Defect polynomials and Tutte polynomials of some asymmetric graphs

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Abstract: We give explicit expressions of the Tutte polynomial of asymmetric complete flower graph and asymmetric incomplete flower graph. We then express these Tutte polynomials as generating functions and decode some valuable information about the asymmetric complete flower graph and asymmetric incomplete flower graph. Furthermore, we convert the Tutte polynomials into coboundary polynomials and give explicit expressions of the k-defect polynomials of these structures. Finally, we conclude that nonisomorphic graphs in this class have the same Tutte polynomials, the same chromatic polynomials, and the same defect polynomials.

Key words: Tutte polynomial, cycle graph, flower graph, coboundary polynomials, k-defect polynomials

1. Introduction

There are several polynomials associated with a graph G; we refer the reader to [4] for a detailed background. Polynomials play an important role in the study of graphs as they encode various information about a graph. Chromatic polynomials of graphs are sometimes easy to compute. However, Tutte polynomials of such graphs seem harder to find, and if known they are complicated. For example, the chromatic polynomial of K_n is $\lambda \prod_{i=1}^{n-1}(\lambda - i)$, but the Tutte polynomial of the same structure as described by Tutte [10] and Welsh [11] is complicated. There are several methods that are used to compute the Tutte polynomial of a graph; just to sample a few methods, we refer to [1, 6].

The coboundary polynomial $B(G; \lambda, S)$ of a graph G is a polynomial in two independent variables λ and S. It was originally defined and studied by Crapo [2] as a generating function in S as

$$B(G; \lambda, S) = \sum S^k \phi_k(G; \lambda),$$

where $\phi_k(G; \lambda)$ is a polynomial in λ. $\phi_k(G; \lambda)$ is called the k-defect polynomial of the graph.

Let G a vertex colored graph. An edge is called bad if it joins two vertices of the same color. The k-defect polynomial, $\phi_k(G; \lambda)$, counts the number of ways of coloring G with k bad edges. It is easily seen that the chromatic polynomial of a graph G, $\chi(G; \lambda)$, is equal to $\phi_0(G; \lambda)$.

The coboundary polynomial of a graph G, $B(G; \lambda, S)$ can be obtained from the Tutte polynomial by the following transformation:

$$B(G; \lambda, S) = (S-1)^r T(G; \frac{S + \lambda - 1}{S - 1}, S).$$

Chromatic polynomials of graphs have been studied widely; we refer the reader to [3]. On the other hand, little is known about the properties of the k-defect polynomials. The importance of the k-defect polynomials should not be underestimated, since if the k-defect polynomials are known it is equivalent to finding the Tutte polynomial. In addition, the improper coloring is applicable in network theory and time tabling just as is the proper coloring.

Flower graphs form a class of graphs that is highly symmetric. Some classes of flower graphs have attractive simple formulas for the chromatic polynomial; see [8]. In this paper, we study the asymmetric complete flower graph and asymmetric incomplete flower graph.

2. Asymmetric flower graphs

In this section we give a definition and an example of an asymmetric complete flower graph and an asymmetric incomplete flower graph.

A graph G is called a *complete flower graph* if it has n vertices that form an n-cycle and n sets of $m-2$ vertices which form m-cycles around the n cycle so that each m-cycle uniquely intersects with the n-cycle on a single edge. This graph is denoted by F_{n_m}. It is clear that F_{n_m} has $n(m-1)$ vertices and nm edges. The m-cycles are called the *petals* and the n-cycle is called the center of F_{n_m}. The n vertices that form the center are all of degree 4 and all the other vertices have degree 2. Note that $m \geq 2$ and $n \geq 2$.

Removing a petal p of F_{n_m} is to take one m-cycle and delete all the vertices of degree 2 and their adjacent edges. We define an *incomplete flower graph with i petals* to be a complete flower graph with $n-i$ petals removed for $i \in \{1, 2, \cdots, n-1\}$. This graph is denoted by $F_{n_m}^i$. It should be noted that the positions of petals removed may be relevant and $F_{n_m} = F_{n_m}^n$. Thus, we have several nonisomorphic graphs represented by $F_{n_m}^i$. An *asymmetric complete flower graph* is a flower graph with center C_n and n petals of different sizes, m_k where $k \in \{1, 2, \cdots, n\}$. If an asymmetric complete flower graph has j_k petals of size m_k it is denoted by $F_{n_{m_{1_{j_1}}, m_{2_{j_2}}, \cdots m_{t_{j_t}}}}$ where $j_1 + j_2 + \cdots j_t = n$. An *asymmetric incomplete flower graph* with i petals is isomorphic to an asymmetric complete flower graph with $n-i$ petals missing. If an asymmetric incomplete flower graph has i petals where j_k petals are of size m_k it is denoted by $F_{n_{m_{1_{j_1}}, m_{2_{j_2}}, \cdots m_{t_{j_t}}}}^i$ where $j_1 + j_2 + \cdots j_t = i$.

The diagrams in Figure are examples of an asymmetric complete flower graph and an asymmetric incomplete flower graph.

3. Tutte polynomial

The Tutte polynomial $T(G; x, y)$ of a graph G, is a polynomial in two independent variables x and y and is defined as

$$F_{6_{2_1, 3_1, 4_2, 5_1, 6_1}} \qquad F_{6_{3_1, 4_2, 6_1}}^4$$
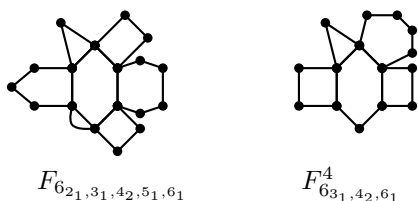
Figure. Asymmetric complete flower graph and asymmetric incomplete flower graph.

$$T(G; x, y) = \sum_{X \subseteq E} (x-1)^{r(E)-r(X)} (y-1)^{|X|-r(X)},$$

where $r(X)$ denotes the rank of the graph (V, X), V the set of vertices, and X the set of edges. Recall that $G \backslash e$ is the graph obtained by deleting an edge e of G and G/e is the graph obtained by contracting an edge e of G. The Tutte polynomial can also be computed using the following deletion and contraction formula:

T1. $T(I; x, y) = x$ and $T(L; x, y) = y$ where I is an isthmus and L is a loop.

T2. If e is an edge of the graph G and e is neither a loop nor an isthmus, then

$$T(G; x, y) = T(G \backslash e; x, y) + T(G/e; x, y).$$

T3. If e is a loop or an isthmus of the graph G, then

$$T(G; x, y) = T(e; x, y) T(G/e; x, y).$$

The following theorem follows from the deletion and contraction formula of the Tutte polynomials.

Theorem 3.1 *Let t_n be a tree on n vertices and let C_n be an n-cycle. Then*

(i) $T(t_n; x, y) = x^{n-1}$.

(ii) $T(C_n; x, y) = y + \sum_{q=1}^{n-1} x^q$.

Some well-known evaluations of the Tutte polynomial are given in the following theorems; we refer to [5, 7, 9].

Theorem 3.2 *Let G be a graph of order n. Then the chromatic polynomial of G follows:*

$$\chi(G; \lambda) = (-1)^{r(G)} \lambda^{n-r(G)} T(\Gamma; 1 - \lambda, 0).$$

Theorem 3.3 *If $G = (V, E)$ is a 2-connected graph, then each of the following statements holds:*

(a) $T(G; -2, 0)$ *corresponds to the number of Eulerian orientations.*

(b) $T(G; -1, -1)$ *corresponds to the dimension of the bicycle space of binary codes.*

(c) $T(G; 0, -1)$ *corresponds to the characteristic function of Eulerian graphs.*

(d) $T(G; 0, 0)$ *corresponds to the characteristic function of the empty graph.*

(e) $T(G; 0, 2)$ *corresponds to the number of orientations of a bridgeless graph G such that each edge is contained in an oriented cycle.*

(f) $T(G; 1, 0)$ *corresponds to the number of acyclic orientations with one fixed source vertex.*

(g) $T(G; 1, 2)$ *corresponds to the number of connected spanning subgraphs.*

(h) $T(G; 2, 0)$ *corresponds to the number of acyclic orientations of* G.

(i) $T(G; 2, 1)$ *corresponds to the number of acyclic subgraphs.*

(j) $T(G; 2, 2)$ *corresponds to the number of spanning subgraphs.*

To ease notation in this paper, if G_1 and G_2 are disjoint graphs, the graph obtained by the operation of merging one vertex of G_1 and one vertex of G_2 will be denoted by $G_1 \bullet G_2$. For any flower graph $F_{n_m}^i$, the edges of a petal will be labeled in the following sequence: $e_1, e_2, \cdots, e_m, e_1$, where e_1 is the edge in both the center and the petal, e_i is adjacent to e_{i+1}, and e_m is adjacent to e_1. We are interested in the edge e_2 to ease our clarification in the following lemmas and theorems. Note that e_2 is an edge of the petal with one vertex in the center and if we delete e_2 from $F_{n_m}^i$ we get $F_{n_m}^{i-1} \bullet P_{m-2}$ where P_{m-2} is a tree with $m-2$ edges, isomorphic to a path.

In the following lemmas and theorems, we shall use the edge e_2 in the deletion and contraction formula when applied to a petal unless otherwise stated.

Lemma 3.4 *Let* $F_{n_m}^1$ *be a flower graph with 1 petal of size* m *and center* C_n. *Then*

$$T(F_{n_m}^1; x, y) = \left(\sum_{q=0}^{m-2} x^q \right) T(C_n; x, y) + y T(C_{n-1}; x, y).$$

Proof We use induction on m and the deletion and contraction method of the Tutte polynomial. Let $m = 2$; thus, we have a flower graph with center C_n and a 2-cycle petal. Thus, $F_{n_2}^1$ is just C_n with one pair of parallel edges. Let e be one of the parallel edges. If we delete e we get C_n and if we contract e we get C_{n-1} with a loop. Hence,

$$\begin{aligned} T(F_{n_m}^1; x, y) &= T(C_n; x, y) + y T(C_{n-1}; x, y) \\ &= \left(\sum_{q=0}^{m-2} x^q \right) T(C_n; x, y) + y T(C_{n-1}; x, y), \end{aligned}$$

therefore true for the base case. Assume it is true when $m = k$ and $k > 2$. Now let $m = k+1$ and let e be an edge e_2 of the petal P. By deletion and contraction method,

$$T(F_{n_{k+1}}^1; x, y) = T(F_{n_{k+1}}^1 \backslash e; x, y) + T(F_{n_{k+1}}^1 / e; x, y). \tag{3.1}$$

However, $F_{n_{k+1}}^1 \backslash e$ is isomorphic to $C_n \bullet P_{k-1}$. Recall that P_{k-1} is a tree isomorphic to a path with $k-1$ edges. Thus, the $k-1$ edges of P_{k-1} are isthmuses in $F_{n_{k+1}}^1 \backslash e$. Hence,

$$T(F_{n_{k+1}}^1 \backslash e; x, y) = x^{k-1} T(C_n; x, y).$$

$F_{n_{k+1}}^1 / e$ is isomorphic to $F_{n_k}^1$. Thus,

$$T(F_{n_{k+1}}^1 / e; x, y) = T(F_{n_k}^1; x, y).$$

By induction hypothesis,

$$T(F^1_{n_{k+1}}/e; x, y) = \left(\sum_{q=0}^{k-2} x^q\right) T(C_n; x, y) + yT(C_{n-1}; x, y).$$

If we substitute in Equation 3.1 we get the required result. □

Recalling the following notation, $F^i_{n_{m_{1_{j_1}}, m_{2_{j_2}}, \cdots, m_{t_{j_t}}}}$ is an asymmetric incomplete flower graph with center C_n and i petals where j_k petals are of size m_k. The following lemma gives the recursive method for the Tutte polynomial of an asymmetric incomplete flower graph with center C_n and i petals, $F^i_{n_{m_{1_{j_1}}, m_{2_{j_2}}, \cdots, m_{t_{j_t}}}}$.

Lemma 3.5 *Let G be an asymmetric incomplete flower graph with i petals, $F^i_{n_{m_{1_{j_1}}, \cdots, m_{t_{j_t}}}}$, where $i \in \{1, 2, \cdots, n-1\}$. If we pick one petal, say of size m_t, then*

$$T(G; x, y) = \left[\sum_{q=0}^{m_t-2} x^q\right] T(F^{i-1}_{n_{m_{1_{j_1}}, \cdots, m_{t_{j_t}-1}}}; x, y) + yT(F^{i-1}_{(n-1)_{m_{1_{j_1}}, \cdots, m_{t_{j_t}-1}}}; x, y).$$

Proof Let G be $F^i_{n_{m_{1_{j_1}}, m_{2_{j_2}}, \cdots, m_{t_{j_t}}}}$. We fix and consider one petal of G of size m_t and call this petal P. Let e be an edge e_2 of petal P. Now by deletion and contraction formula of the Tutte polynomial,

$$T(G; x, y) \quad = \quad T(G\backslash e; x, y) + T(G/e; x, y). \tag{3.2}$$

Now $G\backslash e$ is isomorphic to $F^{i-1}_{n_{m_{1_{j_1}}, \cdots, m_{t_{j_t}-1}}} \bullet P_{m_t-2}$ and the $m_t - 2$ edges of P_{m_t-2} are isthmuses $G\backslash e$. Hence,

$$T(G\backslash e; x, y) = x^{m_t-2} T(F^{i-1}_{n_{m_{1_{j_1}}, \cdots, m_{t_{j_t}-1}}}; x, y).$$

G/e is isomorphic to $F^i_{n_{m_{1_{j_1}}, \cdots, m_{t_{j_t}-1}, (m_t-1)_1}}$. Now petal P is of size m_t-1. Repeat the deletion and contraction formula on petal P until this petal is removed and get $T(G/e; x, y)$. Then substitute back $T(G\backslash e; x, y)$ and $T(G/e; x, y)$ in Equation 3.2 to get the required result. □

Lemma 3.6 *Let $F^i_{n_{m_{1_{j_1}}, \cdots, m_{t_{j_t}}, m_{t+1_{j_{t+1}}}}}$ be an asymmetric flower graph with i petals, where $i \in \{1, 2, \cdots, n\}$ and let $j_{t+1} = 1$. Then*

$$\sum_{\substack{\phi_1 + \cdots + \phi_{t+1} = i+1-k \\ 0 \le \phi_r \le j_r}} \left[\prod_{r=1}^{t+1} \binom{j_r}{\phi_r} \left(\sum_{q=0}^{m_r-2} x^q\right)^{\phi_r}\right]$$

$$= \left(\sum_{q=0}^{m_{t+1}-2} x^q\right) \sum_{\substack{\phi_1 + \cdots + \phi_t = i-k \\ 0 \le \phi_r \le j_r}} \left[\prod_{r=1}^{t} \binom{j_r}{\phi_r} \left(\sum_{q=0}^{m_r-2} x^q\right)^{\phi_r}\right]$$

$$+ \sum_{\substack{\phi_1 + \cdots + \phi_t = i+1-k \\ 0 \le \phi_r \le j_r}} \left[\prod_{r=1}^{t} \binom{j_r}{\phi_r} \left(\sum_{q=0}^{m_r-2} x^q\right)^{\phi_r}\right].$$

Proof   Since $j_{t+1} = 1$, then $\phi_{t+1} \in \{0, 1\}$. Thus the sum

$$\sum_{\substack{\phi_1 + \cdots + \phi_{t+1} = i+1-k \\ 0 \leq \phi_r \leq j_r}} \left[\prod_{r=1}^{t+1} \binom{j_r}{\phi_r} \left(\sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right]$$

can be split into two sums where one sum takes the case $\phi_{t+1} = 1$ and the other sum takes the case $\phi_{t+1} = 0$.

If $\phi_{t+1} = 1$ then $\phi_1 + \cdots + \phi_{t+1} = i+1-k$ implies $\phi_1 + \cdots + \phi_t = i-k$ and $\binom{j_{t+1}}{\phi_{t+1}} = \binom{1}{1}$. Hence,

$$\sum_{\substack{\phi_1 + \cdots + \phi_{t+1} = i+1-k \\ 0 \leq \phi_r \leq j_r \\ \phi_{t+1} = 1}} \left[\prod_{r=1}^{t+1} \binom{j_r}{\phi_r} \left(\sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \tag{3.3}$$

$$= \left(\sum_{q=0}^{m_{t+1}-2} x^q \right) \sum_{\substack{\phi_1 + \cdots + \phi_t = i-k \\ 0 \leq \phi_r \leq j_r}} \left[\prod_{r=1}^{t} \binom{j_r}{\phi_r} \left(\sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right].$$

If $\phi_{t+1} = 0$ then $\phi_1 + \cdots + \phi_{t+1} = i+1-k$ implies $\phi_1 + \cdots + \phi_t = i+1-k$ and $\binom{j_{t+1}}{\phi_{t+1}} = \binom{1}{0}$. Hence,

$$\sum_{\substack{\phi_1 + \cdots + \phi_{t+1} = i+1-k \\ 0 \leq \phi_r \leq j_r \\ \phi_{t+1} = 0}} \left[\prod_{r=1}^{t+1} \binom{j_r}{\phi_r} \left(\sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] = \sum_{\substack{\phi_1 + \cdots + \phi_t = i+1-k \\ 0 \leq \phi_r \leq j_r}} \left[\prod_{r=1}^{t} \binom{j_r}{\phi_r} \left(\sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right]. \tag{3.4}$$

Hence the result if we sum both cases when $\phi_{t+1} = 0$ and $\phi_{t+1} = 1$. □

The following theorem gives an explicit expression of the Tutte polynomial of an asymmetric incomplete flower graph.

Theorem 3.7 *Let G be an asymmetric incomplete flower graph with i petals, $F^i_{n_{m_{1_{j_1}}, m_{2_{j_2}}, \cdots, m_{t_{j_t}}}}$, where $i \in \{1, 2, \cdots, n-1\}$. Then*

$$T(G; x, y) = \sum_{k=0}^{i} y^k \left(\sum_{\substack{\phi_1 + \cdots + \phi_t = i-k \\ 0 \leq \phi_r \leq j_r}} \left[\prod_{r=1}^{t} \binom{j_r}{\phi_r} \left(\sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \right) \left(y + \sum_{q=1}^{n-1-k} x^q \right).$$

Proof   We use induction on the number of petals i. Let $i = 2$, and let G be $F^2_{n_{m_{1_1}, m_{2_1}}}$, an asymmetric incomplete flower graph with 2 petals, one of size m_1, the other of size m_2 and center C_n. Then, by Lemma 3.5,

$$T(G; x, y) = \left(\sum_{q=0}^{m_1-2} x^q \right) T(F^1_{n_{m_{2_1}}}; x, y) + y T(F^1_{(n-1)_{m_{2_1}}}; x, y). \tag{3.5}$$

By Lemma 3.4,

$$T(F^1_{n_{m_2}};x,y) = \left(\sum_{q=0}^{m_2-2} x^q\right) T(C_n;x,y) + yT(C_{n-1};x,y),$$

$$T(F^1_{(n-1)_{m_2}};x,y) = \left(\sum_{q=0}^{m_2-2} x^q\right) T(C_{n-1};x,y) + yT(C_{n-2};x,y).$$

Hence, if we substitute in Equation 3.5, we get

$$
\begin{aligned}
T(G;x,y) &= \left(\sum_{q=0}^{m_1-2} x^q\right)\left(\sum_{q=0}^{m_2-2} x^q\right)\left(y+\sum_{q=1}^{n-1} x^q\right)\\
&+ y\left(\sum_{q=0}^{m_1-2} x^q\right)\left(y+\sum_{q=1}^{n-2} x^q\right) + y\left(\sum_{q=0}^{m_2-2} x^q\right)\left(y+\sum_{q=1}^{n-2} x^q\right) + y^2\left(y+\sum_{q=1}^{n-3} x^q\right)\\
&= \sum_{k=0}^{2} y^k\left(\sum_{\substack{\phi_1+\cdots+\phi_t=2-k\\ 0\le\phi_r\le j_r}}\left[\prod_{r=1}^{2}\binom{j_r}{\phi_r}\left(\sum_{q=0}^{m_r-2} x^q\right)^{\phi_r}\right]\right)\left(y+\sum_{q=1}^{n-1-k} x^q\right).
\end{aligned}
$$

Hence, it is true for an asymmetric incomplete flower graph with 2 petals. Now we assume it is true for $i=l$ where $l>2$ and $l<n-1$. Let $i=l+1$ where $l+1\le n-1$. Let G be $F^{l+1}_{n_{m_{1_{j_1}},m_{2_{j_2}},\cdots,m_{t_{j_t}},m_{t+1_1}}}$ such that it has one petal of size m_{t+1} more than $F^l_{n_{m_{1_{j_1}},m_{2_{j_2}},\cdots,m_{t_{j_t}}}}$. Then by Lemma 3.5,

$$T(G;x,y) = \left(\sum_{q=0}^{m_{t+1}-2} x^q\right) T(F^l_{n_{m_{1_{j_1}},m_{2_{j_2}},\cdots,m_{t_{j_t}}}};x,y) + yT(F^l_{(n-1)_{m_{1_{j_1}},m_{2_{j_2}},\cdots,m_{t_{j_t}}}};x,y).$$

By induction hypothesis,

$$\left(\sum_{q=0}^{m_{t+1}-2} x^q\right) T(F^l_{n_{m_{1_{j_1}},m_{2_{j_2}},\cdots,m_{t_{j_t}}}};x,y) \tag{3.6}$$

$$= \left(\sum_{q=0}^{m_{t+1}-2} x^q\right)\sum_{k=0}^{l} y^k\left(\sum_{\substack{\phi_1+\cdots+\phi_t=l-k\\ 0\le\phi_r\le j_r}}\left[\prod_{r=1}^{t}\binom{j_r}{\phi_r}\left(\sum_{q=0}^{m_r-2} x^q\right)^{\phi_r}\right]\right) T(C_{n-k};x,y)$$

and

$$yT(F^l_{(n-1)_{m_{1_{j_1}},m_{2_{j_2}},\cdots,m_{t_{j_t}}}};x,y) \tag{3.7}$$

$$= y\sum_{k=0}^{l} y^k\left(\sum_{\substack{\phi_1+\cdots+\phi_t=l-k\\ 0\le\phi_r\le j_r}}\left[\prod_{r=1}^{t}\binom{j_r}{\phi_r}\left(\sum_{q=0}^{m_r-2} x^q\right)^{\phi_r}\right]\right) T(C_{n-1-k};x,y).$$

Hence,

$$T(G; x, y) \tag{3.8}$$

$$= \left(\sum_{q=0}^{m_{t+1}-2} x^q \right) \sum_{k=0}^{l} y^k \left(\sum_{\substack{\phi_1+\cdots+\phi_t=l-k \\ 0 \le \phi_r \le j_r}} \left[\prod_{r=1}^{t} \binom{j_r}{\phi_r} \left(\sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \right) T(C_{n-k}; x, y)$$

$$+ y \sum_{k=0}^{l} y^k \left(\sum_{\substack{\phi_1+\cdots+\phi_t=l-k \\ 0 \le \phi_r \le j_r}} \left[\prod_{r=1}^{t} \binom{j_r}{\phi_r} \left(\sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \right) T(C_{n-1-k}; x, y).$$

Let s be a fixed integer in the set $\{0, 1, \cdots, l\}$. Then track the term $y^s T(C_{n-s}; x, y)$ in Equation 3.6; it occurs when $k = s$. In Equation 3.7, the term $y^s T(C_{n-s}; x, y)$ occurs when $k - 1 = s$. Thus, in Equation 3.8, we have the term $y^s T(C_{n-s}; x, y)$ with coefficient

$$\left(\sum_{q=0}^{m_{t+1}-2} x^q \right) \sum_{\substack{\phi_1+\cdots+\phi_t=l-k \\ 0 \le \phi_r \le j_r}} \left[\prod_{r=1}^{t} \binom{j_r}{\phi_r} \left(\sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \tag{3.9}$$

$$+ \sum_{\substack{\phi_1+\cdots+\phi_t=l-(k-1) \\ 0 \le \phi_r \le j_r}} \left[\prod_{r=1}^{t} \binom{j_r}{\phi_r} \left(\sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right]$$

$$= \sum_{\substack{\phi_1+\cdots+\phi_{t+1}=i+1-k \\ 0 \le \phi_r \le j_r}} \left[\prod_{r=1}^{t+1} \binom{j_r}{\phi_r} \left(\sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \quad \text{by Lemma 3.6,}$$

hence true for any $i \in \{1, 2, \cdots, n-1\}$. □

In order to state our next result we need the following notation. We denote the coefficient of z^m a power series $f(z)$ by $[z^m]f(z)$. Define

$$C_{i,k}(d) = C_{i,k}(d; m_1, j_1, \ldots, m_t, j_t)$$

$$= \sum_{\ell=0}^{d} \binom{i-k+d-\ell}{d-\ell} \sum_{\substack{\phi_1+\cdots+\phi_t=i+1-k \\ (m_1-1)\varphi_1+\cdots+(m_r-1)\varphi_t=\ell \\ 0 \le \varphi_r \le \phi_r \le j_r}} (-1)^{\sum_{j=1}^{t} \varphi_j} \prod_{r=1}^{t} \binom{j_r}{\phi_r} \binom{\phi_r}{\varphi_r}.$$

Corollary 3.8 *Let G be an asymmetric incomplete flower graph with i petals, $F^i_{n_{m_1}_{j_1}, m_2_{j_2}, \cdots, m_t_{j_t}}$, where $i \in \{1, 2, \cdots, n-1\}$. Then the coefficient of $x^d y^k$ in $T(G; x, y)$ is given by*

$$[x^d y^k] T(G; x, y) = C_{i,k}(d) + C_{i,k}(d-1) - C_{i,k}(d-(n-k)).$$

Proof By Theorem 3.7, we have

$$[y^k]T(G;x,y) = \frac{1}{(1-x)^{i+1-k}} \sum_{\substack{\phi_1+\cdots+\phi_t=i+1-k \\ 0 \leq \phi_r \leq j_r}} \left[\prod_{r=1}^{t} \binom{j_r}{\phi_r} \left(1-x^{m_r-1}\right)^{\phi_r} \right]$$

$$+ \frac{x}{(1-x)^{i+1-k}} \sum_{\substack{\phi_1+\cdots+\phi_t=i-k \\ 0 \leq \phi_r \leq j_r}} \left[\prod_{r=1}^{t} \binom{j_r}{\phi_r} \left(1-x^{m_r-1}\right)^{\phi_r} \right] \left(1-x^{n-1-k}\right),$$

which implies that $[x^d y^k]T(G;x,y) = C_{i,k}(d) + C_{i,k}(d-1) - C_{i,k}(d-(n-k))$, which completes the proof. □

Theorem 3.9 *Define* $C'_{i,k}(d,n) = C_{i,k}(d) + C_{i,k}(d-1) - C_{i,k}(d-(n-k))$. *Let* G *be an asymmetric incomplete flower graph with* i *petals,* $F^i_{n_{m_{1_{j_1}}, m_{2_{j_2}}, \cdots, m_{t_{j_t}}}}$, *where* $i \in \{1, 2, \cdots, n-1\}$. *Then the* ℓ-*defect polynomial of* G *is*

$$\phi_\ell(G;\lambda) = \sum_{d,k \geq 0} \sum_{a=0}^{r-d} C'_{i,k}(d)(-1)^{r-d-a}(\lambda-1)^{d-(\ell-a-k)} \binom{r-d}{a} \binom{d}{\ell-a-k}.$$

Proof By Corollary 3.8, we have that

$$B(G;\lambda,S) = (S-1)^r T(G; 1+\lambda/(S-1); S) = \sum_{d,k \geq 0} C'_{i,k}(d,n)(S-1)^{r-d} S^k (S+\lambda-1)^d,$$

which implies that the coefficient of S^ℓ in $B(G;\lambda,S)$ is given by

$$\phi_\ell(G;\lambda) = \sum_{d,k \geq 0} \sum_{a=0}^{r-d} C'_{i,k}(d)(-1)^{r-d-a}(\lambda-1)^{d-(\ell-a-k)} \binom{r-d}{a} \binom{d}{\ell-a-k},$$

which completes the proof. □

We need the following lemmas before we give the explicit expression of the Tutte polynomial of an asymmetric complete flower graph.

Lemma 3.10 *Let* G *be* $F_{2_{m_{1_1}, m_{2_1}}}$, *an asymmetric complete flower graph with* 2 *petals of sizes* m_1 *and* m_2. *Then*

$$T(G;x,y) = \left(\sum_{q=0}^{m_1-2} x^q \right) T(F^1_{2_{m_{2_1}}}; x, y) + y^2 T(C_{m_2-1}; x, y).$$

Proof We consider the petal of size m_1 and call this petal P, so we use deletion and contraction formula of the Tutte polynomial on the edges of petal P that are not in the center. Remove isthmuses and loops to get the required result. □

The following lemma gives the recursive method of the Tutte polynomial of an asymmetric complete flower graph.

Lemma 3.11 *Let G be $F_{n_{m_{1_{j_1}},m_{2_{j_2}},\cdots,m_{t_{j_t}}}}$, an asymmetric complete flower graph. Then*

$$T(G;x,y) = \left(\sum_{q=0}^{m_t-2} x^q\right) T(F^{n-1}_{n_{m_{1_{j_1}}},\cdots,m_{t_{j_t-1}}};x,y) + yT(F_{(n-1)_{m_{1_{j_1}}},\cdots,m_{t_{j_t-1}}},;x,y).$$

Proof Similar to the proof of Lemma 3.5. □

The following theorem gives an explicit expression of the Tutte polynomial of an asymmetric complete flower graph.

Theorem 3.12 *Let G be the graph $F_{n_{m_{1_{j_1}},m_{2_{j_2}},\cdots,m_{t_{j_t}}}}$, an asymmetric complete flower graph. Then*

$$T(G;x,y) = \left[\sum_{k=0}^{n} y^k \sum_{\substack{\phi_1+\cdots+\phi_t=n-k \\ 0\le\phi_r\le j_r}} \left[\prod_{r=1}^{t} \binom{j_r}{\phi_r}\left(\sum_{q=0}^{m_r-2} x^q\right)^{\phi_r}\right]\left(y+\left[\sum_{q=1}^{n-1-k} x^q\right]\right)\right] - y^n.$$

Proof By induction on the number of vertices of the center n. The base case is $n=2$. By Lemma 3.10,

$$T(F_{2_{m_{1_1}},m_{2_1}};x,y) = \left(\sum_{q=0}^{m_1-2} x^q\right) T(F^1_{2_{m_{2_1}}};x,y) + y^2 T(C_{m_2-1};x,y).$$

Applying Lemma 3.4 and then Theorem 3.1 part (ii), we get

$$T(F_{2_{m_{1_1}},m_{2_1}};x,y)$$

$$= \left(\sum_{q=0}^{m_1-2} x^q\right)\left(\sum_{q=0}^{m_2-2} x^q\right) T(C_2;x,y) + y\left(\sum_{q=0}^{m_1-2} x^q\right) T(C_1;x,y) + y^2 T(C_{m_2}-1;x,y)$$

$$= \left(\sum_{q=0}^{m_1-2} x^q\right)\left(\sum_{q=0}^{m_2-2} x^q\right)(y+x) + y\left(\sum_{q=0}^{m_1-2} x^q\right)y + y^2\left(y+\sum_{q=1}^{m_2-2} x^q\right).$$

Rearranging, we get

$$T(F_{2_{m_{1_1}},m_{2_1}};x,y)$$

$$= \left(\sum_{q=0}^{m_1-2} x^q\right)\left(\sum_{q=0}^{m_2-2} x^q\right)\left(y+\sum_{q=1}^{2-1} x^q\right) + y^2\left(\sum_{q=0}^{m_1-2} x^q\right) + y^2\left(\sum_{q=0}^{m_2-2} x^q\right) + (y^3-y^2)$$

$$= \left[\sum_{k=0}^{1} y^k \sum_{\substack{\phi_1+=2-k \\ 0\le\phi_r\le j_r}} \left[\prod_{r=1}^{2} \binom{j_r}{\phi_r}\left(\sum_{q=0}^{m_r-2} x^q\right)^{\phi_r}\right]\left(y+\left[\sum_{q=1}^{1-k} x^q\right]\right)\right] + [y^3-y^2].$$

Hence, it is true for an asymmetric complete flower with 2-cycle center. Assume it is true for $n=l$ where $l>2$.
Now consider $n=l+1$ and let G be $F^{l+1}_{(l+1)_{m_{1_{j_1}},m_{2_{j_2}},\cdots,m_{t_{j_t}},m_{t+1_1}}}$ such that G has one petal of size m_{t+1} more
than $F^l_{n_{m_{1_{j_1}},m_{2_{j_2}},\cdots,m_{t_{j_t}}}}$. Then by Lemma 3.11,

$$T(G;x,y) = \left(\sum_{q=0}^{m_{t+1}-2} x^q\right) T(F^l_{(l+1)_{m_{1_{j_1}},\cdots,m_{t_{j_t}}}};x,y) + yT(F_{(l)_{m_{1_{j_1}},\cdots,m_{t_{j_t}}}},;x,y). \tag{3.10}$$

By Theorem 3.7,

$$\left(\sum_{q=0}^{m_{t+1}-2} x^q\right) T(F^l_{(l+1)_{m_{1_{j_1}},\cdots,m_{t_{j_t}}}}; x, y) \tag{3.11}$$

$$= \left(\sum_{q=0}^{m_{t+1}-2} x^q\right) \sum_{k=0}^{l} y^k \left(\sum_{\substack{\phi_1+\cdots+\phi_t=l-k \\ 0\le\phi_r\le j_r}} \left[\prod_{r=1}^{t}\binom{j_r}{\phi_r}\left(\sum_{q=0}^{m_r-2} x^q\right)^{\phi_r}\right]\right)\left(y+\sum_{q=1}^{l-k} x^q\right).$$

By induction hypothesis,

$$yT(F_{l_{m_{1_{j_1}},\cdots,m_{t_{j_t}}}}; x, y) \tag{3.12}$$

$$= y\left[\sum_{k=0}^{l-1} y^k \sum_{\substack{\phi_1+\cdots+\phi_t \\ =l-k \\ 0\le\phi_r\le j_r}} \left[\prod_{r=1}^{t}\binom{j_r}{\phi_r}\left(\sum_{q=0}^{m_r-2} x^q\right)^{\phi_r}\right]\left(y+\left[\sum_{q=1}^{l-1-k} x^q\right]\right)\right] + \left[y^{l+1}-y^l\right].$$

Hence, if we substitute Equation 3.11 and Equation 3.12 in Equation 3.10, we get

$$T(G; x, y) \tag{3.13}$$

$$= \left(\sum_{q=0}^{m_{t+1}-2} x^q\right) \sum_{k=0}^{l} y^k \left(\sum_{\substack{\phi_1+\cdots+\phi_t=l-k \\ 0\le\phi_r\le j_r}} \left[\prod_{r=1}^{t}\binom{j_r}{\phi_r}\left(\sum_{q=0}^{m_r-2} x^q\right)^{\phi_r}\right]\right)\left(y+\sum_{q=1}^{l-k} x^q\right)$$

$$= y\left[\sum_{k=0}^{l-1} y^k \sum_{\substack{\phi_1+\cdots+\phi_t \\ =l-k \\ 0\le\phi_r\le j_r}} \left[\prod_{r=1}^{t}\binom{j_r}{\phi_r}\left(\sum_{q=0}^{m_r-2} x^q\right)^{\phi_r}\right]\left(y+\left[\sum_{q=1}^{l-1-k} x^q\right]\right)\right] + \left[y^{l+1}-y^l\right].$$

Let s be a fixed integer in $\{0, 1, \cdots, n-1\}$. Track the term $y^s\left(y+\left[\sum_{q=1}^{l-1-s} x^q\right]\right)$ in Equation 3.11; it occurs when $k=s$. In Equation 3.12, the term $y^s\left(y+\left[\sum_{q=1}^{l-1-s} x^q\right]\right)$ occurs when $k-1=s$. Thus, in Equation 3.13, we have the term $y^s\left(y+\left[\sum_{q=1}^{l-1-s} x^q\right]\right)$ with coefficient

$$\left(\sum_{q=0}^{m_{t+1}-2} x^q\right) \sum_{\substack{\phi_1+\cdots+\phi_t=l-k \\ 0\le\phi_r\le j_r}} \left[\prod_{r=1}^{t}\binom{j_r}{\phi_r}\left(\sum_{q=0}^{m_r-2} x^q\right)^{\phi_r}\right] \tag{3.14}$$

$$+ \sum_{\substack{\phi_1+\cdots\phi_t=l-(k-1) \\ 0\le\phi_r\le j_r}} \left[\prod_{r=1}^{t}\binom{j_r}{\phi_r}\left(\sum_{q=0}^{m_r-2} x^q\right)^{\phi_r}\right]$$

$$= \sum_{\substack{\phi_1+\cdots+\phi_{t+1}=i+1-k \\ 0\le\phi_r\le j_r}} \left[\prod_{r=1}^{t+1}\binom{j_r}{\phi_r}\left(\sum_{q=0}^{m_r-2} x^q\right)^{\phi_r}\right] \text{ by Lemma 3.6.}$$

Hence,

$$T(G; x, y)$$

$$= \left[\sum_{s=0}^{(l+1)-1} y^s \sum_{\substack{\phi_1+\cdots+\phi_t,\phi_{t+1} \\ =(l+1)-s \\ 0 \le \phi_r \le j_r}} \left[\prod_{r=1}^{t+1} \binom{j_r}{\phi_r} \left(\sum_{q=0}^{m_r-2} x^q \right)^{\phi_r} \right] \left(y + \left[\sum_{q=1}^{(l+1)-1-s} x^q \right] \right) \right] + \left[y^{l+2} y^{l+1} \right],$$

therefore true for any asymmetric complete flower graph with center C_n where $n > 1$. □

By the proof of Corollary 3.8 and Theorem 3.12, we obtain the following result.

Corollary 3.13 *Let G be the graph $F_{n_{m_1_{j_1}}, m_2_{j_2}, \cdots, m_t_{j_t}}$, an asymmetric complete flower graph. Then the coefficient of $x^d y^k$ in $T(G; x, y)$ is given by*

$$[x^d y^k] T(G; x, y) = C_{n,k}(d) + C_{n,k}(d-1) - C_{n,k}(d - (n-k) - \delta_{k,n} \delta_{d,0},$$

where $\delta_{a,b} = 1$ if $a = b$ and $\delta_{a,b} = 0$ otherwise.

Similar techniques as in the proof of Theorem 3.9, Corollary 3.13 lead to the following result.

Theorem 3.14 *Define $C''_{i,k}(d, n) = C_{i,k}(d) + C_{i,k}(d-1) - C_{i,k}(d - (n-k)) - \delta_{n,k} \delta_{d,0}$, where $\delta_{a,b} = 1$ if $a = b$ and $\delta_{a,b} = 0$ otherwise. Let G be an asymmetric incomplete flower graph with i petals, $F^i_{n_{m_1_{j_1}}, m_2_{j_2}, \cdots, m_t_{j_t}}$, where $i \in \{1, 2, \cdots, n-1\}$. Then the ℓ-defect polynomial of G is*

$$\phi_\ell(G; \lambda) = \sum_{d,k \ge 0} \sum_{a=0}^{r-d} C''_{i,k}(d)(-1)^{r-d-a}(\lambda - 1)^{d-(\ell-a-k)} \binom{r-d}{a} \binom{d}{\ell-a-k}.$$

4. Conclusion

Two nonisomorphic graphs are χ-equivalent if they have the same chromatic polynomial. Two nonisomorphic graphs are Tutte-equivalent if they have the same Tutte polynomial. The following theorem highlights classes of graphs that are χ-equivalent and Tutte-equivalent and that have the same k-defect polynomials.

Theorem 4.1 *Let G_1 and G_2 be nonisomorphic asymmetric incomplete flower graphs represented by $F^i_{n_{m_1_{j_1}}, m_2_{j_2}, \cdots, m_t_{j_t}}$. Then*

(i) $T(G_1; x, y) = T(G_2; x, y)$,

(ii) $\chi(G_1; \lambda) = \chi(G_2; \lambda)$,

(iii) $\phi_k(G_1; \lambda) = \phi_k(G_2; \lambda)$.

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