)(Y-(t_2-t_3)X-t_3Z)(Y-\frac{1-t_4}{1-t_1}X-\frac{t_1-t_4}{1-t_1}Z)=0$, where $t_1 = \pm t, t_2 = -3 \pm (4t-t^2), t_3 = 2 \mp t, t_4 = -1 \mp (t^2-3t)$, and t satisfies $t^3 - 5t^2 + 6t - 1 = 0$.

If $L_3 \cap L_6, L_2 \cap L_8$ are on L_9, then L_{10} must pass through one of $\{(L_4 \cap L_6, L_3 \cap L_8, L_1 \cap L_7), (L_4 \cap L_7, L_3 \cap L_8, L_1 \cap L_6), (L_4 \cap L_8, L_3 \cap L_7, L_1 \cap L_6)\}$.

The first equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-(t_3-1)X-Z)(Y-(t_4-t_2)X-t_2Z)(Y-\frac{1-t_4}{1-t_1}X-\frac{t_1-t_4}{1-t_1}Z)=0$, where $t_1 = \pm(t^2+t), t_2 = \pm(t-t^3), t_3 = \pm t^2, t_4 = \pm t$, and t satisfies $t^4 + t^3 - t^2 - t + 1 = 0$.

The second equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-(t_3-1)X-Z)(Y-(t_4-t_3)X-t_3Z)(Y-\frac{1-t_4}{1-t_1}X-\frac{t_1-t_4}{1-t_1}Z)=0$, where $t_1 = \frac{1}{2} \pm \frac{1}{2}t, t_2 = 1 \mp 2t, t_3 = \mp t, t_4 = \pm t$ and t, satisfies $t^2 + 2t - 1 = 0$.

The third equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-(t_3-1)X-Z)(Y-(t_2-t_4)X-t_4Z)(Y-\frac{1-t_4}{1-t_1}X-\frac{t_1-t_4}{1-t_1}Z)=0$, where $t_1 = -1, t_2 = \pm 2t, t_3 = 2, t_4 = 1 \pm t$, and t satisfies $t^2 + t - 1 = 0$.

③ $L_5 \cap L_8$ is on L_9. After a permutation $(7,8)(1,2)$, it is lattice isomorphic to ②.

(II). L_{10} passes through 2 triple points in $(L_6 \cup L_7 \cap L_8)$. We assume that $(L_1 \cap L_8, L_2 \cap L_7, L_3 \cap L_6)$ are on L_{11}, $(L_9 \cap L_{10}, L_{10} \cap L_{11})$ are on L_4, L_5 respectively, and $L_5 \cap L_6$ is on L_9. Then L_9 must pass through one of $\{(L_1 \cap L_7, L_2 \cap L_8), (L_3 \cap L_7, L_2 \cap L_8), (L_1 \cap L_7, L_3 \cap L_8)\}$.

① $(L_1 \cap L_7, L_2 \cap L_8)$ are on L_9, so then up to lattice isomorphism L_{10} must pass through $\{L_4 \cap L_6, L_3 \cap L_7\}$ or $\{L_4 \cap L_7, L_3 \cap L_8\}$.

The first equation can be defined by

$XY(X-Z)(2X+Z)(Y-Z)(2Y+Z)(Y+Z)(Y+2Z)(Y+2X)(Y-2X+Z)(2Y-2X+Z)=0.$

The second equation can be defined by

$XY(X-Z)(X-2Z)(Y-Z)(4Y-3Z)(2Y-3Z)(2Y-Z)(2Y-X)(2Y+X-3Z)(4Y+3X-9Z)=0.$

② $(L_3 \cap L_7, L_2 \cap L_8)$ are on L_9, so then L_{10} must pass through one of $\{(L_4 \cap L_6, L_1 \cap L_7), (L_4 \cap L_7, L_1 \cap L_6), (L_4 \cap L_8, L_1 \cap L_6), (L_4 \cap L_8, L_1 \cap L_7)\}$.

The first equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-t_4X)(Y-\frac{1-t_2}{t_1}X-Z)(Y-(1-t_3)X-t_3Z) = 0$, where $t_1 = t, t_2 = t, t_3 = -1+2t, t_4 = 2t$, and t satisfies $2t^2 - 2t + 1 = 0$.

The second equation can be defined by $XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-t_4X)(Y-\frac{t_2-1}{t_1}X-Z)(Y-(1-t_3)X-t_3Z) = 0$, where $t_1 = 1-t, t_2 = t^2-t+2, t_3 = t, t_4 = t^2-t+1$, and t satisfies $t^3 - 2t^2 + 3t - 1 = 0$.

The third equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-t_4X)(Y-(t_2-1)X-Z)(Y-(1-t_3)X-t_3Z) = 0$, where $t_1 = t^2+t, t_2 = -t^3-t^2+2t+2, t_3 = t, t_4 = -t^3+2t$, and t satisfies $t^4+t^3-2t^2-2t+1 = 0$.

The fourth equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-t_4X)(Y-\frac{t_2-1}{1-t_1}X-t_2Z)(Y-(1-t_3)X-t_3Z) = 0$, where $t_1 = \frac{1}{2}+\frac{1}{2}t^2, t_2 = \frac{1}{2}t^2+t+\frac{1}{2}, t_3 = t, t_4 = t^2+t$, and t satisfies $t^4+t^3-2t^2-2t+1 = 0$.

③ $(L_1 \cap L_7, L_3 \cap L_8)$ are on L_9. After a permutation $(7,8)(1,2)$, it is lattice isomorphic to ②.

Subcase 3. All of $(L_9 \cap L_{10}, L_9 \cap L_{11}, L_{10} \cap L_{11})$ are triple points. Then L_9, L_{10}, and L_{11} pass through 8 triple points in $(L_6 \cup L_7 \cap L_8)$ and at least one of L_9, L_{10}, L_{11} passes through 3 triple points in $(L_6 \cup L_7 \cap L_8)$. We always assume that L_{11} can be such a line. Furthermore, up to a lattice isomorphism, we assume that $(L_1 \cap L_8, L_2 \cap L_7, L_3 \cap L_6)$ are on L_{11}, and $L_9 \cap L_{10}, L_9 \cap L_{11}, L_{10} \cap L_{11}$ are on L_1, L_4, L_5 respectively.

(I). L_9 passes through 3 triple points in $(L_6 \cup L_7 \cap L_8)$ and L_{10} must pass through 1 triple point in $(L_6 \cup L_7 \cap L_8)$. Note that L_9 must pass through one of $\{L_5 \cap L_6, L_5 \cap L_7, L_5 \cap L_8\}$ so that L_5 contains 3 multiple points. Up to a permutation $(6,7)(2,3)$, we assume that $\{L_5 \cap L_6, L_3 \cap L_7, L_2 \cap L_8\}$ or $\{L_5 \cap L_8, L_3 \cap L_7, L_2 \cap L_6\}$ are on L_9.

① $L_5 \cap L_6, L_3 \cap L_7, L_2 \cap L_8$ are on L_9. Then L_{10} must pass through one of $\{L_4 \cap L_6, L_4 \cap L_7, L_4 \cap L_8\}$.

The first equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-t_4X)(Y+t_4(t_2-1)X-t_2Z)(Y-(1-t_3)X-t_3Z) = 0$, where $t_1 = \frac{1}{2}t, t_2 = \frac{1}{4}t+1, t_3 = t, t_4 = 2$, and t satisfies $t^2 - 3t + 4 = 0$.

The second cannot be realized.

The third equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-t_4X)(Y-\frac{t_4(t_2-1)}{t_4-1}X+\frac{t_4-t_2}{t_4-1}Z)(Y-(1-t_3)X-t_3Z) = 0$, where $t_1 = t, t_2 = \frac{1}{2}t-\frac{1}{2}, t_3 = -1, t_4 = 2t-1$, and t satisfies $2t^2 - t + 1 = 0$.

② $L_5 \cap L_8, L_3 \cap L_7, L_2 \cap L_6$ are on L_9. Then L_{10} must pass through one of $\{L_4 \cap L_6, L_4 \cap L_7, L_4 \cap L_8\}$.

The first equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y+t_4X)(Y-\frac{t_2(1-t_3)}{t_3}X-t_2Z)(Y-(1-t_3)X-t_3Z) = 0$, where $t_1 = \frac{5}{2}, t_2 = \frac{3}{2}, t_3 = 3, t_4 = -2$.

The second cannot be realized.

The third equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-t_4X)(Y+t_2(t_3-1)X-t_2t_3Z)(Y-(1-t_3)X-t_3Z) = 0$, where $t_1 = -t+2, t_2 = t, t_3 = 2t, t_4 = -2t+1$, and t satisfies $2t^2 - t + 1 = 0$.

(II). Both L_9 and L_{10} pass through 2 triple points in $(L_6 \cup L_7 \cap L_8)$. Up to a lattice isomorphism, assume that $L_5 \cap L_6, L_3 \cap L_7$ or $L_5 \cap L_8, L_3 \cap L_7$ are on L_9.

① $L_5 \cap L_6, L_3 \cap L_7$ are on L_9. Then L_{10} must pass through one of $\{(L_4 \cap L_6, L_2 \cap L_8), (L_4 \cap L_7, L_2 \cap L_6), (L_4 \cap L_7, L_2 \cap L_8), (L_4 \cap L_8, L_2 \cap L_6)\}$.

The first equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(t_1Y-t_3X)(Y-(t_4-t_2)X-t_2Z)(Y-(1-t_3)X-t_3Z)=0$, where $t_1=t, t_2=-2t+4, t_3=2t, t_4=2t-2$, and t satisfies $2t^2-t-2=0$.

The second case cannot be realized.

The third equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(t_1Y-t_3X)(Y-\frac{t_2-t_4}{1-t_1}X+(\frac{t_2-t_4}{1-t_1}-t_4)Z)(Y-(1-t_3)X-t_3Z)=0$, where $t_1=-\frac{2}{5}t^3+\frac{1}{5}t^2+t-\frac{2}{5}, t_2=-4t^3+t^2+7t-12, t_3=t, t_4=-t^3+2t-2$, and t satisfies $t^4+t^3-2t^2+t+4=0$.

The fourth equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(t_1Y-t_3X)(Y-(t_2-t_4)X-t_4Z)(Y-(1-t_3)X-t_3Z)=0$, where $t_1=t, t_2=-2t+1, t_3=-1, t_4=2t-1$, and t satisfies $2t^2-1=0$.

② $L_5 \cap L_8, L_3 \cap L_7$ are on L_9. Then L_{10} must pass through one of $\{(L_4 \cap L_6, L_2 \cap L_8), (L_4 \cap L_7, L_2 \cap L_6), (L_4 \cap L_7, L_2 \cap L_8), (L_4 \cap L_8, L_2 \cap L_6)\}$.

The first equation can be defined by

$XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-\frac{t_3}{t_1-1}X+\frac{t_3}{t_1-1}Z)(Y-(t_4-t_2)X-t_2Z)(Y-(1-t_3)X-t_3Z)=0$, where $t_1=-t^2-t, t_2=t^4, t_3=t, t_4=t^3$, and t satisfies $t^5+t^4-t^2-t-1=0$.

The second equation can be defined by $XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-\frac{t_3}{t_1-1}X+\frac{t_3}{t_1-1}Z)(Y-\frac{t_2-t_4}{t_1}X-t_4Z)(Y-(1-t_3)X-t_3Z)=0$, where $t_1=\frac{3}{2}-\frac{1}{2}t, t_2=\frac{1}{2}t+\frac{1}{2}, t_3=t, t_4=2$, and t satisfies $t^2-2t-1=0$.

The third equation can be defined by $XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-\frac{t_3}{t_1-1}X+\frac{t_3}{t_1-1}Z)(Y-\frac{t_2-t_4}{1-t_1}X-(t_4-\frac{t_2-t_4}{1-t_1})Z)(Y-(1-t_3)X-t_3Z)=0$, where $t_1=-t, t_2=t+2, t_3=t, t_4=2$, and t satisfies $t^2-2=0$.

The fourth equation can be defined by $XY(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-t_4Z)(Y-\frac{t_3}{t_1-1}X+\frac{t_3}{t_1-1}Z)(Y-(t_2-t_4)X-t_4Z)(Y-(1-t_3)X-t_3Z)=0$, where $t_1=2-t, t_2=t, t_3=t-2, t_4=2$, and t satisfies $t^2-4t+2=0$. □

6.2. All triple points are in the pencil of the quintuple point

Assume that all the triple points are on the lines passing through the quintuple point. We first show that there at most 13 triple points, and at least 10 triple points if the arrangement is nonreductive.

Lemma 6.3 *Let $\mathcal{A}$ be a nonreductive arrangement of 11 lines with 1 quintuple point so that all triple points are on the lines passing through the quintuple point. Then there are at most 13 triple points and at least 10 triple points.*

Proof From Lemma 2.6 and Theorem 2.7, we have the following equations:

$$\begin{cases} n_2 + 3n_3 & +6n_4 + 10n_5 = 55, \\ n_2 + \frac{3}{4}n_3 & \geq 11 + n_5. \end{cases}$$

From the above equations, we compute $n_3 \leq 14$.

Let $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5$ be the quintuple point. Since there are 14 triple points on those 5 lines and we know that each of the 5 lines passes through at least 2 and at most 3 triple points, then we may assume that each of L_1, L_2, L_3, and L_4 passes through 3 triple points. On the other hand, each of the other six lines passes through at least 3 and at most 5 triple points. Let a, b, and c be the numbers of lines in $\{L_6, L_7, L_8, L_9, L_{10}, L_{11}\}$ that pass through 3, 4, and 5 triple points, respectively. Then a and b should satisfy the following system of equations:

$$a + b + c = 6,$$

$$3a + 4b + 5c = 28.$$

From the above equations, we have two solutions:

One is

$$a = 1, b = 0, c = 5.$$

The other is

$$a = 0, b = 2, c = 4.$$

Because there are 14 triple points, one of $\{L_1, L_2, L_3, L_4, L_5\}$ has only two triple points. This fact tells us that $c \leq 4$. The first case does not exist. For the second case, we consider $\mathcal{A}' = \mathcal{A} \setminus L_5$. Now $\mathcal{A}'$ is an arrangement of 10 lines with 1 quadruple point. All triple points are in the pencil of the quadruple point, and $\mathcal{A}'$ has 12 triple points. By [2, Lemma 5.3], $\mathcal{A}'$ cannot be realized, so $\mathcal{A}$ does not exist.

Because each line of $\{L_1, L_2, L_3, L_4, L_5\}$ has at least two triple points, then $n_3 \geq 10$.

□

The classification will run on the numbers of triple points.

Theorem 6.4 *Let $\mathcal{A}$ be a nonreductive arrangement of 11 lines with a quintuple point and 13 triple points such that all triple points are on the 5 lines passing through the quintuple point. Then the moduli space $\mathcal{M}_{\mathcal{A}}$ is irreducible, and in fact is one point.*

Proof Let $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5$ be the quintuple point. Since there are 14 triple points on those 5 lines and we know that each of 5 lines passes through at least 2 and at most 3 triple points, then we may assume that each of L_1, L_2 and L_3 passes through 3 triple points. On the other hand, each of the other six lines passes through at least 3 and at most 5 triple points. Let a, b, and c be the numbers of lines in $\{L_6, L_7, L_8, L_9, L_{10}, L_{11}\}$ that pass through 3, 4, and 5 triple points, respectively. Then a and b should satisfy the following system of equations:

$$a + b + c = 6,$$

$$3a + 4b + 5c = 26.$$

From the above equations, we have three solutions:

$$a = 2, b = 0, c = 4; \quad a = 1, b = 2, c = 3; \quad a = 0, b = 4, c = 2.$$

For the first case, we consider $\mathcal{A}^{'} = \mathcal{A} \setminus L_4$. Now $\mathcal{A}^{'}$ is an arrangement of 10 lines with 1 quadruple point. All triple points are in the pencil of the quadruple point, and $\mathcal{A}^{'}$ has 11 triple points. From [2, Theorem 5.4], $\mathcal{M}_{\mathcal{A}'}$ is irreducible. In fact, it is one point, because every case has only one solution from the proof of [2, Theorem 5.4]. The moduli space $\mathcal{M}_{\mathcal{A}}$ is irreducible.

For the second case, because $b = 2$, one of $\{L_6, L_7, L_8, L_9, L_{10}, L_{11}\}$ must intersect 4 lines, and one of the intersect lines must be L_4 or L_5. We assume it is L_4. Now we consider $\mathcal{A}^{'} = \mathcal{A} \setminus L_4$. The rest of this proof is similar to the first case.

For the third case, we consider $\mathcal{A}^{'} = \mathcal{A} \setminus L_4$. The rest of this proof is similar to the first case.

□

Remark 6.5 *The example of Theorem 6.4 is easy to construct (see Figure 65).*

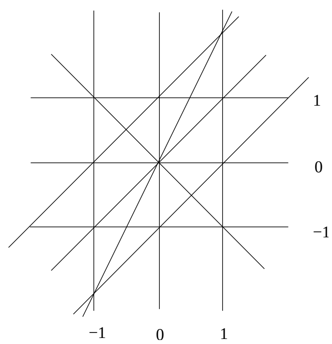

Figure 65.

The equation is defined as follows:

$$(X - Z)X(X + Z)(Y + X)(Y - X)(Y + Z - X)(Y - Z - X)(Y - 2X)(Y - Z)Y(Y + Z) = 0.$$

Theorem 6.6 *Let $\mathcal{A}$ be a nonreductive arrangement of 11 lines with a quintuple point and 12 triple points such that all triple points are on the 5 lines passing through the quintuple point. Then the quotient moduli space $\mathcal{M}^c_{\mathcal{A}}$ is irreducible, and in fact is one or two points.*

Proof Let $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5$ be the quintuple point. On the one hand, since there are 12 triple points on those 5 lines and we know that each of 5 lines passes through at least 2 and at most 3 triple points, then we may assume that each of L_1 and L_2 passes through 3 triple points. On the other hand, each of the other six lines passes through at least 3 and at most 5 triple points. Let a, b, and c be the numbers of lines in $\{L_6, L_7, L_8, L_9, L_{10}, L_{11}\}$ that pass through 3, 4, and 5 triple points, respectively. Then a and b should satisfy the following system of equations:

$$a + b + c = 6,$$

$$3a + 4b + 5c = 24.$$

From the above equations, we have these solutions:

$$a = 0, b = 6, c = 0; \quad a = 1, b = 4, c = 1; \quad a = 2, b = 2, c = 2; \quad a = 3, b = 0, c = 3.$$

For the first case, we consider $\mathcal{A}^{'} = \mathcal{A} \setminus L_4$. Now $\mathcal{A}^{'}$ is an arrangement of 10 lines with 1 quadruple point. All triple points are in the pencil of the quadruple point, and $\mathcal{A}^{'}$ has 10 triple points. From [2, Theorem 5.5], $\mathcal{M}^c_{\mathcal{A}'}$ is irreducible. In fact, it is one point. Because every case has only conjugation solutions from the proof of [2, Theorem 5.5], the moduli space $\mathcal{M}^c_{\mathcal{A}}$ is irreducible.

For the second case, because $a = 1$, we assume that the line is L_6. Two of $\{L_1, L_2, L_3, L_4, L_5\}$ cannot intersect with L_6. We assume that these lines are L_4 and L_5, and L_4 must have two triple points. Now we consider that $\mathcal{A}^{'} = \mathcal{A} \setminus L_4$, and $\mathcal{A}^{'}$ is an arrangement of 10 lines with 1 quadruple point. All triple points are in the pencil of the quadruple point, and $\mathcal{A}^{'}$ has 10 triple points. From [2, Theorem 5.5], $\mathcal{M}^c_{\mathcal{A}'}$ is irreducible, and in fact is one point, so the moduli space $\mathcal{M}^c_{\mathcal{A}}$ is irreducible.

For the third case, because $a = 2$, we assume the lines are L_6 and L_7. One of $\{L_1, L_2, L_3, L_4, L_5\}$ must not intersect L_6 and L_7. We assume that this line is L_4, and L_4 must have two triple points. Now we consider that $\mathcal{A}^{'} = \mathcal{A} \setminus L_4$, and $\mathcal{A}^{'}$ is an arrangement of 10 lines with 1 quadruple point. The rest of this proof is similar to the second case.

For the fourth case, up to lattice isomorphism, there is only one case (see Figure 66).

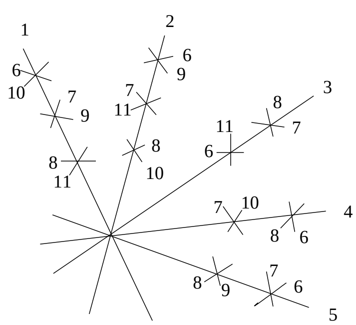

Figure 66.

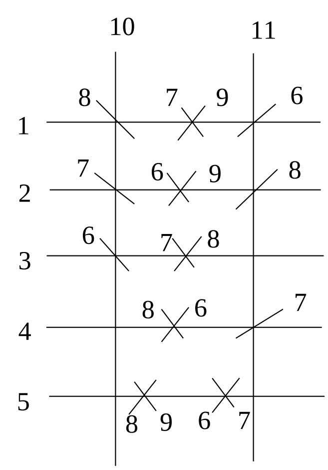

Figure 67.

It is easy to see that line 11 and line 10 must have a double point that is not on the 5 lines passing through the quintuple point, so Figure 66 and Figure 67 are equivalent.

It is easy to compute that $\mathcal{A}$ does not exist. $\square$

From the above discussions, we have the following corollary:

Corollary 6.7 *Let $\mathcal{A}$ be a nonreductive line arrangement of* 11 *lines with $n_5 = 1$, $n_4 = 0$, and $n_r = 0, r \geq 6$. Moreover, all triple points are on the* 5 *lines passing through the quintuple point. If it contains more than* 11 *triple points, then there is no Zariski pair.*

Let $\mathcal{A}$ be a nonreductive arrangement of 11 lines with a quintuple point and all triple points are on the 5 lines passing through the quintuple point. If the number of the triple points is less than 12, then there are many cases in which $\mathcal{M}_{\mathcal{A}}$ is more than 2 points or even one dimension. Now we give two examples.

Example 6.1 *The line arrangements are with* 11 *triple points, and all triple points are on the* 5 *lines passing through the quintuple point (see Figure 68).*

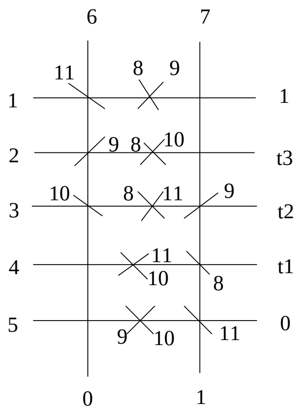

Figure 68.

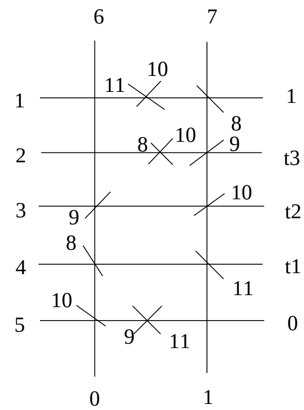

Figure 69.

After some easy computation, we get the equation as follows: $X(X-Z)(Y-Z)(Y-t_3Z)(Y-t_2Z)(Y-t_1Z)Y((t_2-t_3)X-Y+t_3Z)(Y-Z+X)(Y-(t_1-\frac{t_1-t_2}{t_2})Z-(\frac{t_1-t_2}{t_2})X)(Y-t_2Z-\frac{t_1-t_2}{1-t_1}X)=0$, *where* $t_1=\frac{1}{5}t^2-\frac{2}{5}t+\frac{4}{5}$, $t_2=t$, $t_3=1-t$, *and* t *satisfies* $t^2-2t-t^3+1=0$, *so that the moduli space* $\mathcal{M}_{\mathcal{A}}$ *is three points.*

Example 6.2 *The line arrangements are with* 10 *triple points, and all triple points are on the* 5 *lines passing through the quintuple point (see Figure 69).*

After some easy computation, we get the equation as follows: $X(X-Z)(Y-Z)(Y-t_3Z)(Y-t_2Z)(Y-t_1Z)(Y(1-t_1))X-Y+t_1Z)((t_3-t_2)X-Y+t_2Z)(t_2X-Y)(t_2(t_1-1)X-(t_2-1)Y+(t_2-t_1)Z)$, *where* $t_1=\frac{1-3t-t^2}{t}$, $t_2=t$, $andt_3=\frac{t^3-3t^2+t}{1-2t}$, *so the moduli space* $\mathcal{M}_{\mathcal{A}}$ *is of one dimension.*

Acknowledgments

This work was supported by the Emmy Noether Research Institute for Mathematics, the Minerva Foundation (Germany). This work was also supported by the NSFC and China Postdoctoral Science Foundation (No. 2013M541711). The authors thank Dr Fei Ye for his comments and suggestions. The authors would like to thank the referee for many useful suggestions for the correction of original manuscript.

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