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Uniquely strongly clean triangular matrices

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Abstract: A ring R is uniquely (strongly) clean provided that for any $a \in R$ there exists a unique idempotent $e \in R$ $\big(e \in comm(a)\big)$ such that $a - e \in U(R)$. We prove, in this note, that a ring R is uniquely clean and uniquely bleached if and only if R is abelian, $\mathbb{T}_n(R)$ is uniquely strongly clean for all $n \geq 1$, i.e. every $n \times n$ triangular matrix over R is uniquely strongly clean, if and only if R is abelian, and $\mathbb{T}_n(R)$ is uniquely strongly clean for some $n \geq 1$. In the commutative case, more explicit results are obtained.

Key words: Uniquely strongly clean ring, uniquely bleached ring, triangular matrix ring

1. Introduction

Throughout this article, all rings are associative with unity. We write $U(R)$ for the set of all units in R. $\mathbb{T}_n(R)$ stands for the ring of all $n \times n$ triangular matrices over a ring R. Let $a, b \in R$. We denote the map from R to $R : x \mapsto ax - xb$ by $l_a - r_b$. We write $\mathbb{M}_n(R)$ for the ring of all $n \times n$ matrices over the ring R. The *commutant* of an element a in a ring R is defined by $comm(a) = \{x \in R \mid xa = ax\}$. $\mathbb{N}$ is the set of all natural numbers.

A ring R is *strongly clean* provided that for any $a \in R$ there exists an idempotent $e \in comm(a)$ such that $a - e \in U(R)$. Strongly clean triangular matrices are extensively studied by many authors, e.g., [1] and [3]. A ring R is called *uniquely clean* provided that for any $a \in R$ there exists a unique idempotent $e \in R$ such that $a - e \in U(R)$. Many characterizations of such rings are studied in [2, 3, 4, 10] and [11]. Following Chen et al. [5], a ring R is called *uniquely strongly clean* provided that for any $a \in R$ there exists a unique idempotent $e \in comm(a)$ such that $a - e \in U(R)$. Uniquely strong cleanness behaves very differently from the properties of uniquely clean rings (cf. [5]). In general, matrix rings do not have such properties (see [13, Proposition 11.8]). Thus, it is attractive to investigate uniquely strong cleanness of triangular matrices over a ring. Chen et al. proved that if R is commutative, then R is uniquely clean if and only if $\mathbb{T}_n(R)$ is uniquely strongly clean for all $n \geq 1$ if and only if $\mathbb{T}_n(R)$ is uniquely strongly clean for some $n \geq 1$.

[5, Question 12] and [13, Question 11.13] asked if "commutative" in the preceding result can be replaced by "abelian". The motivation of this note is to explore this problem. Following [7], a ring R is *uniquely bleached* provided that for any $a \in J(R)$, $b \in U(R)$, $l_a - r_b$, and $l_b - r_a$ are isomorphism. We prove, in this note, that R is uniquely clean and uniquely bleached if and only if R is abelian, $\mathbb{T}_n(R)$ is uniquely strongly clean for all

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$n \geq 1$ if and only if R is abelian, and $\mathbb{T}_n(R)$ is uniquely strongly clean for some $n \geq 1$. In the commutative case, more explicit results are obtained. These also generalize the main theorems in [5] and [6], and provide many new classes of such rings.

2. The main results

It is well known that every uniquely clean ring is a uniquely strongly clean ring, but the converse is not true. For instance, $\mathbb{T}_2(\mathbb{Z}_{(2)})$ is uniquely strongly clean, while it is not uniquely clean. We are concerned with uniquely strongly clean triangular matrix rings over a uniquely clean base ring. We begin with

Lemma 1 *Let R be a ring. If $\mathbb{T}_2(R)$ is uniquely strongly clean, then R is uniquely bleached.*

Proof In view of [5, Example 5], R is uniquely strongly clean. Let $a \in J(R)$ and $b \in U(R)$, and let $r \in R$. Choose $A = \begin{bmatrix} a & -r \\ & b \end{bmatrix} \in \mathbb{T}_2(R)$. Then there exists a unique idempotent $E = [e_{ij}] \in \mathbb{T}_2(R)$ such that $A - E \in U\big(\mathbb{T}_2(R)\big)$ and $EA = AE$. It can be easily seen that e_{11} and $e_{22} \in R$ are idempotents. Further, $a - e_{11} \in U(R)$ and $b - e_{22} \in U(R)$. As $a - 0 \in U(R)$ and $b - 1 \in U(R)$, by the uniquely strong cleanness of R, we get $e_{11} = 0$ and $e_{22} = 1$. Thus, $E = \begin{bmatrix} 0 & x \\ & 1 \end{bmatrix}$ for some $x \in R$. It follows from $EA = AE$ that $ax - xb = r$. Assume that $ay - yb = r$. Then we have an idempotent $F = \begin{bmatrix} 0 & y \\ & 1 \end{bmatrix}$ such that $A - F \in U\big(\mathbb{T}_2(R)\big)$ and $AF = FA$. By the uniqueness of E, we get $x = y$. Therefore, $l_a - r_b : R \to R$ is an isomorphism. Likewise, $l_b - r_a : R \to R$ is an isomorphism. Accordingly, R is uniquely bleached, as asserted. □

The following theorem is a generalization of Theorem 1 in [6].

Theorem 2 *Let R be a ring. Then the following are equivalent:*

(1) *R is uniquely clean and uniquely bleached.*

(2) *R is abelian, and $\mathbb{T}_n(R)$ is uniquely strongly clean for all $n \in \mathbb{N}$.*

(3) *R is abelian, $\mathbb{T}_n(R)$ is uniquely strongly clean for some $n \in \mathbb{N}$.*

Proof (1) $\Rightarrow$ (2) In view of [10, Theorem 20], R is abelian. Clearly, the result holds for $n = 1$. Assume that the result holds for $n (n \geq 1)$. Let $A = \begin{bmatrix} a_{11} & \alpha \\ & A_1 \end{bmatrix} \in \mathbb{T}_{n+1}(R)$ where $a_{11} \in R$, $\alpha \in \mathbb{M}_{1\times n}(R)$, and $A_1 \in \mathbb{T}_n(R)$. Since R is uniquely clean, we can find a unique idempotent $e_{11} \in R$ such that $u_{11} := a_{11} - e_{11} \in U(R)$ and $a_{11}e_{11} = e_{11}a_{11}$. Furthermore, we have a unique idempotent $E_1 \in \mathbb{T}_n(R)$ such that $U_1 := A_1 - E_1 \in U\big(\mathbb{T}_n(R)\big)$ and $A_1E_1 = E_1A_1$; hence, $U_1E_1 = E_1U_1$. Let $E = \begin{bmatrix} e_{11} & x \\ & E_1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & \alpha - x \\ & U_1 \end{bmatrix}$, where $x \in \mathbb{M}_{1\times n}(R)$. Observing that

$$\begin{array}{lcll} E^2 = E & \Leftrightarrow & e_{11}x + xE_1 = x; & (i) \\ UE = EU & \Leftrightarrow & u_{11}x + (\alpha - x)E_1 = e_{11}(\alpha - x) + xU_1, & (ii) \end{array}$$

and then combining (i) with (ii) yields that

$$(u_{11} + 2e_{11} - 1)x - xU_1 = e_{11}\alpha - \alpha E_1. \qquad (*)$$

It is enough to show that there exists a unique $x \in \mathbb{M}_{1\times n}(R)$ such that $(*)$ holds. In view of [10, Theorem 20], $R/J(R)$ is Boolean, and so $2 \in J(R)$. Furthermore, $u_{11} \in 1 + J(R)$. This shows that $u_{11} + 2e_{11} - 1 \in J(R)$.

Write $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$, $e_{11}\alpha - \alpha E_1 = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$, and $U_1 = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ & c_{22} & \cdots & c_{2n} \\ & & \ddots & \\ & & & c_{nn} \end{bmatrix}$ where each $c_{ii} \in U(R)$. The equation $(*)$ is equivalent to the n equations:

$$\begin{aligned} (u_{11} + 2e_{11} - 1)x_1 - x_1c_{11} &= v_1; \\ (u_{11} + 2e_{11} - 1)x_2 - x_2c_{22} &= v_1 + x_1c_{12}; \\ &\vdots \\ (u_{11} + 2e_{11} - 1)x_n - x_nc_{nn} &= v_n + x_1c_{1n} + \cdots + x_{n-1}c_{(n-1)n}. \end{aligned}$$

As R is uniquely bleached, we have a unique $x_i \in R$ $(i = 1, \ldots, n)$, and so there exists a unique x such that $(*)$ holds. Further, we see that

$$\begin{aligned} & (u_{11} + 2e_{11} - 1)x(e_{11}I_n + E_1) - x(e_{11}I_n + E_1)U_1 \\ = \; & \alpha(e_{11}I_n - E_1)(e_{11}I_n + E_1) \\ = \; & \alpha(e_{11}I_n - E_1) \\ = \; & (u_{11} + 2e_{11} - 1)x - xU_1 \end{aligned}$$

because $E_1U_1 = U_1E_1$. Set $y = x(e_{11}I_n + E_1)$. This implies that $(u_{11} + 2e_{11} - 1)(y - x) - (y - x)U_1 = 0$. Write $y - x = \begin{bmatrix} z_1 & \cdots & z_n \end{bmatrix}$. Then

$$\begin{aligned} (u_{11} + 2e_{11} - 1)z_1 - z_1c_{11} &= 0; \\ (u_{11} + 2e_{11} - 1)z_2 - z_2c_{22} &= z_1c_{12}; \\ &\vdots \\ (u_{11} + 2e_{11} - 1)z_n - z_nc_{nn} &= z_1c_{1n} + \cdots + z_{n-1}c_{(n-1)n}. \end{aligned}$$

Since R is uniquely bleached, we get each $z_i = 0$, and so $y = x$. This gives that $e_{11}x + xE_1 = x$. Furthermore, $u_{11}x - (\alpha - x)E_1 = e_{11}(\alpha - x) + xU_1$. Therefore, we have a uniquely strongly clean expression

$$A = \begin{bmatrix} e_{11} & x \\ & E_1 \end{bmatrix} + \begin{bmatrix} a_{11} - e_{11} & \alpha - x \\ & A_1 - E_1 \end{bmatrix}.$$

By induction, $\mathbb{T}_n(R)$ is uniquely strongly clean for all $n \in \mathbb{N}$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) In light of [5, Example 5], both R and $\mathbb{T}_2(R)$ are uniquely strongly clean, hence the result by [10, Theorem 20] and Lemma 1. □

Remark 3 Examples of uniquely bleached rings include the ring with nil Jacobson radical, the ring for which some power of each element in $J(R)$ is central, and commutative rings.

The double commutant of an element a in a ring R is defined by $comm^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in comm(a)\}$. Clearly, $comm^2(a) \subseteq comm(a)$. This concept is closely related to quasipolar, perfectly clean, and pseudopolar elements (for details see [4, 8, 9, 12]). We end this note by a more explicit result than [5, Theorem 10].

Theorem 4 *Let R be a commutative ring, and let $n \in \mathbb{N}$. Then the following are equivalent:*

(1) *R is uniquely clean.*

(2) *For any $A \in \mathbb{T}_n(R)$, there exists a unique idempotent $E \in comm^2(A)$ such that $A - E \in U\big(\mathbb{T}_n(R)\big)$.*

Proof. (1) ⇒ (2) For any $A \in \mathbb{T}_n(R)$, we claim that there exists an idempotent $E \in comm^2(A)$ such that $A - E \in U\big(\mathbb{T}_n(R)\big)$. Suppose that the result holds for $n-1 (n \geq 2)$. Let $A = \begin{bmatrix} a_{11} & \alpha \\ & A_1 \end{bmatrix} \in \mathbb{T}_n(R)$, where $a_{11} \in R, \alpha \in \mathbb{M}_{1\times(n-1)}(R)$ and $A_1 \in \mathbb{T}_{n-1}(R)$. Since R is a commutative uniquely clean ring, there exists a unique idempotent $E = \begin{bmatrix} e_{11} & x \\ & E_1 \end{bmatrix}$ such that $A - E \in U\big(\mathbb{T}_n(R)\big)$ and $E \in comm(A)$ by Theorem 2. Write $A - E = \begin{bmatrix} u_{11} & \alpha - x \\ & U_1 \end{bmatrix}$. According to the proof of Theorem 2, we know that $\alpha(E_1 - e_{11}I_{n-1}) = x\big(U_1 - (u_{11} + 2e_{11} - 1)I_{n-1}\big)$ by $(*)$ and A_1 is uniquely strongly clean with E_1. This implies that $E_1 \in comm^2(A_1)$ by induction. For any $X = \begin{bmatrix} x_{11} & \beta \\ & X_1 \end{bmatrix} \in comm(A)$, we have $x_{11}\alpha + \beta A_1 = a_{11}\beta + \alpha X_1$, and so $\alpha(X_1 - x_{11}I_{n-1}) = \beta(A_1 - a_{11}I_{n-1})$. We check that

$$\begin{aligned} & \beta(A_1 - a_{11}I_{n-1})(E_1 - e_{11}I_{n-1}) \\ = & \alpha(X_1 - x_{11}I_{n-1})(E_1 - e_{11}I_{n-1}) \\ = & \alpha(E_1 - e_{11}I_{n-1})(X_1 - x_{11}I_{n-1}) \\ = & x\big(U_1 - (u_{11} + 2e_{11} - 1)I_{n-1}\big)(X_1 - x_{11}I_{n-1}) \\ = & x(X_1 - x_{11}I_{n-1})\big(U_1 - (u_{11} + 2e_{11} - 1)I_{n-1}\big) \end{aligned}$$

because $E_1 \in comm^2(A_1)$ and $X_1 \in comm(A_1)$. Moreover,

$$\begin{aligned} & \beta(A_1 - a_{11}I_{n-1})(E_1 - e_{11}I_{n-1}) \\ = & \beta(E_1 - e_{11}I_{n-1})\big(E_1 + U_1 - (e_{11} + u_{11})I_{n-1}\big) \\ = & \beta(E_1 - e_{11}I_{n-1})\big(E_1 + e_{11}I_{n-1} + (U_1 - 2e_{11} - u_{11})I_{n-1}\big) \\ = & \beta\big(E_1 - e_{11}I_{n-1} + (E_1 - e_{11}I_{n-1})(U_1 - 2e_{11} - u_{11})I_{n-1}\big) \\ = & \beta(E_1 - e_{11}I_{n-1})\big(U_1 + (1 - 2e_{11} - u_{11})I_{n-1}\big) \\ = & \beta(E_1 - e_{11}I_{n-1})\big(U_1 - (u_{11} + 2e_{11} - 1)I_{n-1}\big). \end{aligned}$$

This shows that $\beta(E_1 - e_{11}I_{n-1}) = x(X_1 - x_{11}I_{n-1})$ since $U_1 - (u_{11} + 2e_{11} - 1)I_{n-1} \in U\big(\mathbb{T}_{n-1}(R)\big)$. Thus, we get $e_{11}\beta + xX_1 = x_{11}x + \beta E_1$; hence, $EX = XE$. That is, $E \in comm^2(A)$, as claimed.

(2) ⇒ (1) Let $a \in R$. Then $A = diag(a, a, \dots, a) \in \mathbb{T}_n(R)$. Hence, we can find a unique idempotent $E = [e_{ij}] \in comm^2(A)$ such that $A - E \in U\big(\mathbb{T}_n(R)\big)$. This implies that $e_{11} \in R$ is an idempotent and $a - e_{11} \in U(R)$. Suppose that $a - e \in U(R)$ with an idempotent $e \in R$. Then $F = diag(e, e, \dots, e) \in \mathbb{T}_n(R)$ is an idempotent. Further, $F \in comm^2(A)$, and that $A - F \in U\big(\mathbb{T}_n(R)\big)$. By the uniqueness, we get $E = F$, and then $e = e_{11}$. Therefore R is uniquely clean, as asserted. □

The next result showed that if R is commutative uniquely clean, then both A and $-A$ are uniquely strongly clean for any $A \in \mathbb{T}_n(R)$.

Corollary 5 *Let R be a commutative uniquely clean ring. Then for any $A \in \mathbb{T}_n(R)$, there exists a unique idempotent $E \in comm(A)$ such that $A - E, A + E \in U\big(\mathbb{T}_n(R)\big)$.*

Proof In view of Theorem 4, we have a unique idempotent $E \in comm^2\big(A^2\big)$ such that $A^2 - E$ is invertible. Obviously, $EA = AE$. Then $A - E$ and $A + E$ are invertible. On the other hand, if $A - F$ and $A + F$ are invertible for some idempotent F that commutes with A, then $A^2 - F$ is invertible. Then $A - E, A - F \in U\big(\mathbb{T}_n(R)\big)$ and $EA = AE, FA = AF$. By Theorem 2, we have $E = F$, as desired. □

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