Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2015) 39: 698 – 705
© TÜBİTAK
doi:10.3906/mat-1412-23

Hilbert series of the finite dimensional generalized Hecke algebras

Zaffar IQBAL*
Department of Mathematics, University of Gujrat, Gujrat, Pakistan

Received: 09.12.2014 • **Accepted/Published Online:** 07.05.2015 • **Printed:** 30.09.2015

Abstract: It is known from the early results of Coxeter that the generalized Hecke algebras $H(Q_m, 3)$, $m \in \{2, 3, 4, 5\}$, are finite dimensional. In this paper we compute the Hilbert series of these finite-type group algebras.

Key words: Braid group, generalized Hecke algebras, Hilbert series

1. Introduction

The braid group B_{n+1} admits the following classical presentation given by Artin [1]:

$$B_{n+1} = \left\langle x_1, x_2, \ldots, x_n \;\middle|\; \begin{array}{l} x_i x_j = x_j\, x_i \ \text{ if } \mid i-j \mid \geq 2 \\ x_{i+1}\, x_i\, x_{i+1} = x_i\, x_{i+1}\, x_i \ \text{if } 1 \leq i \leq n-1 \end{array} \right\rangle.$$

Elements of B_{n+1} are words expressed in the generators $x_1, x_2, \ldots, x_n$ and their inverses. The braid monoid MB_{n+1} has the same presentation as B_{n+1}.

The *generalized Hecke algebras* [5] are defined as the quotients

$$H(Q, n+1) = \mathbb{C}[B_{n+1}]/\big(Q(b_i); i = 1, \ldots, n\big)$$

of the group algebras of the braid group by the ideal generated by a polynomial $Q(b_i)$, having $Q(0) \neq 0$. If the degree of Q equals 3 we call them cubic Hecke algebras.

Coxeter [3] computed the cardinalities of the quotients of B_3 by relations $x_i^n = 1$, namely 24 for $n = 3$, 96 for $n = 4$, and 600 for $n = 5$. Then the algebras

$$H(Q_m, 3) = \Big\langle b_1, b_2 : b_2 b_1 b_2 = b_1 b_2 b_1, b_1^m = 1, b_2^m = 1 \Big\rangle$$

are finite dimensional of these dimensions, where $Q_m = x^m - 1$, $m \in \{2, 3, 4, 5\}$. For $n \geq 6$ these algebras are infinite dimensional. This motivated us to compute the Hilbert series in the finite dimensional case.

In [6] we constructed a linear system for the braid monoid B_{n+1} and computed the Hilbert series for the braid monoids MB_3 and MB_4. In [7] we computed the Hilbert series of braid monoid MB_4 in band generators. In this paper we construct a similar kind of linear system to compute the Hilbert series.

*Correspondence: zaffar.iqbal@uog.edu.pk
2010 *AMS Mathematics Subject Classification:* 20F36, 20F05, 13D40.

2. Hilbert series of the group algebras $H(Q_m, 3), m = 3, 4, 5$

Definition 2.1 *[4] Let G be a finitely generated group and S be a finite set of generators of G. The word length $l_S(g)$ of an element $g \in G$ is the smallest integer n for which there exists $s_1, \ldots, s_n \in S \cup S^{-1}$ such that $g = s_1 \cdots s_n$.*

Definition 2.2 *[4] Let G be a finitely generated group and S be a finite set of generators of G. The growth function of the pair (G, S) associates to an integer $k \geq 0$ the number $a(k)$ of elements $g \in G$ such that $l_S(g) = k$ and the corresponding spherical growth series or the Hilbert series is given by $P_G(t) = \sum_{k=0}^{\infty} a(k)t^k$.*

To get a canonical form of a word in an algebra the diamond lemma by Bergman [2] is extremely useful. To understand the notions of reductions and ambiguities we start with his terminology.

Let k be a commutative associative ring with unity, X a set, $\langle X \rangle$ the free monoid on X, and $k\langle X \rangle$ the free associative k-algebra on X. Let S be a set of pairs of the form $\sigma = (W_\sigma, f_\sigma)$, where $W_\sigma \in \langle X \rangle$ and $f_\sigma \in k\langle X \rangle$. For any $\sigma \in S$ and $A, B \in \langle X \rangle$, let $r_{A\sigma B} : k\langle X \rangle \to k\langle X \rangle$ be a k-module endomorphism such that this morphism sends $AW_\sigma B$ to $Af_\sigma B$ (r_* fixes all other elements of $\langle X \rangle$). The maps r_* are said to *reductions*. Let $\sigma, \tau \in S$ and $A, B, C \in \langle X \rangle - 1$ such that $W_\sigma = AB$, $W_\tau = BC$, and then ABC is said to an *ambiguity* of S. An element $a \in k\langle X \rangle$ is called *irreducible* (or *canonical*) if a involves none of the monomials $AW_\sigma B$; otherwise, a is called *reducible* (for more details see [2]).

If all relations in $k\langle X \rangle$ as a module are defined then we say that we have a complete set of relations in $k\langle X \rangle$. The diamond lemma [2] says that a set of relations is complete if all the ambiguities are solved. We call a complete set of relations in $H(Q_m, 3)$ a *complete presentation* of $H(Q_m, 3)$. The other names for the complete presentation being used are Gröbner bases, presentation with solvable ambiguities, rewriting systems, etc.

We made the terminology more understandable for the words in the braid monoids and for the algebra $H(Q_m, 3)$. Applying the above reductions we solve the ambiguities in $H(Q_m, 3)$. The solution of all the ambiguities gives us a complete set of relations in $H(Q_m, 3)$.

Let us have a few more words about the ambiguity and its solution. Let U, V, and w be nonempty words; then we denote $Uw \times_w wV$ by the word UwV. In a relation in $H(Q_m, 3)$ we place the equivalent words on the left-hand side that are greater in length-lexicographic ordering (we choose a natural order $b_1 < b_2 < \cdots < b_n$ between the generators). For example, the words $b_2 b_1 b_2$ and $b_1 b_2 b_1$ are equivalent in the braid monoid MB_3. Hence, we write $b_2 b_1 b_2 = b_1 b_2 b_1$ as the basic braid relation. Let Uw and wV be two words consisting of the left-hand sides of two relations in MB_3; then (defined as above, too) the word $W = Uw \times_w wV$ is an ambiguity. In this case W has two resolutions, namely $(Uw)V$ and $U(wV)$. If we apply finite reductions on $(Uw)V$ and $U(wV)$ and both give exactly the same word then we say that the ambiguity W is solvable. If $(Uw)V$ and $U(wV)$ differ by lexicographically then W gives a new relation in MB_3.

For example, consider a word $b_2 b_1 b_2$ from the left-hand side of a relation in $H(Q_3, 3)$. Then a word $b_2 b_1 b_2 b_1 b_2$ is an ambiguity and it has two resolutions $b_2 b_1 (b_2 b_1 b_2)$ and $(b_2 b_1 b_2) b_1 b_2$. Applying a reduction on the first we have $b_2 b_1^2 b_2 b_1$ and on second we get $b_1 b_2 b_1^2 b_2$. Hence, we have the relation

$$b_2 b_1^2 b_2 b_1 = b_1 b_2 b_1^2 b_2 \tag{2.1}$$

in a complete presentation of $H(Q_3, 3)$. In this way we solve all ambiguities in $H(Q_m, 3)$. Funar [5] gave a complete presentation of $H(Q_3, 3)$:

$$H(Q_3,3)=\Big\langle b_1,b_2: b_2b_1b_2=b_1b_2b_1, b_1^3=1, b_2^3=1, b_2b_1^2b_2b_1=b_1b_2b_1^2b_2, b_2b_1^2b_2^2=b_1^2b_2^2b_1, b_2^2b_1^2b_2=b_1b_2^2b_1^2\Big\rangle.$$

As defined earlier (here in the form of relations), in a complete presentation of $H(Q_m,3)$, a word containing a subword consisting of L.H.S. of any relation of $H(Q_m,3)$ will be called a *reducible word* and a word that does not contain a subword consisting of L.H.S. of any relation will be called an *irreducible word*.

In general we denote by $B_*^{(m)}$ the set of reducible words and by $A_*^{(m)}$ the set of irreducible words in $H(Q_m,3)$. The words $Ub_2\times_2 b_2V$ and $Ub_2b_1\times_{21} b_2b_1V$ denote the products Ub_2V and Ub_2b_1V, respectively.

Let us denote by $B_1^{(m)}=\{b_2b_1b_2\}$ and $B_{k+1}^{(m)}=\{b_2b_1^{k+1}b_2b_1\}$ the set of reducible words of $H(Q_m,3)$. For the irreducible words we use the following notations: $A_k^{(m)}$ denote the set of irreducible words starting with b^k and $A_{2^k\cdot 1}^{(m)}$ denote the set of irreducible words starting with $b_2^kb_1$. The Hilbert series of $A_k^{(m)}$, $A_{2^k\cdot 1}^{(m)}$, and $H(Q_m,3)$ are denoted by $P_k^{(m)}$, $P_{2^k\cdot 1}^{(m)}$, and $P_H^{(m)}(t)$, respectively, where

$$P_H^{(m)}(t)=1+P_1^{(m)}+P_2^{(m)}.$$

For the computations of the Hilbert series of $H(Q_3,3)$, we have the following linear system for the irreducible words.

Proposition 2.3 *The following equalities hold for the Hilbert series of irreducible words in $H(Q_3,3)$:*

1) $P_1^{(3)}=t+t^2+(t+t^2)P_2^{(3)}$,

2) $P_2^{(3)}=t+t^2+P_{2\cdot 1}^{(3)}+P_{2^2\cdot 1}^{(3)}$,

3) $P_{2\cdot 1}^{(3)}=tP_1^{(3)}-t^2P_2^{(3)}-t^3P_{2\cdot 1}^{(3)}-t^5-t^6-t^7$,

4) $P_{2^2\cdot 1}^{(3)}=tP_{2\cdot 1}^{(3)}-t^5$.

Proof 1) The set $A_1^{(3)}$ is decomposed as $A_1^{(3)}=\{b_1,b_1^2\}\cup\big(\{b_1,b_1^2\}\times A_2^{(3)}\big)$. This implies Relation 1).

2) It follows immediately from $A_2^{(3)}=\{b_1,b_1^2\}\cup A_{2\cdot 1}^{(3)}\cup A_{2^2\cdot 1}^{(3)}$.

3) The decomposition of $A_{2\cdot 1}^{(3)}$ is given by

$A_{2\cdot 1}^{(3)}=\big(\{b_2\}\times A_1^{(3)}\big)\setminus\Big(\big(B_1^{(3)}\times_2 A_2^{(3)}\big)\cup\big(B_2^{(3)}\times_{21} A_{2\cdot 1}^{(3)}\big)\cup\{b_2b_1^2b_2^2, b_2b_1^2b_2^2b_1, b_2b_1^2b_2^2b_1^2\}\Big)$ and hence we have 3)).

4) It follows from $A_{2^2\cdot 1}^{(3)}=\big(\{b_2\}\times A_{2\cdot 1}^{(3)}\big)\setminus\{b_2^2b_1^2b_2\}$. □

For the computation of Hilbert series the complete presentation of the algebra is very important. By the diamond lemma [2], the set of relations is complete in the complete presentation. Here we give the complete presentation of $H(Q_4,3)$ in the following:

Proposition 2.4 *A complete presentation of $H(Q_4,3)$ is given as*

$$H(Q_4,3)=\Big\langle b_1,\, b_2: b_2b_1b_2=b_1b_2b_1, b_1^4=1, b_2^4=1,\, b_2b_1^2b_2b_1=b_1b_2b_1^2b_2,\, b_2b_1^3b_2b_1=b_1b_2b_1^2b_2^2,$$
$$b_2b_1^2b_2^2b_1^2=b_1^2b_2^2b_1^2b_2, b_2b_1^2b_2^3=b_1^3b_2^2b_1,\, b_2b_1^3b_2^3=b_1^3b_2^3b_1,\, b_2b_1^3b_2^2b_1^3=b_1^3b_2^2b_1^3b_2,\, b_2b_1^3b_2^2b_1^2b_2=b_1b_2b_1^3b_2^2b_1^2,$$
$$b_2^2b_1^2b_2^2b_1=b_1b_2^2b_1^2b_2^2,\, b_2^2b_1^3b_2^2b_1=b_1b_2^2b_1^3b_2^2,\, b_2^3b_1^2b_2=b_1b_2^2b_1^3,\, b_2^3b_1^3b_2=b_1b_2^3b_1^3\Big\rangle.$$

(An outline of the proof of Proposition 2.4 is given in the Appendix.)

Now we develop a linear system for the irreducible words in $H(Q_4, 3)$.

Proposition 2.5 *The following equalities hold for the Hilbert series of irreducible words in* $H(Q_4, 3)$*:*

1) $P_1^{(4)} = t + t^2 + t^3 + (t + t^2 + t^3)P_2^{(4)}$,

2) $P_2^{(4)} = t + t^2 + t^3 + P_{2\cdot 1}^{(4)} + P_{2^2\cdot 1}^{(4)} + P_{2^3\cdot 1}^{(4)}$,

3) $P_{2\cdot 1}^{(4)} = tP_1^{(4)} - t^2 P_2^{(4)} - t^3 P_{2\cdot 1}^{(4)} - t^4 P_{2\cdot 1}^{(4)} - t^6 - 3t^7 - 4t^8 - 6t^9 - 4t^{10} - t^{11}$,

4) $P_{2^2\cdot 1}^{(4)} = tP_{2\cdot 1}^{(4)} - t^7 - t^8 - t^9$,

5) $P_{2^3\cdot 1}^{(4)} = tP_{2^2\cdot 1}^{(4)} - t^6 - 2t^7 - t^8$.

Proof 1) The set $A_1^{(4)}$ is decomposed as $A_1^{(4)} = \{b_1, b_1^2, b_1^3\} \cup \big(\{b_1, b_1^2, b_1^3\} \times A_2^{(4)}\big)$. This gives Relation 1).

2) It follows immediately from $A_2^{(4)} = \{b_2, b_2^2, b_2^3\} \cup A_{2\cdot 1}^{(4)} \cup A_{2^2\cdot 1}^{(4)} \cup A_{2^3\cdot 1}^{(4)}$.

3) The decomposition of $A_{2\cdot 1}^{(4)}$ is given by

$A_{2\cdot 1}^{(4)} = \big(\{b_2\} \times A_1^{(4)}\big) \setminus \Big(\big(B_1^{(4)} \times_2 A_2^{(4)}\big) \cup \big(B_2^{(4)} \times_{21} A_{2\cdot 1}^{(4)}\big) \cup \big(B_3^{(4)} \times_{21} A_{2\cdot 1}^{(4)}\big) \cup \{b_2b_1^2b_2^3, b_2b_1^2b_2^3b_1, b_2b_1^2b_2^3b_1^2, b_2b_1^2b_2^3b_1^3,$
$b_2b_1^3b_2^3, b_2b_1^3b_2^3b_1, b_2b_1^3b_2^3b_1^2, b_2b_1^3b_2^3b_1^3, b_2b_1^2b_2^2b_1^2, b_2b_1^2b_2^2b_1^2b_2, b_2b_1^2b_2^2b_1^2b_2^2, b_2b_1^2b_2^2b_1^3, b_2b_1^2b_2^2b_1^3b_2, b_2b_1^2b_2^2b_1^3b_2^2, b_2b_1^3b_2^2b_1^3,$
$b_2b_1^3b_2^2b_1^3b_2, b_2b_1^3b_2^2b_1^3b_2^2, b_2b_1^3b_2^2b_1^2b_2, b_2b_1^3b_2^2b_1^2b_2^2\}\Big)$ and hence we have 3).

4) It follows from $A_{2^2\cdot 1}^{(4)} = \big(\{b_2\} \times A_{2\cdot 1}^{(4)}\big) \setminus \{b_2^2b_1^2b_2^2b_1, b_2^2b_1^3b_2^2b_1, b_2^2b_1^3b_2^2b_1^2\}$.

5) It follows from $A_{2^3\cdot 1}^{(4)} = \big(\{b_2\} \times A_{2^2\cdot 1}^{(4)}\big) \setminus \{b_2^3b_1^2b_2, b_2^3b_1^2b_2^2, b_2^3b_1^3b_2, b_2^3b_1^3b_2^2\}$. □

Proposition 2.6 *A complete presentation of* $H(Q_5, 3)$ *is given by*

$H(Q_5, 3) = \Big\langle b_1, b_2 : b_2b_1b_2 = b_1b_2b_1, b_1^5 = 1, b_2^5 = 1, R_i, 1 \leq i \leq 41\Big\rangle$, *where the relations* R_i *are given by*

$R_1 : b_2b_1^2b_2b_1 = b_1b_2b_1^2b_2,\ R_2 : b_2b_1^3b_2b_1 = b_1b_2b_1^2b_2^2,\ R_3 : b_2b_1^4b_2b_1 = b_1b_2b_1^2b_2^3$,

$R_4 : b_2b_1^2b_2^4 = b_1^4b_2^2b_1,\ R_5 : b_2b_1^3b_2^4 = b_1^4b_2^3b_1,\ R_6 : b_2b_1^4b_2^4 = b_1^4b_2^4b_1$,

$R_7 : b_2^4b_1^2b_2 = b_1b_2^2b_1^4$, $R_8 : b_2^4b_1^3b_2 = b_1b_2^3b_1^4,\ R_9 : b_2^4b_1^4b_2 = b_1b_2^4b_1^4$,

$R_{10} : b_2b_1^2b_2^2b_1^3 = b_1^3b_2^2b_1^2b_2, R_{11} : b_2b_1^2b_2^3b_1^3 = b_1^3b_2^3b_1^2b_2, R_{12} : b_2b_1^3b_2^2b_1^4 = b_1^4b_2^2b_1^3b_2$,

$R_{13} : b_2b_1^4b_2^2b_1^4 = b_1^4b_2^3b_1^3b_2, R_{14} : b_2b_1^3b_2^3b_1^4 = b_1^4b_2^2b_1^4b_2, R_{15} : b_2b_1^4b_2^3b_1^4 = b_1^4b_2^3b_1^4b_2$,

$R_{16} : b_2^3b_1^2b_2^2b_1 = b_1b_2^2b_1^2b_2^3, R_{17} : b_2^3b_1^3b_2^2b_1 = b_1b_2^2b_1^3b_2^3, R_{18} : b_2^3b_1^4b_2^2b_1 = b_1b_2^2b_1^4b_2^3$,

$R_{19} : b_2^3b_1^4b_2^3b_1 = b_1b_2^3b_1^4b_2^3, R_{20} : b_2b_1^2b_2^2b_1^2b_2^2 = b_1^2b_2^2b_1^2b_2^2b_1$,

$R_{21} : b_2b_1^2b_2^3b_1^2b_2^3 = b_1^3b_2^2b_1^3b_2^2b_1$, $R_{22} : b_2b_1^3b_2^2b_1^3b_2^3 = b_1^3b_2^3b_1^2b_2^3b_1$,

$R_{23} : b_2b_1^3b_2^3b_1^3b_2^3 = b_1^3b_2^3b_1^3b_2^3b_1, R_{24} : b_2b_1^4b_2^2b_1^2b_2 = b_1b_2b_1^3b_2^2b_1^3$,

$R_{25} : b_2b_1^4b_2^3b_1^2b_2 = b_1b_2b_1^3b_2^3b_1^3, R_{26} : b_2^2b_1^2b_2^2b_1^2b_2 = b_1b_2^2b_1^2b_2^2b_1^2$,

$R_{27} : b_2^2b_1^3b_2^2b_1^2b_2 = b_1b_2^2b_1^2b_2^3b_1^2, R_{28} : b_2^2b_1^4b_2^2b_1^3b_2 = b_1b_2^2b_1^3b_2^3b_1^3$,

$R_{29} : b_2^2b_1^4b_2^3b_1^3b_2 = b_1b_2^2b_1^4b_2^3b_1^3, R_{30} : b_2^3b_1^2b_2^3b_1^2b_2 = b_1b_2^2b_1^3b_2^2b_1^3$,

$R_{31} : b_2^3b_1^3b_2^3b_1^2b_2 = b_1b_2^2b_1^4b_2^2b_1^3, R_{32} : b_2^3b_1^3b_2^3b_1^3b_2 = b_1b_2^3b_1^3b_2^3b_1^3$,

$R_{33} : b_2b_1^3b_2^2b_1^2b_2^2b_1 = b_1b_2b_1^3b_2^2b_1^2b_2^2, R_{34} : b_2b_1^3b_2^2b_1^2b_2^3b_1^2 = b_1^3b_2^2b_1^2b_2^3b_1^2b_2$,

$R_{35} : b_2b_1^3b_2^2b_1^3b_2^2b_1^3 = b_1^3b_2^2b_1^3b_2^2b_1^3b_2, R_{36} : b_2b_1^4b_2^2b_1^3b_2^2b_1 = b_1b_2b_1^3b_2^3b_1^2b_2^3$,

$R_{37}: b_2b_1^4b_2^2b_1^3b_2^3b_1^3 = b_1^4b_2^2b_1^3b_2^3b_1^3b_2, R_{38}: b_2b_1^4b_2^3b_1^3b_2^3b_1 = b_1b_2b_1^4b_2^3b_1^3b_2^3,$
$R_{39}: b_2^2b_1^3b_2^2b_1^3b_2^2b_1 = b_1b_2^2b_1^3b_2^2b_1^3b_2^2,\ R_{40}: b_2^2b_1^3b_2^3b_1^2b_2^3b_1 = b_1b_2^2b_1^3b_2^3b_1^2b_2^3,$
$R_{41}: b_2b_1^4b_2^2b_1^3b_2^3b_1^2b_2 = b_1b_2b_1^4b_2^2b_1^3b_2^3b_1^2.$

(An outline of the proof of Proposition 2.6 is given in the Appendix.)

Now we develop a linear system for the irreducible words in $H(Q_5,3)$.

Proposition 2.7 *The following equalities hold for the Hilbert series of irreducible words in* $H(Q_5,3)$:

1) $P_1^{(5)} = t+t^2+t^3+t^4+(t+t^2+t^3+t^4)P_2^{(5)},$

2) $P_2^{(5)} = t+t^2+t^3+t^4+P_{2\cdot1}^{(5)}+P_{2^2\cdot1}^{(5)}+P_{2^3\cdot1}^{(5)}+P_{2^4\cdot1}^{(5)},$

3) $P_{2\cdot1}^{(5)} = tP_1^{(5)} - t^2P_2^{(5)} - (t^3+t^4+t^5)P_{2\cdot1}^{(5)} - t^7 - 3t^8 - 7t^9 - 11t^{10} - 17t^{11} - 19t^{12} - 21t^{13} - 20t^{14} - 19t^{15} - 14t^{16} - 7t^{17} - 2t^{18},$

4) $P_{2^2\cdot1}^{(5)} = tP_{2\cdot1}^{(5)} - t^9 - t^{10} - t^{11} - 2t^{12} - 4t^{13} - 4t^{14} - 3t^{15} - t^{16},$

5) $P_{2^3\cdot1}^{(5)} = tP_{2^2\cdot1}^{(5)} - t^8 - 2t^9 - 2t^{10} - 4t^{11} - 5t^{12} - 4t^{13} - 2t^{14},$

6) $P_{2^4\cdot1}^{(5)} = tP_{2^2\cdot1}^{(5)} - t^7 - 2t^8 - 3t^9 - 3t^{10} - 3t^{11} - t^{12} - t^{13}.$

Proof 1) The set $A_1^{(5)}$ is decomposed as

$A_1^{(5)} = \{b_1, b_1^2, b_1^3, b_1^4\} \cup \big(\{b_1, b_1^2, b_1^3, b_1^4\} \times A_2^{(5)}\big)$. This gives Relation 1).

2) It follows immediately from

$A_2^{(5)} = \{b_2, b_2^2, b_2^3, b_2^4\} \cup A_{2\cdot1}^{(5)} \cup A_{2^2\cdot1}^{(5)} \cup A_{2^3\cdot1}^{(5)} \cup A_{2^4\cdot1}^{(5)}.$

3) The decomposition of $A_{2\cdot1}^{(5)}$ is given by

$A_{2\cdot1}^{(5)} = \big(\{b_2\} \times A_1^{(5)}\big) \setminus \Big(\big(B_1^{(5)} \times_2 A_2^{(5)}\big) \cup \big(B_2^{(5)} \times_{21} A_{2\cdot1}^{(5)}\big) \cup \big(B_3^{(5)} \times_{21} A_{2\cdot1}^{(5)}\big) \cup \big(B_4^{(5)} \times_{21} A_{2\cdot1}^{(5)}\big) \cup \big\{b_2b_1^2b_2^4,$
$b_2b_1^2b_2^4b_1, b_2b_1^2b_2^4b_1^2, b_2b_1^2b_2^4b_1^3, b_2b_1^2b_2^4b_1^4, b_2b_1^3b_2^4, b_2b_1^3b_2^4b_1, b_2b_1^3b_2^4b_1^2, b_2b_1^3b_2^4b_1^3, b_2b_1^3b_2^4b_1^4, b_2b_1^4b_2^4, b_2b_1^4b_2^4b_1,$
$b_2b_1^4b_2^4b_1^2, b_2b_1^4b_2^4b_1^3, b_2b_1^4b_2^4b_1^4, b_2b_1^2b_2^2b_1^3, b_2b_1^2b_2^2b_1^3b_2, b_2b_1^2b_2^2b_1^3b_2^2, b_2b_1^2b_2^2b_1^3b_2^2b_1, b_2b_1^2b_2^2b_1^3b_2^2b_1^2, b_2b_1^2b_2^2b_1^3b_2^2b_1^3,$
$b_2b_1^2b_2^2b_1^3b_2^2b_1^3b_2, b_2b_1^2b_2^2b_1^3b_2^2b_1^3b_2^2, b_2b_1^2b_2^2b_1^3b_2^3, b_2b_1^2b_2^2b_1^3b_2^3b_1, b_2b_1^2b_2^2b_1^3b_2^3b_1^2, b_2b_1^2b_2^2b_1^3b_2^3b_1^2b_2, b_2b_1^2b_2^2b_1^3b_2^3b_1^2b_2^2,$
$b_2b_1^2b_2^2b_1^3b_2^3b_1^2b_2^3, b_2b_1^2b_2^2b_1^3b_2^3b_1^3, b_2b_1^2b_2^2b_1^3b_2^3b_1^3b_2, b_2b_1^2b_2^2b_1^3b_2^3b_1^3b_2^2, b_2b_1^2b_2^2b_1^4, b_2b_1^2b_2^2b_1^4b_2, b_2b_1^2b_2^2b_1^4b_2^2,$
$b_2b_1^2b_2^2b_1^4b_2^2b_1, b_2b_1^2b_2^2b_1^4b_2^2b_1^2, b_2b_1^2b_2^2b_1^4b_2^2b_1^3, b_2b_1^2b_2^2b_1^4b_2^3, b_2b_1^2b_2^2b_1^4b_2^3b_1, b_2b_1^2b_2^2b_1^4b_2^3b_1^2, b_2b_1^2b_2^2b_1^4b_2^3b_1^3,$
$b_2b_1^2b_2^3b_1^3, b_2b_1^2b_2^3b_1^3b_2, b_2b_1^2b_2^3b_1^3b_2^2, b_2b_1^2b_2^3b_1^3b_2^3, b_2b_1^2b_2^3b_1^3b_2^3b_1, b_2b_1^2b_2^3b_1^3b_2^3b_1^2, b_2b_1^2b_2^3b_1^3b_2^3b_1^3, b_2b_1^2b_2^3b_1^4,$
$b_2b_1^2b_2^3b_1^4b_2, b_2b_1^2b_2^3b_1^4b_2^2, b_2b_1^2b_2^3b_1^4b_2^3, b_2b_1^3b_2^2b_1^4, b_2b_1^3b_2^2b_1^4b_2, b_2b_1^3b_2^2b_1^4b_2^2, b_2b_1^3b_2^2b_1^4b_2^2b_1, b_2b_1^3b_2^2b_1^4b_2^2b_1^2,$
$b_2b_1^3b_2^2b_1^4b_2^2b_1^3, b_2b_1^3b_2^2b_1^4b_2^3, b_2b_1^3b_2^2b_1^4b_2^3b_1, b_2b_1^3b_2^2b_1^4b_2^3b_1^2, b_2b_1^3b_2^2b_1^4b_2^3b_1^3, b_2b_1^4b_2^2b_1^4, b_2b_1^4b_2^2b_1^4b_2, b_2b_1^4b_2^2b_1^4b_2^2,$
$b_2b_1^4b_2^2b_1^4b_2^2b_1, b_2b_1^4b_2^2b_1^4b_2^2b_1^2, b_2b_1^4b_2^2b_1^4b_2^2b_1^3, b_2b_1^4b_2^2b_1^4b_2^3, b_2b_1^4b_2^2b_1^4b_2^3b_1, b_2b_1^4b_2^2b_1^4b_2^3b_1^2, b_2b_1^4b_2^2b_1^4b_2^3b_1^3, b_2b_1^3b_2^3b_1^4,$
$b_2b_1^3b_2^3b_1^4b_2, b_2b_1^3b_2^3b_1^4b_2^2, b_2b_1^3b_2^3b_1^4b_2^3, b_2b_1^4b_2^3b_1^4, b_2b_1^4b_2^3b_1^4b_2, b_2b_1^4b_2^3b_1^4b_2^2, b_2b_1^4b_2^3b_1^4b_2^3, b_2b_1^2b_2^2b_1^2b_2^2,$
$b_2b_1^2b_2^2b_1^2b_2^2b_1, b_2b_1^2b_2^2b_1^2b_2^2b_1^2, b_2b_1^2b_2^2b_1^2b_2^3, b_2b_1^2b_2^2b_1^2b_2^3b_1, b_2b_1^2b_2^2b_1^2b_2^3b_1^2, b_2b_1^2b_2^2b_1^2b_2^3b_1^2b_2, b_2b_1^2b_2^2b_1^2b_2^3b_1^2b_2^2, b_2b_1^2b_2^3b_1^2b_2^3,$
$b_2b_1^2b_2^3b_1^2b_2^3b_1, b_2b_1^2b_2^3b_1^2b_2^3b_1^2, b_2b_1^3b_2^2b_1^3b_2^3, b_2b_1^3b_2^2b_1^3b_2^3b_1, b_2b_1^3b_2^2b_1^3b_2^3b_1^2, b_2b_1^3b_2^2b_1^3b_2^3b_1^2b_2, b_2b_1^3b_2^2b_1^3b_2^3b_1^2b_2^2, b_2b_1^3b_2^2b_1^3b_2^3b_1^2b_2^3,$
$b_2b_1^3b_2^2b_1^3b_2^3b_1^3, b_2b_1^3b_2^2b_1^3b_2^3b_1^3b_2, b_2b_1^3b_2^2b_1^3b_2^3b_1^3b_2^2, b_2b_1^3b_2^3b_1^3b_2^3, b_2b_1^3b_2^3b_1^3b_2^3b_1, b_2b_1^3b_2^3b_1^3b_2^3b_1^2, b_2b_1^3b_2^3b_1^3b_2^3b_1^3, b_2b_1^4b_2^2b_1^2b_2,$
$b_2b_1^4b_2^2b_1^2b_2^2, b_2b_1^4b_2^2b_1^2b_2^2b_1, b_2b_1^4b_2^2b_1^2b_2^2b_1^2, b_2b_1^4b_2^2b_1^2b_2^3, b_2b_1^4b_2^2b_1^2b_2^3b_1, b_2b_1^4b_2^2b_1^2b_2^3b_1^2, b_2b_1^4b_2^2b_1^2b_2^3b_1^2b_2, b_2b_1^4b_2^2b_1^2b_2^3b_1^2b_2^2,$
$b_2b_1^4b_2^3b_1^2b_2, b_2b_1^4b_2^3b_1^2b_2^2, b_2b_1^4b_2^3b_1^2b_2^3, b_2b_1^4b_2^3b_1^2b_2^3b_1, b_2b_1^4b_2^3b_1^2b_2^3b_1^2, b_2b_1^3b_2^2b_1^2b_2^2b_1, b_2b_1^3b_2^2b_1^2b_2^2b_1^2, b_2b_1^3b_2^2b_1^2b_2^3b_1^2,$
$b_2b_1^3b_2^2b_1^2b_2^3b_1^2b_2, b_2b_1^3b_2^2b_1^2b_2^3b_1^2b_2^2, b_2b_1^3b_2^2b_1^3b_2^2b_1^3, b_2b_1^3b_2^2b_1^3b_2^2b_1^3b_2, b_2b_1^3b_2^2b_1^3b_2^2b_1^3b_2^2, b_2b_1^4b_2^2b_1^3b_2^2b_1, b_2b_1^4b_2^2b_1^3b_2^2b_1^2,$

$b_2b_1^4b_2^2b_1^3b_2^2b_1^3, b_2b_1^4b_2^2b_1^3b_2^2b_1^3b_2, b_2b_1^4b_2^2b_1^3b_2^2b_1^3b_2^2, b_2b_1^4b_2^2b_1^3b_2^3b_1^3, b_2b_1^4b_2^2b_1^3b_2^3b_1^3b_2, b_2b_1^4b_2^2b_1^3b_2^3b_1^3b_2^2, b_2b_1^4b_2^3b_1^3b_2^3b_1, b_2b_1^4b_2^3b_1^3b_2^3b_1^2,$
$b_2b_1^4b_2^3b_1^3b_2^3b_1^3, b_2b_1^4b_2^2b_1^3b_2^3b_1^2b_2, b_2b_1^4b_2^2b_1^3b_2^3b_1^2b_2^2, b_2b_1^4b_2^2b_1^3b_2^3b_1^2b_2^3, \}\Big)$ and hence we have 3).

4) It follows from

$A^{(5)}_{2^2\cdot 1} = \big(\{b_2\} \times A^{(5)}_{2\cdot 1}\big) \setminus \{b_2^2b_1^2b_2^2b_1^2b_2, b_2^2b_1^3b_2^2b_1^2b_2, b_2^2b_1^3b_2^2b_1^2b_2^2, b_2^2b_1^3b_2^2b_1^2b_2^3, b_2^2b_1^3b_2^2b_1^2b_2^3b_1, b_2^2b_1^4b_2^2b_1^3b_2, b_2^2b_1^4b_2^2b_1^3b_2^2,$
$b_2^2b_1^4b_2^2b_1^3b_2^3, b_2^2b_1^4b_2^2b_1^3b_2^3b_1, b_2^2b_1^4b_2^2b_1^3b_2^3b_1^2, b_2^2b_1^4b_2^3b_1^3b_2, b_2^2b_1^4b_2^3b_1^3b_2^2, b_2^2b_1^4b_2^3b_1^3b_2^3, b_2^2b_1^3b_2^3b_1^2b_2^3b_1, b_2^2b_1^3b_2^3b_1^2b_2^3b_1^2, b_2^2b_1^3b_2^2b_1^3b_2^2b_1,$
$b_2^2b_1^3b_2^2b_1^3b_2^2b_1^2\}.$

5) It follows from $A^{(5)}_{2^3\cdot 1} = \big(\{b_2\} \times A^{(5)}_{2^2\cdot 1}\big) \setminus \{b_2^3b_1^2b_2^2b_1, b_2^3b_1^2b_2^2b_1^2, b_2^3b_1^3b_2^2b_1, b_2^3b_1^3b_2^2b_1^2, b_2^3b_1^3b_2^2b_1^3, b_2^3b_1^3b_2^2b_1^3b_2, b_2^3b_1^3b_2^2b_1^3b_2^2,$
$b_2^3b_1^4b_2^2b_1, b_2^3b_1^4b_2^2b_1^2, b_2^3b_1^4b_2^2b_1^3, b_2^3b_1^4b_2^3b_1, b_2^3b_1^4b_2^3b_1^2, b_2^3b_1^4b_2^3b_1^3, b_2^3b_1^2b_2^3b_1^2b_2, b_2^3b_1^2b_2^3b_1^2b_2^2, b_2^3b_1^3b_2^3b_1^2b_2, b_2^3b_1^3b_2^3b_1^2b_2^2, b_2^3b_1^3b_2^3b_1^2b_2^3,$
$b_2^3b_1^3b_2^3b_1^3b_2, b_2^3b_1^3b_2^3b_1^3b_2^2\}.$

6) It follows from $A^{(5)}_{2^4\cdot 1} = \big(\{b_2\} \times A^{(5)}_{2^3\cdot 1}\big) \setminus \{b_2^4b_1^2b_2, b_2^4b_1^2b_2^2, b_2^4b_1^2b_2^3, b_2^4b_1^2b_2^3b_1, b_2^4b_1^2b_2^3b_1^2, b_2^4b_1^3b_2, b_2^4b_1^3b_2^2, b_2^4b_1^3b_2^3, b_2^4b_1^3b_2^3b_1,$
$b_2^4b_1^3b_3^3b_1^2, b_2^4b_1^3b_3^3b_1^3, b_2^4b_1^4b_2, b_2^4b_1^4b_2^2, b_2^4b_1^4b_2^3\}.$ $\square$

At the end we give our main result.

Theorem 2.8 *The Hilbert series of the group algebras* $H(Q_m, 3), m = 3, 4, 5$ *are given by*

1) $P^{(3)}_H(t) = 1 + 2t + 4t^2 + 5t^3 + 6t^4 + 4t^5 + 2t^6.$

2) $P^{(4)}_H(t) = 1 + 2t + 4t^2 + 7t^3 + 10t^4 + 14t^5 + 17t^6 + 16t^7 + 13t^8 + 8t^9 + 3t^{10} + t^{11}.$

3) $P^{(5)}_H(t) = 1 + 2t + 4t^2 + 7t^3 + 14t^4 + 18t^5 + 27t^6 + 38t^7 + 50t^8 + 59t^9 + 67t^{10} + 70t^{11} + 68t^{12} + 59t^{13} + 48t^{14} + 34t^{15} + 21t^{16} + 10t^{17} + 4t^{18} + t^{19}.$

Proof 1) Solving (using any suitable software) the system given in Proposition 2.3 we have $P^{(3)}_1 = t + 2t^2 + 3t^3 + 4t^4 + 4t^5 + 2t^6$ and $P^{(3)}_2 = t + 2t^2 + 2t^3 + 2t^4$. Hence,

$$\begin{aligned} P^{(3)}_H(t) &= 1 + P^{(3)}_1 + P^{(3)}_2 \\ &= 1 + 2t + 4t^2 + 5t^3 + 6t^4 + 4t^5 + 2t^6. \end{aligned}$$

2) Solving the linear system given in Proposition 2.5 we have

$P^{(4)}_1 = t+2t^2+4t^3+6t^4+9t^5+12t^6+14t^7+12t^8+8t^9+3t^{10}+t^{11}$ and $P^{(4)}_2 = t+2t^2+3t^3+4t^4+5t^5+5t^6+2t^7+t^8$.
Hence,

$$\begin{aligned} P^{(4)}_H(t) &= 1 + P^{(4)}_1 + P^{(4)}_2 \\ &= 1 + 2t + 4t^2 + 7t^3 + 10t^4 + 14t^5 + 17t^6 + 16t^7 + 13t^8 + 8t^9 \\ &\quad + 3t^{10} + t^{11}. \end{aligned}$$

3) Solving the linear system given in Proposition 2.7 we have

$$\begin{aligned} P^{(5)}_1 &= t + 2t^2 + 4t^3 + 7t^4 + 11t^5 + 17t^6 + 25t^7 + 35t^8 + 45t^9 + 52t^{10} + 57t^{11} + 57t^{12} \\ &\quad + 53t^{13} + 45t^{14} + 33t^{15} + 21t^{16} + 10t^{17} + 4t^{18} + t^{19} \text{ and} \end{aligned}$$

$$P^{(5)}_2 = t + 2t^2 + 3t^3 + 5t^4 + 7t^5 + 10t^6 + 13t^7 + 15t^8 + 14t^9 + 15t^{10} + 13t^{11} + 11t^{12}$$

$+ 6t^{13} + 3t^{14} + t^{15}$. Therefore, we have

$$\begin{aligned} P_H^{(5)}(t) &= 1 + P_1^{(5)} + P_2^{(5)} \\ &= 1 + 2t + 4t^2 + 7t^3 + 12t^4 + 18t^5 + 27t^6 + 38t^7 + 50t^8 + 59t^9 \\ &\quad +67t^{10} + 70t^{11} + 68t^{12} + 59t^{13} + 48t^{14} + 34t^{15} + 21t^{16} + 10t^{17} \\ &\quad +4t^{18} + t^{19}. \end{aligned}$$

□

Remark 2.9 *Note that the degrees* $d(P_H^{(m)})$ *of the polynomial of the Hilbert series of* $H(Q_m, 3)$, $m \in \{1, \ldots, 5\}$ *are 1, 3, 6, 11, 19. One can see that* $d(P_H^{(m)})$ *is related with Fibonacci numbers* $F_m = F_{m-1} + F_{m-2}$ ($F_0 = 1$, $F_1 = 1$) *by the relation* $d(P_H^{(m)}) = F_{m+2} - 2$.

Remark 2.10 *We believe that the result holds also when* Q_m *is an arbitrary polynomial of degree* m, *with* $Q_m(0) \neq 0$, *for* $m = 3, 4, 5$. *This is known to be true for* $m = 3$ *([5]).*

Acknowledgments

This work was done using the support of an HEC-Pakistan grant during the author's visit at the Institute Fourier, University of Grenoble, France. The author is grateful to Louis Funar for useful suggestions and advice.

Appendix

Proof of Proposition 2.4 (outline). We solve all the ambiguities to find out all the relations given in these propositions. The computations are too long, so we give here a few computations to find out the relations.

- Relation 2.1 has been explained before.
- The ambiguity $b_2b_1^2b_2b_1b_2$ has two resolutions $b_2b_1^2(b_2b_1b_2)$ and $(b_2b_1^2b_2b_1)b_2$. Applying reductions on both we get $b_2b_1^3b_2b_1$ and $b_1b_2b_1^2b_2^2$, respectively. We say that the ambiguity $b_2b_1^2b_2b_1b_2$ gives a relation $b_2b_1^3b_2b_1 = b_1b_2b_1^2b_2^2$.
- Similarly the ambiguity $b_2b_1b_2b_1^3b_2b_1$ gives a relation $b_1^2b_2b_1^2b_2^3 = b_1b_2^2b_1$. This relation is nonhomogeneous (i.e. the degrees are not same on both sides). By left multiplication to this relation by b_1^2 we make it homogeneous and get another relation, $b_2b_1^2b_2^3 = b_1^3b_2^2b_1$.
- The ambiguity $b_2b_1b_2b_1^2b_2^3$ gives a relation $b_1b_2b_1^3b_2^3 = b_2^3b_1$. Again this relation is not homogeneous. Left multiplication by b_1^3 gives $b_2b_1^3b_2^3 = b_1^3b_2^3b_1$.

By continuing solving the ambiguities we get new relations. In the process some ambiguities are solvable and give no new relations. Hence, solving all the ambiguities, we have the presentation of the Proposition 2.4.

Proof of Proposition 2.6 (outline). Proof of this proposition is much longer than that of Proposition 2.4. As above, here we give the proof of a few relations. Proofs of others are based on similar computations.

- R_1 and R_2 are the relations of $H(Q_4, 3)$.
- The ambiguity $b_2b_1^3b_2b_1b_2$ gives R_3.
- $b_2b_1^4b_2b_1b_2$ gives a nonhomogeneous relation $b_1b_2b_1^2b_2^4 = b_2^2b_1$. Left multiplication by b_1^4 gives R_4.
- $b_2b_1b_2b_1^2b_2^4$ gives $b_1b_2b_1^3b_2^4 = b_2^3b_1$. Left multiplication by b_1^4 gives R_5.

- $b_2b_1b_2b_1^3b_2^4$ gives $b_1b_2b_1^4b_2^4 = b_2^4b_1$. Left multiplication by b_1^4 gives R_6.
- $b_2b_1^4b_2^4b_1b_2$ gives $b_1^4b_2^4b_1^2b_2 = b_2^2b_1^4$. Left multiplication by b_1 gives R_7.
- $b_2b_1^2b_2^4b_1^2b_2$ gives R_{12}.
- $b_2b_1^3b_2^4b_1^2b_2$ gives R_{13}.
- $b_2b_1^3b_2^4b_1^3b_2$ gives R_{15}.
- $b_2b_1b_2b_1^4b_2^2b_1^4$ gives R_8.
- $b_2b_1b_2b_1^4b_2^3b_1^4$ gives R_9 and so on. By following this procedure and checking all the ambiguities and after a lot of computations we have all the relations. This process terminates when all the ambiguities are solvable and gives no further new relations.

References

[1] Artin E. Theory of braids. Ann Math 1947; 48: 101–126.

[2] Bergman G. The diamond lemma for ring theory. Adv Math 1978; 29: 178–218.

[3] Coxeter HSM. Regular Complex Polytopes. 2nd ed. Cambridge, UK: Cambridge University Press, 1991.

[4] Harpe PD. Topics in Geometric Group Theory. Chicago, IL, USA: University of Chicago Press, 2000.

[5] Funar L. On the quotients of cubic Hecke algebras. Commun Math Phys 1995; 173: 513–558.

[6] Iqbal Z. Hilbert series of positive braids. Algebr Colloq 2011; 18: 1017–1028.

[7] Iqbal Z, Yousaf S. Hilbert series of braid monoid MB_4 in band generators. Turk J Math 2014; 38: 977–984.