Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2015) 39: 650 – 664
© TÜBİTAK
doi:10.3906/mat-1410-4

On isophote curves and their characterizations

Fatih DOĞAN[1,*], **Yusuf YAYLI**[2]
[1]Department of Mathematics, Faculty of Science, Bartın University, Bartın, Turkey
[2]Department of Mathematics, Faculty of Science, Ankara University, Ankara, Turkey

Received: 02.10.2014 • **Accepted/Published Online:** 01.05.2015 • **Printed:** 30.09.2015

Abstract: An isophote curve comprises a locus of the surface points whose normal vectors make a constant angle with a fixed vector. The main objective of this paper is to find the axis of an isophote curve via its Darboux frame and afterwards to give some characterizations about the isophote curve and its axis in Euclidean 3-space. Particularly, for isophote curves lying on a canal surface other characterizations are obtained.

Key words: Isophote curve, silhouette curve, geodesic, general helix, slant helix, canal surface

1. Introduction

An isophote curve is one of the characteristic curves on a surface such as parameter, geodesic, and asymptotic curves or lines of curvature.

An isophote curve on a surface is a nice consequence of Lambert's cosine law in the optics branch of physics. Lambert's law states that the intensity of illumination on a diffuse surface is proportional to the cosine of the angle generated between the surface normal vector N and the light vector d. According to this law the intensity is irrespective of the actual viewpoint; hence the illumination is the same when viewed from any direction [9]. In other words, isophotes of a surface are curves with the property that their points have the same light intensity from a given source (curves of constant illumination intensity). When the source light is at infinity, we may consider that the light flow consists of parallel lines. Hence, we can give a geometric description of isophote curves on surfaces, namely, they are curves such that the surface normal vectors in points of the curve make a constant angle with a fixed direction (which represents the light direction). These curves are successfully used in computer graphics but also it is interesting to study them for geometry.
Then to find an isophote curve on a surface we use the formula

$$\frac{\langle N(u,v),d\rangle}{\|N(u,v)\|}=\cos\theta,\ 0\leq\theta\leq\frac{\pi}{2}.$$

In the special case, an isophote curve is called a silhouette curve if

$$\frac{\langle N(u,v),d\rangle}{\|N(u,v)\|}=\cos\frac{\pi}{2}=0,$$

where d is the direction vector of the line of sight from infinity.

*Correspondence: mathfdogan@hotmail.com
2010 *AMS Mathematics Subject Classification:* 53A04; 53A05.

Koenderink and van Doorn [6] studied the field of constant image brightness contours (isophote curves). They showed that the spherical image (the Gauss map) of an isophote curve is a latitude circle on the unit sphere S^2 and the problem was reduced to that of obtaining the inverse Gauss map of these circles. By means of this, they defined two kind singularities of the Gauss map: folds (curves) and simple cusps (apex, antapex points), and there are structural properties of the field of isophote curves that bear an invariant relation to geometric features of the object.

Poeschl [7] used isophote curves in car body construction via detecting irregularities along these curves on a free form surface. These irregularities (discontinuity of a surface or of the Gaussian curvature) emerge by differentiating of the equation $\langle N(u,v), l\rangle = \cos\theta = c$ (constant)

$$\langle N_u, l\rangle\, du + \langle N_v, l\rangle\, dv = 0$$

$$\frac{dv}{du} = -\frac{\langle N_u, l\rangle}{\langle N_v, l\rangle}, \ \ \langle N_v, l\rangle \neq 0,$$

where l (d) is the light vector.

Sara [8] researched local shading of a surface through isophote curve properties. By using the fundamental theory of surfaces, he focused on accurate estimation of surface normal tilt and on qualitatively correct Gaussian curvature recovery.

Kim and Lee [5] parameterized isophote curves for a surface of rotation and a canal surface. They utilized the fact that both these surfaces decompose into a set of circles where the surface normal vectors at points on each circle construct a cone. Again the vectors that make a constant angle with the fixed vector d construct another cone and thus the tangential intersection of these cones gives the parametric range of the connected component isophote curve. Similarly, the same authors [4] parameterized the perspective silhouette of a canal surface by solving the problem that characteristic circles meet each other tangentially.

Izumiya and Takeuchi [3] defined a slant helix as a space curve whose principal normal lines make a constant angle with a fixed direction. They showed that a certain slant helix is also a geodesic on the tangent developable surface of a general helix. As an amazing consequence in our paper, we see that the curve, which is both a geodesic and a slant helix on a surface, is an isophote curve.

Dogan [1] has studied isophote curves on timelike surfaces in Minkowski space E_1^3. Recently, Dogan and Yayli [2] have also investigated isophote curves on spacelike surfaces in E_1^3. In both papers, they observed that there is a close relation between isophote curves and special curves on the surfaces.

A canal surface is the envelope of a family of one parameter spheres and is useful to represent various objects, e.g., pipes, hoses, ropes, or intestines of a body. A canal surface is an important instrument in surface modeling for CAD/CAM such as tubular surfaces, tori, and Dupin cyclides.

In this paper, we give some basic facts and concepts concerning curve and surface theory in section 2. In section 3, we concentrate on finding the axis of an isophote curve and also to characterize it in different ways. Finally, in section 4, for an isophote curve lying on a canal surface, we obtain interesting results as to a moving sphere that generates such a canal surface and then we find isophote curves as some v-parameter curves on the tube.

2. Preliminaries

Firstly, we give some basic notions about curves and surfaces. The differential geometry of curves starts with a smooth map of s; let us call it $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ that parameterizes a spatial curve and it will be denoted again with α. We say that the curve is parameterized by arc-length if $\left\|\alpha'(s)\right\| = 1$ (unit-speed), where $\alpha'(s)$ is the first derivative of α with respect to s. Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ be a regular curve with an arc-length parameter s and $\kappa(s) = \left\|\alpha''(s)\right\| > 0$, where κ is the curvature of α and α'' is the second derivative of α with respect to s. Since the curvature κ is nonzero, the Frenet frame $\{T,\ n,\ b\}$ is well-defined along the curve α and as follows:

$$T(s) = \alpha'(s),$$
$$n(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|},$$
$$b(s) = T(s) \times n(s),$$

where T, n, and b are the tangent, the principal normal, and the binormal of α, respectively. For a unit-speed curve with $\kappa > 0$, the derivatives of the Frenet frame (Frenet-Serret formulas) are given by

$$T'(s) = \kappa(s)n(s),$$
$$n'(s) = -\kappa(s)T(s) + \tau(s)b(s),$$
$$b'(s) = -\tau(s)n(s),$$

where $\tau(s) = \dfrac{\left\langle \alpha'(s) \times \alpha''(s), \alpha'''(s) \right\rangle}{\kappa^2(s)}$ is the torsion of α and "$\times$" is the cross product on $\mathbb{R}^3$.

Let M be a regular surface and $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a unit-speed curve on the surface. Then the Darboux frame $\{T,\ B,\ N\}$ is well-defined along the curve α, where T is the tangent of α and N is the unit normal of M, and $B = N \times T$. Darboux equations of this frame are given by

$$\begin{aligned} T' &= k_g B + k_n N, \\ B' &= -k_g T + \tau_g N, \\ N' &= -k_n T - \tau_g B, \end{aligned} \tag{2.1}$$

where " $'$ " denotes the derivative of $T,\ B,$ and N with respect to s along the curve α; k_n, k_g, and τ_g are the normal curvature, the geodesic curvature, and the geodesic torsion of α, respectively. With the above notations, let ϕ denote the angle between the surface normal N and the binormal b. Using equations in (2.1), we get

$$\begin{aligned} \kappa^2 &= k_g^2 + k_n^2, \\ k_g &= \kappa \cos\phi, \\ k_n &= \kappa \sin\phi, \\ \tau_g &= \tau - \phi'. \end{aligned} \tag{2.2}$$

If we rotate the Darboux frame $\{T,\ B,\ N\}$ by ϕ about T, we obtain the Frenet frame $\{T,\ n,\ b\}$.

$$\begin{bmatrix} T \\ n \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} T \\ B \\ N \end{bmatrix}$$

$$\begin{aligned} T &= T, \\ n &= \cos(\phi)B + \sin(\phi)N, \\ b &= -\sin(\phi)B + \cos(\phi)N. \end{aligned}$$

From the above equations, we obtain

$$\begin{aligned} N &= \sin(\phi)n + \cos(\phi)b, \\ B &= \cos(\phi)n - \sin(\phi)b. \end{aligned} \tag{2.3}$$

3. The axis of an isophote curve

In this section, we will get the fixed vector d of an isophote curve via its Darboux frame. Let M be a regular surface and let $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a unit-speed isophote curve. Then from the definition of the isophote curve

$$\langle N(u,v), d\rangle = \cos\theta = constant, \tag{3.1}$$

where $N(u,v)$ is the unit normal vector of the surface $S(u,v)$ (a parameterization of M) and d is the unit fixed vector on the axis of isophote curve.
Now, we begin to find the fixed vector d. Since $\alpha : I \subset \mathbb{R} \longrightarrow M$ is a unit-speed isophote curve, the Darboux frame can be defined as $\{T,\ B,\ N\}$ along the curve α. If we differentiate Eq. (3.1) with respect to s along the curve, then we have

$$\left\langle N^{'}, d\right\rangle = 0. \tag{3.2}$$

From Eq. (2.1), it follows that

$$\langle -k_n T - \tau_g B, d\rangle = 0$$

$$-k_n \langle T, d\rangle - \tau_g \langle B, d\rangle = 0$$

$$\langle T, d\rangle = -\frac{\tau_g}{k_n}\langle B, d\rangle .$$

Because the Darboux frame $\{T,\ B,\ N\}$ is an orthonormal basis, if we say $\langle B, d\rangle = a$ in the last equation, then d can be written as

$$d = -\frac{\tau_g}{k_n} aT + aB + \cos\theta N.$$

Since $\|d\| = 1$, we get

$$a = \mp\frac{k_n}{\sqrt{k_n^2+\tau_g^2}}\sin\theta.$$

Thus, the vector d is obtained as

$$d = \pm\frac{\tau_g}{\sqrt{k_n^2+\tau_g^2}}\sin\theta T \mp \frac{k_n}{\sqrt{k_n^2+\tau_g^2}}\sin\theta B + \cos\theta N \tag{3.3}$$

or from Eq. (2.2) and Eq. (2.3) in terms of the Frenet frame,

$$d = \pm\frac{\tau_g}{\sqrt{k_n^2+\tau_g^2}}\sin\theta T + \left[\mp\frac{k_n k_g}{\kappa\sqrt{k_n^2+\tau_g^2}}\sin\theta + \frac{k_n}{\kappa}\cos\theta\right]n$$
$$+\left[\pm\frac{k_n^2}{\kappa\sqrt{k_n^2+\tau_g^2}}\sin\theta + \frac{k_g}{\kappa}\cos\theta\right]b. \tag{3.4}$$

Here, in fact, d is a constant vector. Let us see this. If we differentiate $N^{'}$ and Eq. (3.2) with respect to s, we get

$$N^{''} = (-k_n^{'} + k_g\tau_g)T - (k_n k_g + \tau_g^{'})B - (k_n^2+\tau_g^2)N.$$

and thus

$$\left\langle N^{''}, d\right\rangle = \frac{\mp(k_n^{'}\tau_g - k_n\tau_g^{'}) \pm k_g(k_n^2+\tau_g^2)}{\sqrt{k_n^2+\tau_g^2}}\sin\theta - (k_n^2+\tau_g^2)\cos\theta = 0,$$

where d is form of Eq. (3.3). As a result, we have

$$\cot\theta = \pm\left[\frac{k_n^2}{(k_n^2+\tau_g^2)^{\frac{3}{2}}}\left(\frac{\tau_g}{k_n}\right)^{'} + \frac{k_g}{(k_n^2+\tau_g^2)^{\frac{1}{2}}}\right],$$

$$\tan\theta = \frac{(k_n^2+\tau_g^2)^{\frac{3}{2}}}{\pm k_g(k_n^2+\tau_g^2) \pm (k_n\tau_g^{'} - k_n^{'}\tau_g)}. \tag{3.5}$$

By Eq. (2.1) and Eq. (3.3), the derivative of d with respect to s is that

$$\begin{aligned} d^{'} &= \pm\sin\theta\left[\left(\frac{\tau_g}{\sqrt{k_n^2+\tau_g^2}}\right)^{'}T + \frac{\tau_g}{\sqrt{k_n^2+\tau_g^2}}(k_g B + k_n N)\right] \\ &\quad \mp\sin\theta\left[\left(\frac{k_n}{\sqrt{k_n^2+\tau_g^2}}\right)^{'}B + \frac{k_n}{\sqrt{k_n^2+\tau_g^2}}(-k_g T + \tau_g N)\right] + \cos\theta(-k_n T - \tau_g B). \end{aligned}$$

If we arrange this equality, we obtain

$$d^{'} = \left(\pm\sin\theta\left[\left(\frac{\tau_g}{\sqrt{k_n^2+\tau_g^2}}\right)^{'} + \frac{k_g k_n}{\sqrt{k_n^2+\tau_g^2}}\right] - k_n\cos\theta\right)T$$
$$+\left(\pm\sin\theta\left[-\left(\frac{k_n}{\sqrt{k_n^2+\tau_g^2}}\right)^{'} + \frac{k_g\tau_g}{\sqrt{k_n^2+\tau_g^2}}\right] - \tau_g\cos\theta\right)B. \tag{3.6}$$

Furthermore, from Eq. (3.5) we have

$$\cos\theta = \pm\sin\theta\frac{k_g(k_n^2+\tau_g^2) + k_n\tau_g^{'} - k_n^{'}\tau_g}{(k_n^2+\tau_g^2)^{\frac{3}{2}}}.$$

If the last equality is replaced in Eq. (3.6), we get

$$d^{'} = \pm\sin\theta\left(\begin{array}{c}\dfrac{\tau_g^{'}(k_n^2+\tau_g^2)-\tau_g(k_nk_n^{'}+\tau_g\tau_g^{'})+k_gk_n(k_n^2+\tau_g^2)}{(k_n^2+\tau_g^2)^{\frac{3}{2}}}\\ +\dfrac{k_nk_n^{'}\tau_g-k_n^2\tau_g^{'}-k_gk_n(k_n^2+\tau_g^2)}{(k_n^2+\tau_g^2)^{\frac{3}{2}}}\end{array}\right)T$$

$$\pm\sin\theta\left(\begin{array}{c}\dfrac{-\kappa_n^{'}(k_n^2+\tau_g^2)+k_n(k_nk_n^{'}+\tau_g\tau_g^{'})+k_g\tau_g(k_n^2+\tau_g^2)}{(k_n^2+\tau_g^2)^{\frac{3}{2}}}\\ +\dfrac{k_n^{'}\tau_g^2-k_n\tau_g\tau_g^{'}-k_g\tau_g(k_n^2+\tau_g^2)}{(k_n^2+\tau_g^2)^{\frac{3}{2}}}\end{array}\right)B.$$

As can be directly seen above, the coefficients of T and B are zero. Therefore, $d^{'}=0$, namely, d is a constant vector. Then the axis of an isophote curve is the line in the fixed direction d. From this time, for the axis of an isophote curve will be also used d.

Theorem 1 *A unit-speed curve α on a surface is an isophote curve if and only if*

$$\cot\theta=\mu(s)=\pm\left(\frac{k_n^2}{(k_n^2+\tau_g^2)^{\frac{3}{2}}}\left(\frac{\tau_g}{k_n}\right)^{'}+\frac{k_g}{(k_n^2+\tau_g^2)^{\frac{1}{2}}}\right)(s)$$

is a constant function where $k_n\neq 0$.

Proof As α is an isophote curve, the Gauss map along the curve α is a circle on the unit sphere S^2. Hence, if we compute the Gauss map $N_{|\alpha}:I\longrightarrow S^2$ along the curve α, the geodesic curvature of $N_{|\alpha}$ becomes $\mu(s)$ as shown below.

$$\begin{aligned} N_{|\alpha}^{'} &= -k_nT-\tau_gB,\\ N_{|\alpha}^{''} &= (-k_n^{'}+k_g\tau_g)T-(k_nk_g+\tau_g^{'})B-(k_n^2+\tau_g^2)N,\\ N_{|\alpha}^{'}\times N_{|\alpha}^{''} &= \tau_g(k_n^2+\tau_g^2)T+\left(k_g(k_n^2+\tau_g^2)+k_n^2\left(\frac{\tau_g}{k_n}\right)^{'}\right)N-k_n(k_n^2+\tau_g^2)B.\end{aligned}$$

Therefore, we get the curvature $\bar{\kappa}$ of $N_{|\alpha}$

$$\begin{aligned}\bar{\kappa} &= \frac{\left\|N_{|\alpha}^{'}\times N_{|\alpha}^{''}\right\|}{\left\|N_{|\alpha}^{'}\right\|^3}\\ &= \frac{\sqrt{\tau_g^2(k_n^2+\tau_g^2)^2+\left(k_g(k_n^2+\tau_g^2)+k_n^2\left(\frac{\tau_g}{k_n}\right)^{'}\right)^2+k_n^2(k_n^2+\tau_g^2)^2}}{\sqrt{(k_n^2+\tau_g^2)^3}}\\ &= \sqrt{1+\frac{\left(k_g(k_n^2+\tau_g^2)+k_n^2\left(\frac{\tau_g}{k_n}\right)^{'}\right)^2}{(k_n^2+\tau_g^2)^3}}.\end{aligned}$$

Let $\bar{k}_g$ and $\bar{k}_n$ be the geodesic curvature and the normal curvature of the Gauss map $N_{|\alpha}$, respectively. Since the normal curvature $\bar{k}_n = 1$ on S^2, if we substitute $\bar{k}_n$ and $\bar{\kappa}$ in the following equation, we obtain the geodesic curvature $\bar{k}_g$ as follows:

$$(\bar{\kappa})^2 = (\bar{k}_g)^2 + (\bar{k}_n)^2$$

$$\bar{k}_g(s) = \mu(s) = \cot\theta = \pm\left(\frac{k_n^2}{(k_n^2+\tau_g^2)^{\frac{3}{2}}}\left(\frac{\tau_g}{k_n}\right)' + \frac{k_g}{(k_n^2+\tau_g^2)^{\frac{1}{2}}}\right)(s).$$

Then the spherical images of isophote curves are circles if and only if $\mu(s)$ is a constant function. $\square$

Lemma 1 ([3]) *Let α be a unit-speed space curve with $\kappa(s) \neq 0$. Then α is a slant helix if and only if $\sigma(s) = \left(\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)'\right)(s)$ is a constant function.*

Lemma 2 (The Lancret Theorem) *Let α be a unit-speed space curve with $\kappa(s) \neq 0$. Then α is a general helix if and only if $\left(\frac{\tau}{\kappa}\right)(s)$ is a constant function.*

Theorem 2 *Let α be a unit-speed isophote curve on the surface M. In that case, we have the following:*
(1) α is a geodesic on M if and only if α is a slant helix with the fixed vector

$$d = \pm\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\sin\theta T \pm \cos\theta n \pm \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\sin\theta b.$$

(2) α is an asymptotic curve on M if and only if α is a general helix with the fixed vector $d = \pm\sin\theta T \pm \cos\theta b$.
(3) If α is a line of curvature, then α is a plane curve and the angle $\theta = \mp\phi$ or $\theta = \mp(\pi - \phi)$.

Proof (1) Since α is a geodesic (i.e. the surface normal N concurs with the principal normal n along the curve α), we have $k_g = 0$ and therefore from Eq. (2.2) it follows that $k_n = \pm\kappa$ and $\tau_g = \tau$. By substituting k_g and k_n in the expression of $\mu(s)$, we obtain that

$$\mu(s) = \pm\left(\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)'\right)(s)$$

is a constant function. Then, by Lemma 1, α is a slant helix. Using Eq. (3.4), the fixed vector of the slant helix is obtained as

$$d = \pm\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\sin\theta T \pm \cos\theta n \pm \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\sin\theta b.$$

In contrast, let α be a slant helix with the fixed vector

$$d = \pm\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\sin\theta T \pm \cos\theta n \pm \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\sin\theta b.$$

Then from Eq. (3.4) the geodesic curvature k_g must be zero, that is to say, α is a geodesic on M.
(2) Since α is an asymptotic curve on M, we have $k_n = 0$ and consequently from Eq. (2.2) it follows that $k_g = \pm\kappa$ and $\tau_g = \tau$. If we replace k_n, k_g, and τ_g in Eq. (3.5), we obtain that

$$\tan\theta = \pm(\frac{\tau_g}{k_g})(s) = \pm(\frac{\tau}{\kappa})(s)$$

is a constant function. Then, from Lemma 2 and Eq. (3.4) it follows that α is a general helix with the fixed vector

$$d = \pm\sin\theta T \pm \cos\theta b.$$

For the converse, let α be a general helix with the fixed vector $d = \pm\sin\theta T \pm \cos\theta b$. By applying Eq. (3.4) again, we get $k_n = 0$; in other words α is an asymptotic curve on M.
(3) Since α is a line of curvature on M, we have $\tau_g = 0$. Accordingly, from Eq. (3.5) we conclude that

$$\tan\theta = \pm\frac{k_n}{k_g} = \pm\frac{\kappa\sin\phi}{\kappa\cos\phi} = \pm\tan\phi.$$

In this situation, we conclude that $\phi = \pm\theta$ or $\phi = \pi \pm \theta$. Because ϕ is a constant, by $\tau_g = \tau - \phi^{'} = 0$, we obtain $\tau = 0$. Then α is a plane curve. □

We now illustrate Theorem 2 case 1 in the following example.

Example 1 ([3]) *We consider a space curve defined by*

$$\begin{aligned}\gamma(\theta) \quad = \quad (& -\frac{(a^2-b^2)}{2a}\left(\frac{\cos((a+b)\theta)}{(a+b)^2} + \frac{\cos((a-b)\theta)}{(a-b)^2}\right), \\ & -\frac{(a^2-b^2)}{2a}\left(\frac{\sin((a+b)\theta)}{(a+b)^2} + \frac{\sin((a-b)\theta)}{(a-b)^2}\right), -\frac{\sqrt{a^2-b^2}}{ab}\cos(b\theta)\ ).\end{aligned}$$

We can calculate that

$$\kappa(\theta) = \sqrt{a^2-b^2}\cos(b\theta), \qquad \tau(\theta) = \sqrt{a^2-b^2}\sin(b\theta),$$

$$\sigma(\theta) = \frac{b}{\sqrt{a^2-b^2}}, \quad \frac{\tau}{\kappa}(\theta) = \tan(b\theta),$$

$$\left(\frac{\tau}{\kappa}(\theta)\right)^{'} = \frac{b}{\cos^2(b\theta)} \neq 0, \ \textit{and} \ \left(\frac{\tau}{\kappa}(\theta)\right)^{''} = \frac{2b^2\tan(b\theta)}{\cos^2(b\theta)} \neq 0.$$

Therefore, $\gamma(\theta)$ is a slant helix and it is not a cylindrical helix. By Theorem 4.2 [3]*, it is a geodesic of the tangent developable surface of a cylindrical helix. In fact, the corresponding tangent developable surface is the rectifying developable surface of $\gamma(\theta)$ by Proposition 4.1* [3]*. We now draw the picture of $\gamma(\theta)$ ($a = 2, b = 1$) in Figure a[3]. We also draw the rectifying developable surface of $\gamma(\theta)$ in Figure b[3].*

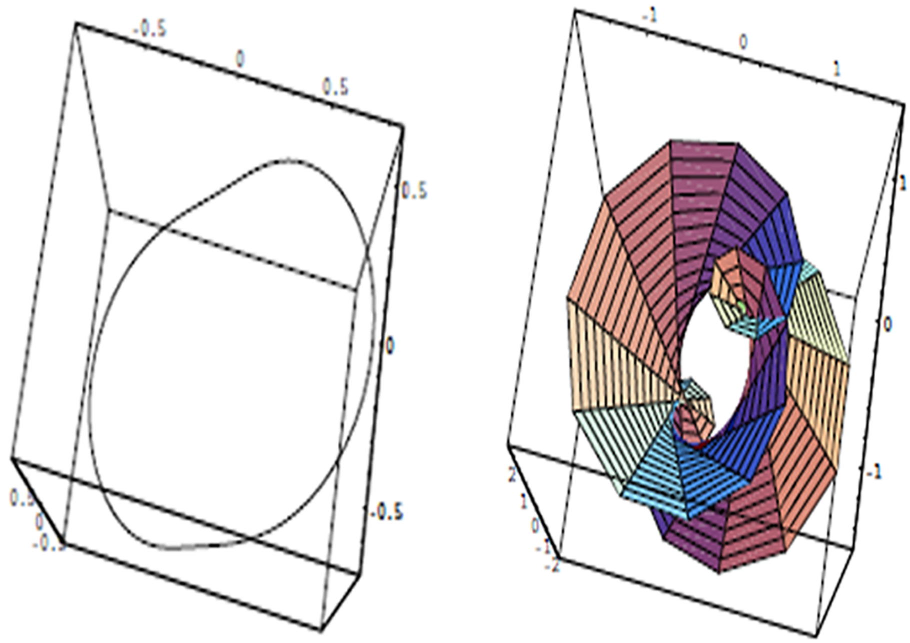

Figure. a) [3] The slant helix $\gamma(\theta)$, which is also a geodesic (isophote curve) of the tangent developable surface of a clylindrical helix, b) The rectifying developable surface of $\gamma(\theta)$ corresponds to the tangent developable surface of a cylindrical helix.

In that case, we can say that there is a curve that is both a slant helix and a geodesic on a surface. This example is a consequence of Theorem 2 case 1, that is to say, such a curve on a surface is an isophote curve. As a simpler example, a helix on a circular cylinder is a geodesic. Moreover, each helix is also a slant helix. Thus, the helix on a circular cylinder is both a slant helix and a geodesic, namely, it is an isophote curve.

Theorem 3 *Let α be a unit-speed isophote curve on the surface M. Then we have the following:*
(1) The axis d is perpendicular to the tangent line of α if and only if α is a line of curvature on M.
(2) The axis d is perpendicular to the principal normal line of α if and only if α is an asymptotic curve on M or $\dfrac{\tau_g}{k_n}$ is a constant function.

Proof (1) Let α be a unit-speed isophote curve. Then from Eq. (3.4) it follows that

$$\langle T, d\rangle = \pm\frac{\tau_g}{\sqrt{k_n^2+\tau_g^2}}\sin\theta.$$

Therefore, the axis d is perpendicular to the tangent line of α if and only if α is a line of curvature on M.
(2) Let the axis d be perpendicular to the principal normal line of α. Then by Eq. (3.4) we have

$$\begin{aligned}\langle n, d\rangle &= \mp\frac{k_n k_g}{\kappa\sqrt{k_n^2+\tau_g^2}}\sin\theta+\frac{k_n}{\kappa}\cos\theta\\ &= \frac{k_n}{\kappa}\left(\mp\frac{k_g}{\sqrt{k_n^2+\tau_g^2}}\sin\theta+\cos\theta\right)\\ &= 0.\end{aligned}$$

By solving this equation, we get $k_n = 0$ or $\cot\theta = \pm\dfrac{k_g}{\sqrt{k_n^2+\tau_g^2}}$. From Eq. (3.5), we gather that

$$\cot\theta = \frac{k_g}{\sqrt{k_n^2+\tau_g^2}} = \frac{k_n^2}{(k_n^2+\tau_g^2)^{\frac{3}{2}}}\left(\frac{\tau_g}{k_n}\right)' + \frac{k_g}{(k_n^2+\tau_g^2)^{\frac{1}{2}}}$$

$$\frac{k_n^2}{(k_n^2+\tau_g^2)^{\frac{3}{2}}}\left(\frac{\tau_g}{k_n}\right)' = 0.$$

In the last equation, for $k_n \neq 0$, $(\dfrac{\tau_g}{k_n})'(s) = 0$ and hence $(\dfrac{\tau_g}{k_n})(s)$ is a constant function. In the same way, the proof of sufficiency is clear. □

Corollary 1 *Let α be a silhouette curve on M. Then α is a line of curvature if and only if it is a plane geodesic curve.*

Proof Let α be a line of curvature. Because α is both a silhouette curve and a line of curvature, we possess $\tau_g = 0$ and $\phi = \theta = \pm\dfrac{\pi}{2}$. Therefore, $k_g = 0$, namely, α is a geodesic. Now that ϕ is a constant and $\tau_g = 0$, from Eq. (2.2) it follows that τ must be zero. Eventually, α is a plane geodesic curve.
Conversely, let α be a plane geodesic curve. In this case, k_g and τ need to be zero. From this, $\tau_g = 0$, in other words, α is a line of curvature. This completes the proof.
Furthermore, from this corollary and Eq. (2.1) it follows that $B' = 0$, i.e., by Eq. (3.3) the axis of silhouette curve d becomes B. □

Corollary 2 *Let α be a silhouette curve with arc-length parameter on M. Then*
(1) The axis d lies in the plane spanned by T and B,
(2) α is a geodesic if and only if the axis d lies on the rectifying plane of the silhouette curve α.

Proof (1) Since α is a silhouette curve, the surface normal vectors are orthogonal to the axis d, that is, $\theta = \dfrac{\pi}{2}$. Then by Eq. (3.3) we get

$$d = \pm\frac{\tau_g}{\sqrt{k_n^2+\tau_g^2}}T \mp \frac{k_n}{\sqrt{k_n^2+\tau_g^2}}B.$$

We see that d lies in the plane spanned by T and B.
(2) Since α is a geodesic, by Theorem 2, α is a slant helix with the fixed vector

$$d = \pm\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\sin\theta T + \cos\theta n \pm \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\sin\theta b.$$

Moreover, as α is a silhouette curve, we have $\theta = \dfrac{\pi}{2}$ and

$$d = \pm\frac{\tau}{\sqrt{\kappa^2+\tau^2}}T \mp \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}b.$$

In other words, the axis d lies on the rectifying plane of the silhouette curve α. By contrast, suppose that the axis d lies on the rectifying plane of the silhouette curve α. Applying Eq. (3.4) we see that $k_g = 0$, namely, α is a geodesic on M. □

4. Some characterizations for isophote curves on a canal surface

In this section, we introduce canal surfaces and tubes and then give some characterizations for isophote curves on them. Firstly, we define a canal surface. A canal surface is defined as the envelope of a family of one parameter spheres. Alternatively, a canal surface is the envelope of a moving sphere with varying radius, defined by the trajectory $C(t)$ (spine curve) of its centers and a radius function $r(t)$. If the radius function $r(t) = r$ is a constant, then the canal surface is called a tube or pipe surface.

Now we give parametric representation of a canal surface. Since a canal surface is the envelope of one parameter spheres with center $C(t)$ and radius $r(t)$, a surface point $p \in \mathbb{E}^3$ satisfies the following equations.

$$\|p - C(t)\| = r(t),$$

$$(p - C(t)) \boldsymbol{.} C^{'}(t) + r(t)r^{'}(t) = 0, \tag{4.1}$$

where "$\boldsymbol{.}$" is the dot product on $\mathbb{E}^3$. If the spine curve $C(t)$ has arc-length parametrization, then the canal surface (equations in (4.1)) is parametrized as

$$K(s,v) = C(s) - r(s)r^{'}(s)T(s) \mp r(s)\sqrt{1 - r'(s)^2}\,(\cos v\ n(s) + \sin v\ b(s))\,, \tag{4.2}$$

where $0 \leq v < 2\pi$; T, n, and b are the tangent, principal normal, and binormal of $C(s)$, respectively.
Since the moving sphere with center $C(s)$ is tangent to the canal surface, the sphere and canal surface have the same tangent plane at arbitrary canal surface points. We know that the normal vector of the sphere is the position vector of it. Then the unit normal vector of the canal surface can be computed as follows:

$$N(s,v) = K(s,v) - C(s)$$

$$N(s,v) = -r(s)r^{'}(s)T(s) \mp r(s)\sqrt{1 - r'(s)^2}\,(\cos v\ n(s) + \sin v\ b(s))\,.$$

As $\|K(s,v) - C(s)\| = r(s)$, norm of the vector $N(s,v)$ is $r(s)$. Hence, if we normalize $N(s,v)$, we get

$$\frac{N(s,v)}{\|N(s,v)\|} = -r^{'}(s)T(s) \mp \sqrt{1 - r'(s)^2}\,(\cos v\ n(s) + \sin v\ b(s))\,. \tag{4.3}$$

In the previous section, we mentioned the axis of an isophote curve and obtained some characterizations of it. This time we shall give other characterizations as regards isophote curves lying on a canal surface. More precisely,

Theorem 4 *If α is a unit-speed isophote curve with the axis d on the canal surface $K(s,v)$, then*

$$-r^{'}\langle T,d\rangle \mp \sqrt{1-r'^2}\cos(v+\phi)\langle B,d\rangle + \left(\sqrt{1-r'^2}\sin(v+\phi) - 1\right)\langle N,d\rangle = 0,$$

where ϕ is the angle between the surface normal N and the binormal b.

Proof Inner product of the axis d and the unit normal vector N in Eq. (4.3) is that

$$\langle N, d\rangle = -r^{'} \langle T, d\rangle \mp \sqrt{1-r^{'2}}\left[\cos v \ \langle n, d\rangle + \sin v \ \langle b, d\rangle\right].$$

If we substitute $\langle n, d\rangle$ and $\langle b, d\rangle$ above, we get

$$\begin{aligned}\langle N, d\rangle &= -r^{'} \langle T, d\rangle \mp \sqrt{1-r^{'2}}\left[\mp\frac{k_n k_g}{\kappa\sqrt{k_n^2+\tau_g^2}}\sin\theta + \frac{k_n}{\kappa}\cos\theta\right]\cos v \\ &\quad \mp\sqrt{1-r^{'2}}\left[\pm\frac{k_n^2}{\kappa\sqrt{k_n^2+\tau_g^2}}\sin\theta + \frac{k_g}{\kappa}\cos\theta\right]\sin v.\end{aligned}$$

By arranging this equation, we obtain

$$\begin{aligned}\langle N, d\rangle &= -r^{'} \langle T, d\rangle \mp \sqrt{1-r^{'2}}\left[\frac{k_n \ (k_g\cos v - k_n\sin v)}{\kappa\sqrt{k_n^2+\tau_g^2}}\right]\sin\theta \\ &\quad +\sqrt{1-r^{'2}}\left[\frac{k_n\cos v + k_g\sin v}{\kappa}\right]\cos\theta.\end{aligned}$$

If we substitute $k_g = \kappa\cos\phi$, $k_n = \kappa\sin\phi$, $\langle B, d\rangle = \dfrac{k_n}{\sqrt{k_n^2+\tau_g^2}}$, and $\langle N, d\rangle = \cos\theta$ above, and use addition formulas for sine and cosine, we obtain

$$-r^{'} \langle T, d\rangle \mp \sqrt{1-r^{'2}}\cos(v+\phi)\langle B, d\rangle + \left(\sqrt{1-r^{'2}}\sin(v+\phi) - 1\right)\langle N, d\rangle = 0.$$

□

Corollary 3 *Let α be a unit-speed isophote curve on the canal surface $K(s,v)$. Then we have the following:*
(1) If the axis d is orthogonal to the tangent line of α, then the canal surface is generated by a moving sphere with linear radius function $r(s) = \lambda s + c$, where

$$\lambda = \frac{\sqrt{\left(\tan\theta\cos(v+\theta)\mp\sin(v+\theta)\right)^2 - 1}}{\tan\theta\cos(v+\theta)\mp\sin(v+\theta)} \quad \textit{and}$$
$$|\sin(v+2\theta)| > \cos\theta, \ \ \cos\theta < \sin v < -\cos\theta.$$

(2) If α is a silhouette curve and the spine curve $C(s)$ is a general helix with the axis d, then the canal surface is generated by a moving sphere with the radius function

$$r(s) = \int \frac{\tan\beta}{\sqrt{\tan^2\beta + \cos^2(v+\phi)}}ds + c; \ c > 0.$$

Proof (1) Assume that the axis d is orthogonal to the tangent line of α, namely, d lies in the plane spanned by N and B. Because $\langle N, d\rangle = \cos\theta$, by the preceding theorem we get

$$\begin{aligned}\sqrt{1-r'^2}\cos(v+\phi)\langle B,d\rangle \mp \left(\sqrt{1-r'^2}\sin(v+\phi)-1\right)\langle N,d\rangle &= 0\\ \sqrt{1-r'^2}\cos(v+\phi)\sin\theta \mp \left(\sqrt{1-r'^2}\sin(v+\phi)-1\right)\cos\theta &= 0.\end{aligned}$$

Since $\langle T, d\rangle = 0$, from Theorem 2 and Theorem 3, α is a line of curvature with $\phi = \theta$ (constant). Therefore, if the final equation is arranged, it follows that

$$1-r'^2 = \frac{1}{\cos^2(v+\theta)\left(\tan\theta \mp \tan(v+\theta)\right)^2}.$$

If we solve this quadratic equation with unknown r', we obtain

$$r(s) = \left(\frac{\sqrt{\left(\tan\theta\cos(v+\theta)\mp\sin(v+\theta)\right)^2-1}}{\tan\theta\cos(v+\theta)\mp\sin(v+\theta)}\right)s+c;\ c>0.$$

For $\left(\tan\theta\cos(v+\theta)\mp\sin(v+\theta)\right)^2 - 1 > 0$, $r(s) > 0$. By solving this inequality, we have the condition $|\sin(v+2\theta)| > \cos\theta$ and $\cos\theta < \sin v < -\cos\theta$.

(2) Assume that α is a silhouette curve, namely, $\langle N, d\rangle = 0$. Additionally, since $C(s)$ is a general helix, we can write $\langle T, d\rangle = \cos\beta$ where β is acute angle. Then

$$\begin{aligned}-r'\langle T,d\rangle \mp \sqrt{1-r'^2}\cos(v+\phi)\langle B,d\rangle &= 0\\ -r'\cos\beta \mp \sqrt{1-r'^2}\cos(v+\phi)\sin\beta &= 0.\end{aligned}$$

If the last equation is arranged, the solution of the quadratic equation with unknown r' is obtained as follows:

$$\left(\tan^2\beta + \cos^2(v+\phi)\right)r'^2 - \tan^2\beta = 0$$

$$r(s) = \int \frac{\tan\beta}{\sqrt{\tan^2\beta+\cos^2(v+\phi)}}ds + c;\ c>0.$$

Since β is an acute angle, $\tan\beta > 0$. Then $r(s) > 0$. □

Proposition 1 *Let the spine curve $C(s)$ be a general helix. If an isophote curve on the canal surface and the general helix $C(s)$ have the same axis d, then the canal surface is generated by a moving sphere with linear radius function $r(s) = \omega s + c$, where $\omega = \dfrac{-1+\sin^2 v\tan\theta}{1+\sin^2 v\tan^2\theta}$ and $\tan\theta > 1$.*

Proof Since $C(s)$ is a general helix with the axis d, from the definition of general helix $\langle T, d\rangle = \cos\theta$ (constant) and so $\langle n, d\rangle = 0$. Thus,

$$\langle N,d\rangle = \langle T,d\rangle = \cos\theta = \lambda_1,\ \ \langle b,d\rangle = \sin\theta = \lambda_2.$$

If we take inner product of the axis d and unit normal N, we have

$$\langle N, d\rangle = -r' \langle T, d\rangle \mp \sqrt{1-r'^2}\left[\cos v \ \langle n, d\rangle + \sin v \ \langle b, d\rangle\right].$$

Substituting λ_1 and λ_2 above, we get

$$\left(\lambda_1^2 + \lambda_2^2 \sin^2 v\right) r'^2 + 2\lambda_1^2 r' + \lambda_1^2 - \lambda_2^2 \sin^2 v = 0.$$

If we solve this quadratic equation, we obtain

$$r(s) = \left(\frac{-1+\sin^2 v \tan\theta}{1+\sin^2 v \tan^2\theta}\right) s + c; \ c > 0.$$

Since θ is an acute angle, $\tan\theta > 0$. In addition, for $-1+\sin^2 v\tan\theta > 0$, $r(s) > 0$. Therefore, it must be $\tan\theta > 1$. □

From now on, we will give some characterizations for isophote curves on a tube. If the radius function $r(s) = r$ is a constant, by Eq. (4.2) a tube is parametrized as follows:

$$K(s,v) = C(s) \mp r\left(\cos v \ n(s) + \sin v \ b(s)\right).$$

Proposition 2 *Let the spine curve $C(s)$ be a general helix with the axis d. Then $v_0 = \left(\frac{2k+1}{2}\right)\pi$ ($k \in Z$) parameter curves of tube are isophote curves with the axis d.*

Proof Because the normal vector of tube $N(s,v) = K(s,v) - C(s)$, we obtain $N(s,v) = \mp r\left(\cos v \ n(s) + \sin v \ b(s)\right)$. For $v_0 = \left(\frac{2k+1}{2}\right)\pi$,

$$N(s, v_0) = \mp r \ b(s).$$

Since $C(s)$ is a general helix with the axis d, from the definition of general helix $\langle b, d\rangle$ is a constant. Therefore, along the curve $v_0 = \left(\frac{2k+1}{2}\right)\pi$,

$$\langle N(s,v_0), d\rangle = \mp r \langle b, d\rangle = constant.$$

Hence $v_0 = \left(\frac{2k+1}{2}\right)\pi$ parameter curves are isophote curves with the axis d on the tube. □

Proposition 3 *Let the spine curve $C(s)$ be a slant helix with the axis d. Then $v_0 = k\pi$ ($k \in Z$) parameter curves of tube are isophote curves with the axis d.*

Proof Because the normal vector of tube $N(s,v) = K(s,v) - C(s)$, we obtain $N(s,v) = \mp r\left(\cos v \ n(s) + \sin v \ b(s)\right)$. For $v_0 = k\pi$,

$$N(s, v_0) = \mp r \ n(s).$$

Since $C(s)$ is a slant helix with the axis d, from the definition of slant helix $\langle n, d\rangle$ is a constant. Therefore, along the curve $v_0 = k\pi$,

$$\langle N(s,v_0), d\rangle = \mp r \langle n, d\rangle = constant.$$

Thus, $v_0 = k\pi$ parameter curves are isophote curves with the axis d on the tube. □

5. Conclusions

In this paper, we found the axis (fixed vector) of an isophote curve through its Darboux frame. Subsequently, we obtained some characterizations regarding these curves. By using the characterizations, we investigated the relation between special curves on a surface and isophote curves. Finally, we obtained some results for an isophote curve lying on a canal surface and then obtained several isophote curves as special parameter curves on a tube.

Acknowledgement

We are grateful to the two anonymous referees for their constructive suggestions.

References

[1] Dogan F. Isophote curves on timelike surfaces in Minkowski 3-Space. An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.); DOI: 10.2478/aicu-2014-0020.

[2] Dogan F, Yayli Y. Isophote curves on spacelike surfaces in Lorentz-Minkowski Space E_1^3. arXiv:1203.4388.

[3] Izumiya S, Takeuchi N. New special curves and developable surfaces. Turk J Math 2004; 28: 153–163.

[4] Kim KJ, Lee IK. The perspective silhouette of a canal surface. Computer Graphics Forum 2003; 1: 15–22.

[5] Kim KJ, Lee IK. Computing isophotes of surface of revolution and canal surface. Comput-Aided Des 2003; 35: 215–223.

[6] Koenderink JJ, van Doorn AJ. Photometric invariants related to solid shape. J Modern Opt 1980; 27: 981–996.

[7] Poeschl T. Detecting surface irregularities using isophotes. Comput Aided Geom Design 1984; 1: 163–168.

[8] Sara R. Local Shading Analysis via Isophotes Properties. PhD, Johannes Kepler University, Austria, 1994.

[9] http://nccastaff.bournemouth.ac.uk/jmacey/CGF/slides/IlluminationModels4up.pdf.