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On Condition $(PWP)_w$ for S-posets

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Abstract: Golchin and Rezaei (Commun Algebra 2009; 37: 1995–2007) introduced the weak version of Condition (PWP) for S-posets, called Condition $(PWP)_w$. In this paper, we continue to study this condition. We first present a necessary and sufficient condition under which the S-poset $A(I)$ satisfies Condition $(PWP)_w$. Furthermore, we characterize pomonoids S over which all cyclic (Rees factor) S-posets satisfy Condition $(PWP)_w$, and pomonoids S over which all Rees factor S-posets satisfying Condition $(PWP)_w$ have a certain property. Finally, we consider direct products of S-posets satisfying Condition $(PWP)_w$.

Key words: Condition $(PWP)_w$, S-poset, Rees factor S-poset, direct product

1. Introduction and preliminaries

A *partially ordered monoid*, or briefly *pomonoid*, is a monoid S together with a partial order $\leq$ on S such that $s \leq s'$ implies $su \leq s'u$ and $us \leq us'$ for all $s, s', u \in S$. An *ordered right ideal* of a pomonoid S is a nonempty subset I of S such that (1) $IS \subseteq I$ and (2) $s \leq t \in I$ implies $s \in I$, for all $s, t \in S$. In this paper, S always denotes a pomonoid, and a right ideal of S is simply a nonempty subset I of S for which $IS \subseteq I$ (not necessarily an ordered right ideal).

Let S be a pomonoid. A *right S-poset*, usually denoted A_S, is a nonempty set A equipped with a partial order $\leq$ and a right action $A \times S \to A$, $(a, s) \mapsto as$, which satisfies the conditions: (1) the action is monotonic in each variable, (2) $a(st) = (as)t$ and $a1 = a$ for all $a \in A$ and $s, t \in S$. Left S-posets ${}_SB$ are defined analogously, and $\Theta_S = \{\theta\}$ is the one-element right S-poset. All left (resp., right) S-posets form a category, denoted S-POS (resp., POS-S), in which the morphisms are the functions preserving both the action and the order (see [3]).

Preliminary work on flatness properties of S-posets was done by Fakhruddin in the 1980s (see [4, 5]), and continued in recent papers (e.g., [1, 2, 6, 7, 10, 12, 13]).

An S-subposet B_S of an S-poset A_S is called *convex* if, for any $a \in A_S$ and $b, b' \in B_S$, $b' \leq a \leq b$ implies $a \in B$. A pomonoid S is called *weakly right reversible* if, for any $s, s' \in S$, there exist $u, v \in S$ such that $us \leq vs'$. A pomonoid S is called *left collapsible* if, for any $s, s' \in S$, there exists $u \in S$ such that $us = us'$. A pomonoid S is called *weakly left collapsible* if, for any $s, s', z \in S$, $sz = s'z$ implies that there exists $u \in S$ such that $us = us'$.

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In [4], Fakhruddin introduced the concept of order congruence on S-posets. An *order congruence* on an S-poset A_S is an S-act congruence θ such that the factor act A/θ can be equipped with a compatible order so that the natural map $A \to A/\theta$ is an S-poset morphism.

An S-poset A_S is called *cyclic* if $A = aS = \{as \mid s \in S\}$ for some $a \in A_S$. An S-poset A_S is cyclic if and only if it is isomorphic to the factor S-poset of S_S by an S-poset congruence. If K_S is a convex right ideal of a pomonoid S, then there exists an S-poset congruence such that one of its classes is K and all the others are singletons. Moreover, the factor S-poset by this congruence is called the *Rees factor S-poset of S by K*, and denoted S/K. For $s \in S$, the congruence class of s in S/K will be denoted by $[s]_{\rho_K}$, or briefly $[s]$.

The tensor product $A \otimes_S B$ of a right S-poset A_S and a left S-poset ${}_SB$ is a poset that can be constructed in a standard way (see [13] for details) so that the map $A \times B \to A \otimes_S B$ sending (a, b) to $a \otimes b$ is balanced, monotonic in both variables, and universal among balanced, monotonic maps from $A \times B$ into posets. The order relation on $A \otimes_S B$ can be described as follows: $a \otimes b \leq a' \otimes b'$ in $A \otimes_S B$ if and only if there exist $a_1, a_2, \ldots, a_n \in A_S$, $b_2, \ldots, b_n \in {}_SB$, and $s_1, t_1, \ldots, s_n, t_n \in S$ such that

$$\begin{array}{ll} a \leq a_1 s_1 & \\ a_1 t_1 \leq a_2 s_2 & s_1 b \leq t_1 b_2 \\ a_2 t_2 \leq a_3 s_3 & s_2 b_2 \leq t_2 b_3 \\ \vdots & \vdots \\ a_n t_n \leq a' & s_n b_n \leq t_n b'. \end{array}$$

It is easily established, as for S-acts, that $A \otimes_S S$ can be equipped with a natural right S-action, and $A \otimes_S S \cong A$ for any S-poset A_S.

In [1, 12], the properties of po-flatness, po-torsion freeness, and Conditions (P), (P_w), and (E) are introduced. An S-poset A_S is called *po-flat* if, for all $a, a' \in A_S$ and $b, b' \in {}_SB$, $a \otimes b \leq a' \otimes b'$ in $A \otimes_S B$ implies $a \otimes b \leq a' \otimes b'$ in $A \otimes_S (Sb \cup Sb')$. An S-poset A_S is called (*principally*) *weakly po-flat* if, for all (principal) left ideals I of a pomonoid S, and all $s, s' \in I$, $a, a' \in A$, $a \otimes s \leq a' \otimes s'$ in $A \otimes_S S$ implies $a \otimes s \leq a' \otimes s'$ in $A \otimes_S I$. An S-poset A_S is said to satisfy *Condition* (P) if, for all $a, a' \in A_S$ and $s, s' \in S$, $as \leq a's'$ implies $a = a''u$, $a' = a''v$ for some $a'' \in A_S$ and $u, v \in S$ with $us \leq vs'$. An S-poset A_S is said to satisfy *Condition* (E) if, for all $a \in A_S$ and $s, s' \in S$, $as \leq as'$ implies $a = a'u$ for some $a' \in A_S$ and $u \in S$ with $us \leq us'$. An S-poset A_S is called *strongly flat* if it satisfies Conditions (E) and (P). An S-poset A_S is said to satisfy *Condition* (P_w) if, for all $a, a' \in A_S$ and $s, s' \in S$, $as \leq a's'$ implies $a \leq a''u$, $a''v \leq a'$ for some $a'' \in A_S$ and $u, v \in S$ with $us \leq vs'$. An element $c \in S$ is called *right po-cancellable* if, for all $s, s' \in S$, $sc \leq s'c$ implies $s \leq s'$. An S-poset A_S is called *po-torsion free* if, for all $a, a' \in A_S$, and all right po-cancellable elements c of S, $ac \leq a'c$ implies $a \leq a'$.

Recall that an S-poset A_S is said to satisfy *Condition* (E') if, for all $a \in A_S$ and $s, s', z \in S$, $as \leq as'$ and $sz = s'z$ imply $a = a'u$ for some $a' \in A_S$ and $u \in S$ with $us \leq us'$. An S-poset A_S is called *weakly subpullback flat* if it satisfies Conditions (E') and (P).

In [6], Conditions (WP), $(WP)_w$, (PWP), and $(PWP)_w$ were introduced. An S-poset A_S is said to satisfy *Condition* (WP) if the corresponding ϕ is surjective for every subpullback diagram $P(I, I, f, f, S)$, where I is a left ideal of S. An S-poset A_S is said to satisfy *Condition* $(WP)_w$ if, for all $a, a' \in A_S$, $s, t \in S$, and all homomorphisms $f\colon {}_S(Ss \cup St) \to {}_SS$, $af(s) \leq a'f(t)$ implies $a \otimes s \leq a'' \otimes us'$ and $a'' \otimes vt' \leq a' \otimes t$ in

$A\otimes_S(Ss\cup St)$ for some $a''\in A_S$, $u,v\in S$ and $s',t'\in\{s,t\}$ with $f(us')\leq f(vt')$. An S-poset A_S is said to satisfy *Condition* (PWP) if the corresponding ϕ is surjective for every subpullback diagram $P(Ss,Ss,f,f,S)$, $s\in S$. An S-poset A_S is said to satisfy *Condition* $(PWP)_w$ if, for all $a,a'\in A_S$ and $s\in S$,

$$as\leq a's \;\text{ implies }\; a\leq a''u \text{ and } a''v\leq a' \;\text{ for some } a''\in A_S,\; u,v\in S \text{ with } us\leq vs.$$

Moreover, the authors in [6] gave equivalent descriptions of Conditions (PWP), (WP), and $(WP)_w$ for (cyclic, Rees factor) S-posets and obtained the relations between these conditions and properties already studied as follows:

$$\begin{array}{ccccccccc}
\text{free} \Rightarrow \text{projective} \Rightarrow \text{strongly flat} \Rightarrow & (\mathrm{P}) & \Rightarrow & (\mathrm{WP}) & \Rightarrow & (\mathrm{PWP}) & & \\
& \Downarrow & & \Downarrow & & \Downarrow & & \\
& (\mathrm{P_w}) & \Rightarrow & (\mathrm{WP})_\mathrm{w} & \Rightarrow & (\mathrm{PWP})_\mathrm{w} & & \\
& \Downarrow & & \Downarrow & & \Downarrow & & \\
& \text{po-flat} & \Rightarrow & \text{w. po-f.} & \Rightarrow & \text{p. w. po-f.} & \Rightarrow & \text{po-t. f.} \\
& \Downarrow & & \Downarrow & & \Downarrow & & \text{(incomparable)} \\
& \text{flat} & \Rightarrow & \text{w. f.} & \Rightarrow & \text{p. w. f.} & \Rightarrow & \text{t. f.}
\end{array}$$

In this paper, we will continue the work of [6] to study Condition $(PWP)_w$. For S-posets, the definition of the S-poset $A(I)$ was introduced in [3]. Qiao et al. in [9] proved that $A(I)$ fails to satisfy Condition (P). Later, in [10], they further investigated some flatness properties of $A(I)$ and provided an equivalent description of $A(I)$ satisfying Condition $(P)_w$. In fact, observing the proof of [9, Lemma 2.4], we obtain that $A(I)$ also fails to satisfy Condition (PWP). However, the situation for Condition $(PWP)_w$ is markedly different. Thereby, in Section 2, we determine the condition under which $A(I)$ satisfies Condition $(PWP)_w$. In [1, 11], some flatness properties of Rees factor S-posets are discussed. Later, Golchin et al. in [6] gave an equivalent characterization of cyclic (Rees factor) S-posets satisfying Condition (PWP) (Conditions $(WP)_w$ and (WP)). In Section 3, we characterize pomonoids S over which all cyclic (Rees factor) S-posets satisfy Condition $(PWP)_w$. In [7], Khosravi first studied direct products of S-posets satisfying some flatness properties. In Section 4, we investigate pomonoids S over which S-posets satisfying Condition $(PWP)_w$ are preserved under direct products.

2. S-posets satisfying Condition $(PWP)_w$

In this section, we discuss S-posets satisfying Condition $(PWP)_w$ and provide a necessary and sufficient condition under which the S-poset $A(I)$ satisfies Condition $(PWP)_w$.

We first give an alternative description for Condition $(PWP)_w$.

Proposition 2.1 *A right S-poset A_S satisfies Condition $(PWP)_w$ if and only if for all $a,a'\in A_S$, $x,y,s\in S$, and all homomorphisms $f: {}_SSs\to {}_SS$, $af(xs)\leq a'f(ys)$ implies that there exist $a''\in A_S$ and $u,v\in S$ such that $f(us)\leq f(vs)$, $a\otimes xs\leq a''\otimes us$ and $a''\otimes vs\leq a'\otimes ys$ in $A\otimes_S Ss$.*

Proof It follows from the definition of Condition $(PWP)_w$. □

The following proposition shows that right S-posets satisfying Condition $(PWP)_w$ are closed under directed colimits. For more information about directed colimits in the category POS-S the reader is referred to [2, 3].

Proposition 2.2 *Every directed colimit of a direct system of right S-posets that satisfy Condition $(PWP)_w$ satisfies Condition $(PWP)_w$.*

Proof Let $(A_i, \phi_{i,j})$ be a direct system of right S-posets satisfying Condition $(PWP)_w$ over a directed index set I with directed colimit (A, α_i). Suppose that $as \leq a's$ in A. Then there exist $a_i \in A_i$ and $a_j \in A_j$ with $a = \alpha_i(a_i)$, $a' = \alpha_j(a_j)$. Since I is directed, by [2, Proposition 2.5] there exists $k \geq i, j$ such that $\phi_{i,k}(a_i)s \leq \phi_{j,k}(a_j)s$ in A_k. Since A_k satisfies Condition $(PWP)_w$, there exist $a'' \in A_k$ and $u, v \in S$ such that $\phi_{i,k}(a_i) \leq a''u$, $a''v \leq \phi_{j,k}(a_j)$ and $us \leq vs$, but then $a = \alpha_i(a_i) = \alpha_k\phi_{i,k}(a_i) \leq \alpha_k(a'')u$. In a similar way $\alpha_k(a'')v \leq a'$, and this implies that A satisfies Condition $(PWP)_w$. □

It follows from [6] that Condition $(PWP)_w$ implies principally weakly po-flat but not conversely in general. However, for right po-cancellable pomonoids, we have the following corollary, which follows from [6] and [12, Theorems 3.21 and 3.22].

Corollary 2.3 *Let S be a right po-cancellable pomonoid and A_S an S-poset. Then the following statements are equivalent:*

(1) *A_S satisfies Condition (P_w);*

(2) *A_S satisfies Condition $(WP)_w$;*

(3) *A_S satisfies Condition $(PWP)_w$;*

(4) *A_S is weakly po-flat;*

(5) *A_S is principally weakly po-flat;*

(6) *A_S is po-torsion free.*

Now we consider the right S-poset $A(I)$ satisfying Condition $(PWP)_w$. The following definition of $A(I)$ first appeared in [3].

Suppose I is a proper right ideal of a pomonoid S. For any $x, y, z \notin S$, let $A(I) = (\{x, y\} \times (S - I)) \cup (\{z\} \times I)$. Define a right S-action on $A(I)$ by

$$(x, u)s = \begin{cases} (x, us), & if\ us \notin I, \\ (z, us), & if\ us \in I, \end{cases}$$

$$(y, u)s = \begin{cases} (y, us), & if\ us \notin I, \\ (z, us), & if\ us \in I, \end{cases}$$

$$(z, u)s = (z, us).$$

The order on $A(I)$ is defined by

$$(w_1, s) \leq (w_2, t) \Leftrightarrow (w_1 = w_2, s \leq t)\ or\ (w_1 \neq w_2,\ s \leq i \leq t\ for\ some\ i \in I).$$

Then $A(I)$ is a right S-poset (see [10] for details).

Theorem 2.4 *Let I be a proper right ideal of a pomonoid S. Then the right S-poset $A(I)$ satisfies Condition $(PWP)_w$ if and only if for any $u, v, s \in S$ and $i \in I$,*

$$us \leq i \leq vs \Rightarrow (\exists\ j \in I)(us \leq js \wedge j \leq v) \vee (js \leq vs \wedge u \leq j).$$

Proof **Necessity**: If $us \leq i \leq vs$ for any $u, v, s \in S$ and $i \in I$, then $(x,1)us \leq (y,1)vs$. There are four cases to be considered:

Case 1. $u, v \notin I$. Then $(x,u)s \leq (y,v)s$. Since $A(I)$ satisfies Condition $(PWP)_w$, there exist $u', v' \in S$ and $(w,p) \in A(I)$ such that

$$(x,u) \leq (w,p)u',\ (w,p)v' \leq (y,v) \text{ and } u's \leq v's. \tag{1}$$

There are three subcases:

Subcase 1. $w = x$. If $pv' \notin I$, then by (1) we have $(x,u) \leq (x,p)u'$, $(x,pv') \leq (y,v)$, and $u's \leq v's$. Hence, there exists $j \in I$ such that $u \leq pu'$ and $pv' \leq j \leq v$. Since $u's \leq v's$ implies $(pu')s \leq (pv')s$, we have $us \leq (pu')s \leq (pv')s \leq js$. If $pv' \in I$, then we can take $j = pv'$.

Subcase 2. $w = y$. If $pu' \notin I$, then by (1) we have $(x,u) \leq (y,pu')$, $(y,p)v' \leq (y,v)$, and $u's \leq v's$. Hence, there exists $j \in I$ such that $u \leq j \leq pu'$ and $pv' \leq v$. Since $u's \leq v's$ implies $(pu')s \leq (pv')s$, we have $js \leq (pu')s \leq (pv')s \leq vs$. If $pu' \in I$, then we can take $j = pu'$.

Subcase 3. $w = z$. In the case, we may take $j = pu'$ or $j = pv'$.

Case 2. $u \notin I$, $v \in I$. This is analogous to case 1.

Case 3. $u \in I$, $v \notin I$. This is also analogous to case 1.

Case 4. $u \in I$, $v \in I$. Then $(z,u)s \leq (z,v)s$. Since $A(I)$ satisfies Condition $(PWP)_w$, there exist $u', v' \in S$ and $(w,p) \in A(I)$ such that $(z,u) \leq (w,p)u'$, $(w,p)v' \leq (z,v)$, and $u's \leq v's$. We have $(z,us) = (z,u)s \leq (w,p)u's \leq (w,p)v's \leq (z,v)s = (z,vs)$, so $us \leq vs$. Then we can take $j = u$ or $j = v$.

Sufficiency: Suppose that $(w_1,u), (w_2,v) \in A(I)$, and $s \in S$ are such that

$$(w_1,u)s \leq (w_2,v)s. \tag{2}$$

There are three cases to be considered:

Case 1. If $w_1 = w_2 = x$, then by (2) we have $(x,u)s \leq (x,v)s$. Hence $(x,u) \leq (x,1)u$, $(x,1)v \leq (x,v)$, and $us \leq vs$.

Case 2. If $w_1 = x$, $w_2 = y$, then by (2) we have $(x,u)s \leq (y,v)s$. By the definition of $A(I)$, there exists $i \in I$ such that $us \leq i \leq vs$. By assumption, there exists $j \in I$ such that $us \leq js$ and $j \leq v$, or $js \leq vs$ and $u \leq j$. If $us \leq js$, $j \leq v$, then $(x,1)j = (z,j) = (y,1)j \leq (y,1)v = (y,v)$ and $(x,u) \leq (x,1)u$, and if $js \leq vs$, $u \leq j$, then $(x,u) = (x,1)u \leq (x,1)j = (z,j) = (y,1)j$ and $(y,1)v \leq (y,v)$. Therefore, $A(I)$ satisfies Condition $(PWP)_w$.

Case 3. If $w_1 = x$, $w_2 = z$, then (2) means $(x,u)s \leq (z,v)s$. Hence, $(x,u) \leq (x,1)u$, $(x,1)v \leq (z,v)$, and $us \leq vs$.

The other cases can be discussed similarly and we obtain that $A(I)$ satisfies Condition $(PWP)_w$. □

Corollary 2.5 *Let S be a pomonoid and 1 the identity of S, in which 1 is incomparable with every other element of S. Then the following conditions on pomonoids are equivalent:*

(1) *All right S-posets satisfy Condition $(PWP)_w$;*

(2) *All right S-posets satisfying Condition (E) satisfy Condition $(PWP)_w$;*

(3) *All finitely generated right S-posets satisfy Condition $(PWP)_w$;*

(4) *All finitely generated right S-posets satisfying Condition (E) satisfy Condition $(PWP)_w$;*

(5) *S is a pogroup.*

Proof The implications $(1) \Rightarrow (2) \Rightarrow (4)$ and $(1) \Rightarrow (3) \Rightarrow (4)$ are all clear.

$(4) \Rightarrow (5)$. Suppose that I is a proper right ideal of a pomonoid S. It follows from [10, Lemma 2.2] that the right S-poset $A(I)$ satisfies Condition (E). By assumption, $A(I)$ satisfies Condition $(PWP)_w$. Since $i \leq i \leq i$ for every $i \in I$, we take $u = v = 1$, and by Theorem 2.4, there exists $j \in I$ such that $j \leq 1$ or $1 \leq j$. However, 1 is isolated and we obtain $j = 1$, a contradiction. Hence, S has no proper right ideals, and so S is a pogroup.

$(5) \Rightarrow (1)$. It is straightforward to verify. □

At the end of this section, we present an example from [12, Example] that $A(I)$ satisfies Condition $(PWP)_w$. Let $S = \{1, 0\}$ be a monoid with the usual order. We consider the ideal $I = \{0\}$. It follows from Theorem 2.4 that $A(I)$ satisfies Condition $(PWP)_w$. However, if $S = \{1, 0\}$ with the discrete order, and the ideal $I = \{0\}$, then $A(I)$ does not satisfy Condition $(PWP)_w$. This is because, taking $u = v = 1$ and $s = i = 0$ in Theorem 2.4, we have $us \leq i \leq vs$, and there does not exist $j \in I$ such that $j \leq v$, or $u \leq j$.

3. Cyclic (Rees factor) S-posets satisfying $(PWP)_w$

In this section, we will give a description of pomonoids S by Condition $(PWP)_w$ of cyclic (Rees factor) S-posets.

A relation σ on an S-poset A_S is called a *pseudo-order* on A_S if it is transitive, compatible with the S-action, and contains the relation $\leq$ on A_S. The relationship between order congruences and pseudo-orders on A_S was given in [14].

Suppose that ρ is a right order congruence on a pomonoid S. Define a relation $\widehat{\rho}$ by

$$s\widehat{\rho}t \Leftrightarrow [s]_\rho \leq [t]_\rho \text{ in } S/\rho.$$

It is clear that $\widehat{\rho}$ is a pseudo-order on A_S. Below we will describe cyclic S-posets satisfying Condition $(PWP)_w$.

Proposition 3.1 *Let ρ be a right order congruence on a pomonoid S. Then the cyclic right S-poset S/ρ satisfies Condition $(PWP)_w$ if and only if*

$$(\forall x, y, t \in S)([x]_\rho t \leq [y]_\rho t \Rightarrow (\exists u, v \in S)(ut \leq vt \wedge x\widehat{\rho}u \wedge v\widehat{\rho}y)).$$

Proof It is a routine matter. □

The following is a direct corollary of Proposition 3.1.

Corollary 3.2 *Let S be any pomonoid. Then Θ_S satisfies Condition $(PWP)_w$.*

To get the results for Rees factor S-posets we need some more preliminary material.

Lemma 3.3 *([1, Lemma 3]) Let K be a convex, proper right ideal of a pomonoid S. Then for $x, y \in S$,*

$$[x]_{\rho_K} \leq [y]_{\rho_K} \text{ in } S/K \Leftrightarrow (x \leq y) \text{ or } (x \leq k \text{ and } k' \leq y \text{ for some } k, k' \in K).$$

Moreover, $[x]_{\rho_K} = [y]_{\rho_K}$ in S/K if and only if either $x = y$ or else $x, y \in K$.

Recall from [1, 11] that a convex, proper right ideal K of a pomonoid S is *strongly left stabilizing*, if

$$(\forall k \in K)(\forall s \in S)(k \leq s \Rightarrow (\exists k' \in K)(k's \leq s), \text{ and } s \leq k \Rightarrow (\exists k'' \in K)(s \leq k''s)).$$

The following two concepts first appeared in [6]. For convenience, we will define them as follows.

Definition 3.4 A convex, proper right ideal K of a pomonoid S is called *strongly left annihilating,* if

$$(\forall t \in S)(\forall x, y \in S\backslash K)([x]_{\rho_K} t \leq [y]_{\rho_K} t \Rightarrow xt \leq yt).$$

Definition 3.5 A convex, proper right ideal K of a pomonoid S is called *double-strongly left annihilating* (briefly, D*-strongly left annihilating*), if for every $s, t \in S\backslash K$ and homomorphism $f\colon {}_S(Ss \cup St) \to {}_S S$,

$$[f(s)]_{\rho_K} \leq [f(t)]_{\rho_K} \Rightarrow f(s) \leq f(t).$$

Every D-strongly left annihilating convex, proper right ideal of a pomonoid S is strongly left annihilating. Indeed, if $[x]_{\rho_K} t \leq [y]_{\rho_K} t$ for $t \in S$ and $x, y \in S\backslash K$, then $[\rho_t(x)]_{\rho_K} \leq [\rho_t(y)]_{\rho_K}$. (If S is a pomonoid and $t \in S$, then $\rho_t\colon S \to S$ will denote the right translation by t, that is, $\rho_t(s) = st$ for any $s \in S$.) This implies that if K is D-strongly left annihilating, then $\rho_t(x) \leq \rho_t(y)$, that is, $xt \leq yt$. Hence, K is strongly left annihilating. The next example from [8, Example 2] shows that not all strongly left annihilating convex, proper right ideals are D-strongly left annihilating.

Example 3.6 (strongly left annihilating $\not\Rightarrow$ D-strongly left annihilating) Let S be an annihilating chain of semigroup $S_1 = \{1\}$, a right zero semigroup $S_2 = \{s, t\}$, a left zero semigroup $S_3 = \{x, y\}$, and a semigroup $S_4 = \{0\}$ $(1 > 2 > 3 > 4)$. The order of S is discrete. (A chain of semigroups $S_\gamma, \gamma \in \Gamma$ is called an annihilating chain if $x \in S_\alpha$ and $y \in S_\beta$, $\alpha > \beta$ implies $xy = yx = y$.) Consider the right ideal $K = \{x, y, 0\}$. If $[u]_{\rho_K} z \leq [v]_{\rho_K} z$ for $z \in S$ and $u, v \in S\backslash K$, then $uz \leq vz$, proving that K is strongly left annihilating. Define a mapping $f:\ Ss \cup St \to S$ by $f(us) = ux$ and $f(ut) = uy$ for all $u \in S$. It is straightforward to check that f is a homomorphism of left S-posets. Now $[f(s)]_{\rho_K} \leq [f(t)]_{\rho_K}$, but it does not imply $f(s) \leq f(t)$, so K is not D-strongly left annihilating.

Lemma 3.7 *([1, Propositions 10 and 13]) Let K be a convex, proper right ideal of a pomonoid S. Then:*

(1) *S/K is principally weakly po-flat if and only if K is strongly left stabilizing.*

(2) *S/K is weakly po-flat if and only if S is weakly right reversible and K is strongly left stabilizing.*

Lemma 3.8 *([6, Theorem 4.5, Corollary 5.7]) Let K be a convex, proper right ideal of a pomonoid S. Then:*

(1) *S/K satisfies Condition (PWP) if and only if K is strongly left stabilizing and strongly left annihilating.*

(2) *S/K satisfies Condition (WP) if and only if S is weakly right reversible, and K is strongly left stabilizing and D-strongly left annihilating.*

For Rees factor S-posets satisfying Condition $(PWP)_w$, we can give the following description.

Definition 3.9 A convex, proper right ideal K of a pomonoid S is called *w-strongly left annihilating,* if $[x]_{\rho_K}t \leq [y]_{\rho_K}t$ for any $x, y \in S\backslash K$ and $t \in S$, there exist $u, v \in S$, and $k, k', l, l' \in K$ such that one of the following four conditions is satisfied:

(a) *$x \leq u, v \leq y$, and $ut \leq vt$;*

(b) *$x \leq u, v \leq l, l' \leq y$, and $ut \leq vt$;*

(c) *$x \leq k, k' \leq u, v \leq y$, and $ut \leq vt$;*

(d) *$x \leq k, k' \leq u, v \leq l, l' \leq y$, and $ut \leq vt$.*

By the definition, every strongly left annihilating convex, proper right ideal of a pomonoid S is w-strongly left annihilating, but the converse is not true by the following Example 3.11.

Theorem 3.10 *Let K be a convex, proper right ideal of a pomonoid S. Then S/K satisfies Condition $(PWP)_w$ if and only if*

(1) *K is strongly left stabilizing, and*

(2) *K is w-strongly left annihilating.*

Proof **Necessity**: Suppose that S/K satisfies Condition $(PWP)_w$. Then S/K is principally weakly po-flat, so by Lemma 3.7, we have (1).

To prove (2) we suppose that $[x]_{\rho_K}t \leq [y]_{\rho_K}t$ for $x, y \in S\backslash K$ and $t \in S$. Since S/K satisfies Condition $(PWP)_w$, by Proposition 3.1, there exist $u, v \in S$ such that $x\widehat{\rho_K}u, v\widehat{\rho_K}y$ and $ut \leq vt$. Thus, we have $[x]_{\rho_K} \leq [u]_{\rho_K}$ and $[v]_{\rho_K} \leq [y]_{\rho_K}$. By Lemma 3.3, $[x]_{\rho_K} \leq [u]_{\rho_K}$ implies $x \leq u$, or $x \leq k$ and $k' \leq u$ for $k, k' \in K$. Similarly, $[v]_{\rho_K} \leq [y]_{\rho_K}$ implies $v \leq y$, or $v \leq l$ and $l' \leq y$ for $l, l' \in K$. Hence, we get the four possible cases of Definition 3.9, and this implies that K is w-strongly left annihilating.

Sufficiency: Assume (1) and (2) hold. To show that S/K satisfies Condition $(PWP)_w$, where K is a convex, proper right ideal of the pomonoid S, it suffices to show that S/K satisfies the conditions of Proposition 3.1. Now we suppose that $[x]_{\rho_K}t \leq [y]_{\rho_K}t$ for $x, y, t \in S$. Then $[xt]_{\rho_K} \leq [yt]_{\rho_K}$. By Lemma 3.3, we have $xt \leq yt$, or $xt \leq k$ and $k' \leq yt$ for $k, k' \in K$. If $xt \leq yt$, then it suffices in Proposition 3.1 to take $u = x, y = v$. Otherwise, there are the following four cases:

Case 1. $x, y \in K$. We can take $u = v = x$.

Case 2. $x \in K$, $y \notin K$. Since $k' \leq yt$, by assumption (1) there exists $k'' \in K$ such that $k''yt \leq yt$, and so it suffices in Proposition 3.1 to take $u = k''y$ and $v = y$.

Case 3. $x \notin K$, $y \in K$. This is analogous to Case 2.

Case 4. $x, y \notin K$. By (2) of the assumption, there exist $u, v \in S$ and $k, k', l, l' \in K$ such that one of the conditions of Definition 3.9 holds. However, in any condition, we always have $x\widehat{\rho_K}u$, $v\widehat{\rho_K}y$, and $xt \leq yt$. □

The following example illustrates that Condition $(PWP)_w$ does not imply Condition (PWP).

Example 3.11 ($(\mathrm{PWP})_\mathrm{w} \not\Rightarrow$ (PWP)) Let $S = \{1, e, f, 0\}$ denote the monoid with the Cayley table

	1	e	f	0
1	1	e	f	0
e	e	e	0	0
f	f	0	f	0
0	0	0	0	0

and suppose that the only nontrivial order relations are $e < 1$ and $0 < f$. We consider the ideal $K_S = \{e, 0\}$. Then $(S, \leq)$ is a pomonoid, and K is a strongly left stabilizing and w-strongly left annihilating convex, proper right ideal. It follows from Theorem 3.10 that S/K satisfies Condition $(PWP)_w$. On the other hand, since $1, f \in S\backslash K$ and $[1]e \leq [f]e$, but $1e \not\leq fe$. Hence, K is not strongly left annihilating. It follows from Lemma 3.8 that S/K does not satisfy Condition (PWP).

In what follows, we give the homological classification of pomonoids S over which all Rees factor S-posets satisfying Condition $(PWP)_w$ have a certain flatness property. To do this, we require the following results.

Lemma 3.12 *([1, Theorem 1]) Let S be any pomonoid. Then:*

(1) *Θ_S satisfies Condition (E) if and only if S is left collapsible.*

(2) *Θ_S satisfies Condition (E') if and only if S is weakly left collapsible.*

(3) *The following statements are equivalent:*

  (a) *Θ_S satisfies Condition (P);*

  (b) *Θ_S satisfies Condition (WP) (see [6, Corollary 5.4]);*

  (c) *Θ_S is weakly (po-)flat;*

  (d) *S is weakly right reversible.*

(4) *Θ_S is (always) principally weakly (po-) flat and (po-) torsion free.*

Lemma 3.13 *([11, Lemma 1.8]) Let K be a convex, proper right ideal of a pomonoid S. Then the following statements are equivalent:*

(1) *S/K is strongly flat;*

(2) *S/K satisfies Condition (P);*

(3) *$|K| = 1$.*

Theorem 3.14 *For any pomonoid S, the following statements are equivalent:*

(1) *S/K satisfying Condition $(PWP)_w$ is weakly po-flat;*

(2) *S/K satisfying Condition $(PWP)_w$ is weakly flat;*

(3) *S is weakly right reversible.*

Proof (1) $\Rightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (3). Since Θ_S always satisfies Condition $(PWP)_w$ and, by assumption, Θ_S is weakly flat, it follows from Lemma 3.12 that S is weakly right reversible.

(3) $\Rightarrow$ (1). Suppose that K is a convex right ideal of a pomonoid S and S/K satisfies Condition $(PWP)_w$. If K is a proper, convex right ideal, using Theorem 3.10, K is a strongly left stabilizing convex, proper right ideal, since S is weakly right reversible and by Lemma 3.7, S/K is weakly po-flat. However, if $K = S$ and S is weakly right reversible, then by Lemma 3.12, $S/K \cong \Theta_S$ is weakly po-flat. □

Note that Condition $(PWP)_w$ and weakly po-flat are independent notions. Indeed, on the one hand, if a pomonoid S is not weakly right reversible, then by Theorem 3.14, there exists a Rees factor S-poset S/K satisfying Condition $(PWP)_w$ that is not weakly po-flat. Therefore, Condition $(PWP)_w$ does not imply weakly po-flat in general. On the other hand, by [6, Example 6.3], there exists a weakly po-flat Rees factor S-poset that fails to satisfy Condition $(PWP)_w$.

Theorem 3.15 *For any pomonoid S, the following statements are equivalent:*

(1) *S/K satisfying Condition $(PWP)_w$ satisfies Condition (WP);*

(2) *S is weakly right reversible, and every strongly left stabilizing and w-strongly left annihilating convex, proper right ideal K of S is D-strongly left annihilating.*

Proof (1) $\Rightarrow$ (2). Since Θ_S satisfies Condition $(PWP)_w$ and by assumption, Θ_S satisfies Condition (WP), from Lemma 3.8, it follows that S is weakly right reversible. Let K be a strongly left stabilizing and w-strongly left annihilating convex, proper right ideal. From Theorem 3.10, it follows that S/K satisfies Condition $(PWP)_w$. By assumption, S/K satisfies Condition (WP) and so by Lemma 3.8, K is D-strongly left annihilating.

(2) $\Rightarrow$ (1). Let K be a convex right ideal of the pomonoid S and S/K satisfies Condition $(PWP)_w$. If K is a convex, proper right ideal, then by Theorem 3.10, K is a strongly left stabilizing and w-strongly left annihilating convex, proper right ideal, and so by assumption, K is a D-strongly left annihilating right ideal. Since S is weakly right reversible, from Lemma 3.8, it follows that S/K satisfies Condition (WP). However, if $K = S$ and S is weakly right reversible, then by Lemma 3.12, $S/K \cong \Theta_S$ satisfies Condition (WP). □

Applying Lemma 3.8 and Theorem 3.10, we can get:

Theorem 3.16 *For any pomonoid S, the following statements are equivalent:*

(1) *S/K satisfying Condition $(PWP)_w$ satisfies Condition (PWP);*

(2) *Every strongly left stabilizing and w-strongly left annihilating convex, proper right ideal K of S is strongly left annihilating.*

Theorem 3.17 *For any pomonoid S, the following statements are equivalent:*

(1) *S/K satisfying Condition $(PWP)_w$ satisfies Condition (P);*

(2) *S is weakly right reversible, and S has no strongly left stabilizing and w-strongly left annihilating convex, proper right ideal K with $|K| > 1$.*

Proof (1) $\Rightarrow$ (2). Since Θ_S satisfies Condition $(PWP)_w$, by assumption, Θ_S satisfies Condition (P). From Lemma 3.12, we obtain that S is weakly right reversible. Assume S has a strongly left stabilizing and w-strongly left annihilating convex, proper right ideal K with $|K| > 1$. From Theorem 3.10 it follows that S/K satisfies Condition $(PWP)_w$. By assumption, S/K satisfies Condition (P), and so by Lemma 3.13, $|K| = 1$, a contradiction is obtained.

(2) $\Rightarrow$ (1). Let K be a convex right ideal of the pomonoid S. Suppose that S/K satisfies Condition $(PWP)_w$. If K is a convex, proper right ideal of S, it follows from Theorem 3.10 that K is a strongly left stabilizing and w-strongly left annihilating convex, proper right ideal. By assumption, $|K| = 1$, and so S/K satisfies Condition (P). However, if $K = S$, $S/K \cong \Theta_S$ satisfies Condition $(PWP)_w$. Since S is weakly right reversible, by Lemma 3.12, Θ_S satisfies Condition (P). □

The following example shows that Condition $(PWP)_w$ does not imply Condition (P).

Example 3.18 *([11, Example 3.22])* Let S be a left zero semigroup K with 1 adjoined and $|K| > 1$. The order of S is discrete. It is easy to verify that K is strongly left stabilizing and w-strongly left annihilating. It follows from Theorem 3.10 that S/K satisfies Condition $(PWP)_w$. However, by Theorem 3.17, S/K does not satisfy Condition (P).

In what follows we will use the following.

Theorem 3.19 *Let K be a convex right ideal of a pomonoid S. The right Rees factor S-poset S/K is weakly subpullback flat if and only if $K = S$ is weakly right reversible and weakly left collapsible, or $|K| = 1$.*

Proof **Necessity**: If $K = S$ and $S/K \cong \Theta_S$ is weakly subpullback flat, then Θ_S satisfies Conditions (P) and (E'). From Lemma 3.12, we obtain that S is weakly right reversible and weakly left collapsible. Assume S has a convex, proper right ideal K and S/K is weakly subpullback flat. Then S/K satisfies Condition (P), and so by lemma 3.13, $|K| = 1$.

Sufficiency: If K is a convex, proper right ideal of the pomonoid S, then by assumption, we have $|K| = 1$ and $S/K \cong S$ is strongly flat, and it is clear that S/K is weakly subpullback flat. However, if $K = S$ is weakly right reversible and weakly left collapsible, then by Lemma 3.12, $S/K \cong \Theta_S$ satisfies Conditions (P) and (E'). Hence, Θ_S is weakly subpullback flat. □

Theorem 3.20 *For any pomonoid S, the following statements are equivalent:*

(1) *S/K satisfying Condition $(PWP)_w$ is weakly subpullback flat;*

(2) *S is weakly right reversible and weakly left collapsible, and S has no strongly left stabilizing and w-strongly left annihilating convex, proper right ideal K with $|K| > 1$.*

Proof (1) $\Rightarrow$ (2). Since Θ_S satisfies Condition $(PWP)_w$, by assumption, Θ_S is weakly subpullback flat. Applying Theorem 3.19, we obtain that S is weakly right reversible and weakly left collapsible. Assume S has a strongly left stabilizing and w-strongly left annihilating convex, proper right ideal K with $|K| > 1$. From Theorem 3.10, it follows that S/K satisfies Condition $(PWP)_w$. By assumption, S/K is weakly subpullback flat, and so by Theorem 3.19, $|K| = 1$, a contradiction.

$(2) \Rightarrow (1)$. Suppose that K is a convex right ideal of the pomonoid S and S/K satisfies Condition $(PWP)_w$. If K is a convex, proper right ideal of S, by Theorem 3.10, K is a strongly left stabilizing and w-strongly left annihilating convex, proper right ideal. By assumption, $|K| = 1$, and so $S/K \cong S$ is strongly flat. Clearly, S/K is weakly subpullback flat. However, if $K = S$, $S/K \cong \Theta_S$ satisfies Condition $(PWP)_w$. Since S is weakly right reversible and weakly left collapsible, by Lemma 3.12, Θ_S satisfies Conditions (P) and (E'). Hence, Θ_S is weakly subpullback flat. □

Theorem 3.21 *For any pomonoid S, the following statements are equivalent:*

(1) *S/K satisfying Condition $(PWP)_w$ is strongly flat;*

(2) *S is left collapsible, and S has no strongly left stabilizing and w-strongly left annihilating convex, proper right ideal K with $|K| > 1$.*

Proof $(1) \Rightarrow (2)$. Since Θ_S satisfies Condition $(PWP)_w$ and by assumption, Θ_S is strongly flat, thus Θ_S satisfies Condition (E). Using Lemma 3.12, S is left collapsible. Assume S has a strongly left stabilizing and w-strongly left annihilating convex, proper right ideal K with $|K| > 1$. By Theorem 3.10, S/K satisfies Condition $(PWP)_w$, so by assumption, S/K is strongly flat, and by Lemma 3.13, $|K| = 1$, a contradiction is obtained.

$(2) \Rightarrow (1)$. Let K be a convex right ideal of the pomonoid S and S/K satisfies Condition $(PWP)_w$. If K a is convex, proper right ideal, by Theorem 3.10, K is a strongly left stabilizing and w-strongly left annihilating convex, proper right ideal. By assumption, $|K| = 1$, and so $S/K \cong S$ is strongly flat. However, if $K = S$, $S/K \cong \Theta_S$ satisfies Condition $(PWP)_w$. Since S is left collapsible, by Lemma 3.12, Θ_S satisfies Condition (E). Hence, Θ_S is strongly flat. □

Theorem 3.22 *For any pomonoid S, the following statements are equivalent:*

(1) *S/K satisfying Condition $(PWP)_w$ is projective;*

(2) *S has a left zero element, and S has no strongly left stabilizing and w-strongly left annihilating convex, proper right ideal K with $|K| > 1$.*

Proof It is similar to that of Theorem 3.21. □

Theorem 3.23 *For any pomonoid S, the following statements are equivalent:*

(1) *S/K satisfying Condition $(PWP)_w$ is free;*

(2) $|S| = 1$.

Proof It can be easily proved. □

Example 3.24 Let S be a pogroup and $|S| > 1$. Then the Rees factor S-poset Θ_S satisfies Condition $(PWP)_w$, but, by Theorem 3.23, Θ_S is not free.

4. Direct products of S-posets satisfying Condition $(PWP)_w$

In this section, we are going to discuss direct products of any arbitrary nonempty family of S-posets satisfying Condition $(PWP)_w$.

If S is a pomonoid, the Cartesian product S^I is a right and left S-poset equipped with the order and the action componentwise where I is a nonempty set. Moreover, $(s_i)_{i\in I} \in S^I$ is denoted simply by (s_i), and the right S-poset $S\times S$ is called the *diagonal right S-poset of S*, usually denoted $D(S)$. (For more information the reader is referred to [7]).

According to [7], the set $L(s,s) := \{(u,v)\in D(S)|\ us \le vs\}$ is a left S-subposet of $D(S)$. Moreover, for each $(p,q)\in D(S)$, the set $\widehat{S(p,q)} := \{(u,v)\in D(S)|\ u\le wp \text{ and } wq\le v \text{ for some } w\in S\}$ is a left S-poset. Clearly, $\widehat{S(p,q)}$ contains the cyclic S-poset $S(p,q)$.

Theorem 4.1 *Let S be a pomonoid. Then the following statements are equivalent:*

(1) *Any finite product of right S-posets satisfying Condition $(PWP)_w$ satisfies Condition $(PWP)_w$;*

(2) *The diagonal right S-poset $D(S)$ satisfies Condition $(PWP)_w$;*

(3) *For every $s\in S$, the set $L(s,s)$ is either empty or for each 2 elements (u,v), $(u',v')\in L(s,s)$, there exists $(p,q)\in L(s,s)$ such that (u,v), $(u',v')\in \widehat{S(p,q)}$.*

Proof $(1)\Rightarrow(2)$ It is obvious.

$(2)\Rightarrow(3)$. Suppose that $D(S)$ satisfies Condition $(PWP)_w$. Let (u,v), $(u',v')\in L(s,s)$ for any $s\in S$. From the inequalities $us\le vs$ and $u's\le v's$ we obtain $(u,u')s\le (v,v')s$. Since $D(S)$ satisfies Condition $(PWP)_w$, there exist $(w,w')\in D(S)$ and p, $q\in S$ such that $(u,u')\le (w,w')p$, $(w,w')q\le (v,v')$, and $ps\le qs$. Thus, we have $(p,q)\in L(s,s)$ and we are done.

$(3)\Rightarrow(1)$. Suppose that $A_1,\dots,A_n$ are right S-posets satisfying Condition $(PWP)_w$. Suppose $a_i, a_i'\in A_i$ for each $1\le i\le n$, and let $s\in S$ be such that $(a_1,\dots,a_n)s\le (a_1',\dots,a_n')s$ in $A=\prod_{i=1}^n A_i$. For every A_i, applying Condition $(PWP)_w$ to the inequalities $a_is\le a_i's$ $(1\le i\le n)$, we get $a_i''\in A_i$ and $p_i,q_i\in S$ such that $a_i\le a_i''p_i$, $a_i''q_i\le a_i'$ and $p_is\le q_is$. Then $(p_i,q_i)\in L(s,s)$ for each i, and so by assumption, there exists $(p,q)\in L(s,s)$ such that $(p_i,q_i)\in L(p,q)$ for each i. Thus, $p_i\le w_ip$ and $w_iq\le q_i$ for some $w_i\in S$ $(1\le i\le n)$. Thus, we calculate that $(a_1,\dots,a_n)\le (a_1''w_1,\dots,a_n''w_n)p$, $(a_1''w_1,\dots,a_n''w_n)q\le (a_1',\dots,a_n')$, and $ps\le qs$, proving that $A=\prod_{i=1}^n A_i$ satisfies Condition $(PWP)_w$. □

For a right po-cancellable pomonoid, Theorem 4.1 yields the following.

Corollary 4.2 *If the pomonoid S is right po-cancellative, then the diagonal right S-poset $D(S)$ satisfies Condition $(PWP)_w$.*

As an extension of Theorem 4.1, the following result is obtained.

Theorem 4.3 *Let S be a pomonoid. Then the following statements are equivalent:*

(1) *The direct product of every nonempty family of right S-posets satisfying Condition $(PWP)_w$ satisfies Condition $(PWP)_w$;*

(2) $(S^I)_S$ *satisfies Condition* $(PWP)_w$ *for every nonempty set* I*;*

(3) *For every* $s \in S$*, the set* $L(s,s)$ *is either empty or there exists* $(p,q) \in L(s,s)$ *such that* $L(s,s) = \widehat{S(p,q)}$*.*

Proof (1) $\Rightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (3). Let $s \in S$ and $L(s,s) \neq \emptyset$. Write $L(s,s) = \{(u_i, v_i) | \ i \in I\}$. Let u and v be the elements of S^I whose ith components are u_i and v_i, respectively. Then we get $us \leq vs$ in S^I. Since S^I satisfies Condition $(PWP)_w$, we have that $u \leq wp$, $wq \leq v$ and $ps \leq qs$ for some p, $q \in S$ and $w \in S^I$. Then $(p,q) \in L(s,s)$, and for each $i \in I$ we have $u_i \leq w_i p$, $w_i q \leq v_i$ where w_i is the ith component of w. Thus, we have $L(s,s) = \widehat{S(p,q)}$, as desired.

(3) $\Rightarrow$ (1). Let $A = \prod_{j \in J} A_j$ be a direct product of right S-posets satisfying Condition $(PWP)_w$. Suppose that $s \in S$, and $a = (a_j)$, $b = (b_j) \in A$ are such that $as \leq bs$. Then we have $a_j s \leq b_j s$ for each $j \in J$. Since A_j satisfies Condition $(PWP)_w$, there are elements $u_j, v_j \in S$ and $c_j \in A_j$ with $a_j \leq c_j u_j$, $c_j v_j \leq b_j$, and $u_j s \leq v_j s$. Therefore, $(u_j, v_j) \in L(s,s) \neq \emptyset$ and by assumption there exists $(p,q) \in L(s,s)$ such that $L(s,s) = \widehat{S(p,q)}$. Then for each $(u_j, v_j) \in L(s,s)$ there exists $w_j \in S$ with $u_j \leq w_j p$ and $w_j q \leq v_j$. Thus, $ps \leq qs$, and for each $j \in J$ we can calculate that $a_j \leq c_j w_j p$ and $c_j w_j q \leq b_j$. Taking $a' = (c_j w_j)_{j \in J} \in A$, we have $a \leq a'p$ and $a'q \leq b$, as required. □

Note that the fact that not every pomonoid S has a diagonal S-poset $D(S)$ satisfying Condition $(PWP)_w$ is shown by the following example.

Example 4.4 Let $S = \{0, x, 1 | \ x^2 = 0\}$ be a monoid with the nontrivial order relations $0 < x < 1$. Then S is a pomonoid, and the diagonal S-poset $D(S)$ does not satisfy Condition $(PWP)_w$.

Proof It is clear that S is a pomonoid. We use Theorem 4.1 to check that $D(S)$ fails to satisfy Condition $(PWP)_w$. Note that $(1,1)$, $(x,0) \in L(x,x)$ for $x \in S$. However, there is no element $(p,q) \in L(x,x)$ such that $(1,1)$, $(x,0) \in \widehat{S(p,q)}$. □

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