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Optimality criteria for sum of fractional multiobjective optimization problem with generalized invexity

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Abstract: The sum of a fractional program is a nonconvex optimization problem in the field of fractional programming and it is difficult to solve. The development of research is restricted to single objective sums of fractional problems only. The branch and bound methods/algorithms are developed in the literature for this problem as a single objective problem. The theoretical and algorithmic development for sums of fractional programming problems is restricted to single objective problems. In this paper, some new optimality conditions are proposed for the sum of a fractional multiobjective optimization problem with generalized invexity. The optimality conditions are obtained by using a modified objective approach and equivalency with the original problem is established.

Key words: Multiobjective programming, sum of ratio, multiobjective linear fractional programming, optimality and duality, saddle point criteria

1. Introduction

Optimization of the ratio of two functions is called a fractional programming (ratio optimization) problem. If collections of fractional objective functions are optimized simultaneously, then the problem is called multiobjective fractional programming. The general fractional programming is defined as:

$$\begin{aligned} &\text{Optimize} \quad F(x) = \frac{f(x)}{g(x)} \\ &\text{subject to} \\ &x \in S = \{x \in \mathbb{R}^n : h_l(x) \leq 0, \quad x \geq 0, \quad g(x) > 0, \quad l = 1,2,3\ldots m\}. \end{aligned} \tag{1}$$

Fractional programming is classified according to the nature of the functions involved in the ratio. If the ratio functions and constraints are linear, then the fractional programming problem is called linear fractional programming [18]. If any ratio function or any constraint is nonlinear, then the fractional programming problem is known as nonlinear fractional programming.

If the objective function is in the form of the sum of fractional functions, then the fractional is known as a sum of fractional programming (SOFP) problem according to Shaible [24].

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A general SOFP problem is defined in the following way:

$$\begin{aligned}
&\text{Min} \quad F(x) = \text{Minimize} \left\{ \sum_{j=1}^{p} \frac{f_j(x)}{g_j(x)} \right\} \\
&\text{subject to} \\
&x \in S = \{x \in \mathbb{R}^n : h_l(x) \leq 0, \quad x \geq 0, \quad l = 1, 2, 3 \ldots m\}, \qquad (2) \\
&\text{where} \quad p \geq 2,
\end{aligned}$$

where each $h_l(x)$ is a constraint function that constructs the feasible region S. The functions $f_j(x) \geq 0$ and $g_j(x) > 0$ are differentiable functions in feasible region S.

The sum of a fractional program is a difficult nonconvex optimization problem with many applications in managerial sciences. Such problems have been discussed by several authors at various levels of generality. Benson [6] presented an algorithm for solving the sum of ratios fractional programming problem. The branch-and-bound search method is used to solve an equivalent concave minimization problem. The main work of the algorithm is the involvement of a sequence of convex programming problems. Kanno [17] proposed a branch-and-bound algorithm and an integer programming algorithm for a fractional programming problem that are the sum of absolute values of affine functions. Chen et al. [8] proposed various versions of Dinkelbach-type algorithms for solving the generalized fractional programming problem. The convergence analysis of all the algorithms through geometric observations was discussed and fundamental properties of convex functions were given. Benson [7] gave a branch-and-bound outer approximation algorithm for globally solving a sum of ratios fractional programming problem. The main work of the algorithm involved solving a sequence of convex programming problems whose feasible regions were identical to each other except for a certain linear constraint. Hackman and Passy [11] proposed a finite pivoting-type algorithm for a sum of linear ratios problem that solves the maximization problem while computing simultaneously the efficient frontier. Application to multistage efficiency analysis was discussed. Benson [3] gave a method for the construction of test problems for the problem of minimization over a convex set of sums of ratios of linear fractional functions. The method determined a function that is the sum of linear fractional functions and attains a global minimum over the set at a point that is found by convex programming and univariate search. Benson [4] showed that using a suitable transformation, several potential and known methods for globally solving these problems become available and further proposed that these methods are accessible and useful in comparison of customized algorithms. Scott and Jefferson [25] derived a duality theory for sum of ratios of the linear fractional and established their connection with geometric programming. Benson [2] gave a branch-and-bound method for solving the nonlinear sum of ratios problem globally. The steps of the algorithm economize the required computations by conducting the branch-and-bound search in R^p. The algorithm is implemented using a sequence of convex programming problems for which standard algorithms are available. Benson [1] presented an algorithm to find the global optimal solution of the nonlinear sum of ratios problem. The branch-and-bound procedure is used to develop the proposed algorithm. Shen and Lin [28] proposed a global optimization algorithm for maximization of the sum of concave-convex ratios within the convex feasible region. The branch-and-bound scheme is used to develop the algorithm. Wang and Zhang [33] presented a branch-and-bound algorithm for globally solving the nonlinear sum of ratios problem on a nonconvex feasible region. It is claimed that the proposed algorithm is convergent to the global minimum through the refinement of the solution of series of linear programming problems. Shen and Wang [31] proposed a branch-and-bound algorithm for solving the sum of linear ratios problem. It was proved that the proposed algorithm is convergent to the global optimal solution. Jiao and Shen [14] gave a short extension of the work of Wang and Zhang for the nonlinear sum of ratios problem. More general results are proposed by using different equivalent problems. Qu et al. [22] proposed a new branch-and-bound algorithm based on rectangle partition and the Lagrangian relaxation for solving the sum of quadratic ratios problem with nonconvex quadratic convergence of the algorithm. Shen et al. [26] proposed a branch-reduced-bound algorithm for solving sum of quadratic ratios with nonconvex constraints. The problem is modified as a monotonic optimization problem and finds the globally optimal solution. Shen et al. [29] gave a method to solve the problem of minimization of sum of convex-convex ratios problem within a feasible region. It was also established that the method is globally convergent. Shen and Wang [30] gave an algorithm for the sum of general fractional functions using linearization of the method and the branch-bound method. Shen et al. [27] proposed a branch-bound algorithm to solve the sum of convex-convex ratios with non-convex constraints. The branch-bound method and Lagrange duality were used to develop the proposed scheme. Jaberipour and Khorram [13] proposed a harmony search algorithm for solving a sum of ratios problem. They claimed that the solutions obtained using their algorithm were better than those of the other proposed methods in the literature.

Benson [5] presented a branch-and-bound algorithm to solve the sum of ratios problem. To develop this algorithm, Lagrangian duality theory was used and it was claimed that the proposed algorithm is better. Recently, Li and Hou [15] proposed a branch-and-bound algorithm to find the global optimum solution for the sum of ratios problem with the ratio of the absolute value of affine functions with coefficients. To develop this algorithm, rectangular partition and space of small dimension were used. Gao and Jin [10] proposed a branch-and-bound algorithm by transformation of sum ratios programming problem into a bilinear programming problem. This algorithm was developed by using three functions, the linear characteristic of convex envelope, concave envelope of double variables, and linear relaxation programming of the bilinear programming. Freund and Jarre [9] proposed an algorithm for the global minimum solution for the sum of a quasiconvex ratio. The interior point method was used to compute the global minimum solution. Singh and Gupta [32] studied the multiparametric sensitivity analysis for the ratios of linear functions. Shaible and Shi [23] published a review of the sum-of-ratios program. This survey contained the applications, theoretical results, and various algorithmic approaches of this difficult fractional programming. Due to the difficult problem, all the methods/algorithms available in the literature were developed for single objective problems only. The theoretical research developments in the direction of multiobjective optimization theory for the sum of fractional programming problem are fewer in comparison with single objectives. Thus, in this paper we develop optimality conditions that are parallel to those of Kim [18].

Consider the following sum of the fractional multiobjective programming problem:

$$\begin{aligned}
&\text{Minimize}\left\{\sum_{j}^{p}\frac{f_{1j}(x)}{g_{1j}(x)},\quad \sum_{j}^{p}\frac{f_{2j}(x)}{g_{2j}(x)},\ldots,\quad \sum_{j}^{p}\frac{f_{kj}(x)}{g_{kj}(x)}\right\},\\
&\text{subject to}\\
&x\in S=\{x\in\mathbb{R}^n : h_l(x)\le 0,\quad x\ge 0,\quad l=1,2,3\ldots,m\},\\
&\text{where}\ \ p,\,k\ge 2.
\end{aligned}\tag{3}$$

Each $h_l(x)$ is a constraint function that constructs the feasible region S. $f_{ij}(x)$ and $g_{ij}(x)$ and $h_l(x)$ are continuously differentiable over $S\subseteq R^n$, S denotes the set of all feasible solutions and $(f_{11},\, f_{12},\ldots,f_{kp})=$

$f : S \to R^k$ $(-g_{11}, -g_{12}, \ldots, -g_{kp}) = -g : S \to R^k$ are invex, and it is assumed that $f_{ij}(x) \geq 0$ and $g_{ij}(x) > 0 \,\forall\, x,\, y \in S,\, i = 1,\, 2 \ldots, k,\, j = 1,\, 2,\, \ldots, p$.
Throughout this article, the following notations are used in R^n:

$$x < y \Leftrightarrow x_r < y_r$$
$$x \leq y \Leftrightarrow x_r \leq y_r$$
$$\sum_{r=1}^{n} x_r . \sum_{r=1}^{n} y_r \Leftrightarrow \sum_{r=1}^{n} x_r y_r .$$

1.1. Optimality

The concept of Pareto optimality was given by Pareto in 1906 [21]. For solving a particular multiobjective fractional programming problem, the problem formulation serves as optimality conditions for Pareto optimality and it is central to the performance. However, these characterizations are slightly different from their meaning in single objective fractional programming. If the formulation provides a necessary condition, then for a point to be Pareto optimal, it must be a solution to that formulation. However, some solutions obtained using this method are not Pareto optimal. On the other hand, if a method provides a condition of sufficiency, then the solution is always Pareto optimal. A point is said to be weakly Pareto optimal if the feasible region has no point that improves all of the objectives simultaneously. In comparison, a point is Pareto optimal if any other point does not exist that improves at least the single objective function without determining another function. Pareto optimal points are weakly Pareto optimal but a weakly Pareto optimal point is not Pareto optimal.

Definition 1 *Pareto sptimal solution:A solution* $x^o \in S$ *is said to be a Pareto optimal (efficient) solution to the SOFMP(3) problem if and only if there is no other solution* $y \in S$ *such that* $\sum_{j=1}^{p} \dfrac{f_{ij}(y)}{g_{ij}(y)} \leq \sum_{j=1}^{p} \dfrac{f_{ij}(x^o)}{g_{ij}(x^o)}$ $\forall$ *i and j and* $\sum_{j=1}^{p} \dfrac{f_{lj}(y)}{g_{lj}(y)} < \sum_{j=1}^{p} \dfrac{f_{lj}(x^o)}{g_{lj}(x^o)}$ *for at least one* l.

Definition 2 *Weak Pareto optimal solution: A solution point,* $x_0 \in S$, *is said to be weak Pareto optimal if and only if there does not exist another solution point,* $y \in S$, *such that* $\sum_{j=1}^{p} \dfrac{f_{ij}(y)}{g_{ij}(y)} < \sum_{j=1}^{p} \dfrac{f_{ij}(x^o)}{g_{ij}(x^o)}$ $\forall$ $i = 1,\, 2,\, \ldots,\, k$ *and* $j = 1,\, 2,\, \ldots,\, p$.

1.2. Generalized invexity

Generalized invexity plays an important role in the theory of multiobjective optimization. The name 'invex' was given by Craven in 1981 and stands for invariant convex. Henson [12] introduced the concept of invexity as a generalization of convexity for scalar constrained optimization problems. He proved that sufficiency of Kuhn–Tucker optimality conditions and weak duality hold with invexity instead of convexity.

Definition 3 *A differentiable function* $(f_{11}, f_{12}, \ldots, f_{kp}) = f : S \to \mathbb{R}^k$ *on a nonempty open set* $S \subseteq R^n$ *is said to be invex with respect to* η *at* $y \in S$ *if for all* $x \in S$, *there exists* $\eta : S \times S \to \mathbb{R}^n$ *for all* $i = 1,\, 2,\, \ldots,\, k,\quad j = 1,\, 2 \ldots,\, p$ *such that*

$$f_{ij}(x) - f_{ij}(y) \geq \nabla f_{ij}(y)\eta^T(x, y). \qquad \forall\, x,\, y \in S.$$

Definition 4 *The function f_{ij} is said to be strictly invex with respect to η, if there exists $\eta : S \times S \to \mathbb{R}^n$ such that for all $i = 1, 2, \ldots, k, \quad j = 1, 2 \ldots, p$ with $x \neq y$ such that*

$$f_{ij}(x) - f_{ij}(y) > \nabla f_{ij}(y)\eta^T(x, y). \qquad \forall\, x,\, y \in S. \tag{4}$$

Lemma 1 *If real valued functions $f_{ij}(x)$ and $-g_{ij}(x)$ are invex with respect to the same $\eta(x, y)$ then $\dfrac{f_{ij}(x)}{g_{ij}(x)}$ is invex function with respect to $\overline{\eta}(x, y) = \dfrac{g_{ij}(y)}{g_{ij}(x)}\eta(x, y)$.*

Proof

$$\frac{f_{ij}(x)}{g_{ij}(x)} - \frac{f_{ij}(y)}{g_{ij}(y)} = \frac{f_{ij}(x) - f_{ij}(y)}{g_{ij}(x)} - \frac{f_{ij}(y)\left[g_{ij}(x) - g_{ij}(y)\right]}{g_{ij}(x)g_{ij}(y)} \geq \frac{g_{ij}(y)}{g_{ij}(x)}\left[\frac{\nabla f_{ij}(y)}{g_{ij}(y)} - \frac{f_{ij}(y)\nabla g_{ij}(y)}{[g_{ij}(y)]^2}\right]\eta(x, y)$$
$$= \frac{g_{ij}(y)}{g_{ij}(x)}\nabla\left(\frac{f_{ij}(y)}{g_{ij}(y)}\right)\eta(x, y).$$
$$= \nabla\left(\frac{f_{ij}(y)}{g_{ij}(y)}\right)\eta(x, y).$$

Therefore, each ratio $\dfrac{f_{ij}(x)}{g_{ij}(x)}$ is an invex function with respect to $\widetilde{\eta}(x, y) = \dfrac{g_{ij}(y)}{g_{ij}(x)}\eta(x, y) \qquad \forall\ i = 1, 2, \ldots, k,$ $j = 1, 2, \ldots, p$. □

Lemma 2 *If real valued functions $f_{ij}(x)$ and $-g_{ij}(x)$ are invex with respect to the same $\eta(x, y)$, then $\sum_{j=1}^{p} \dfrac{f_{ij}(x)}{g_{ij}(x)}$ is an invex function with respect to $\widetilde{\eta}(x, y) = \sum_{j=1}^{p} \dfrac{g_{ij}(y)}{g_{ij}(x)}\eta(x, y)$.*

Proof

$$\sum_{j=1}^{p}\frac{f_{ij}(x)}{g_{ij}(x)} - \sum_{j=1}^{p}\frac{f_{ij}(y)}{g_{ij}(y)} = \sum_{j=1}^{p}\left(\frac{f_{ij}(x)}{g_{ij}(x)} - \frac{f_{ij}(y)}{g_{ij}(y)}\right) = \sum_{j=1}^{p}\left[\frac{f_{ij}(x) - f_{ij}(y)}{g_{ij}(x)} - \frac{f_{ij}(y)\left[g_{ij}(x) - g_{ij}(y)\right]}{g_{ij}(x)g_{ij}(y)}\right]$$
$$\geq \sum_{j=1}^{p}\frac{g_{ij}(y)}{g_{ij}(x)}\left[\frac{\nabla f_{ij}(y)}{g_{ij}(y)} - \frac{f_{ij}(y)\nabla g_{ij}(y)}{[g_{ij}(y)]^2}\right]\eta(x, y)$$
$$= \sum_{j=1}^{p}\nabla\left(\frac{f_{ij}(y)}{g_{ij}(y)}\right)\frac{g_{ij}(y)}{g_{ij}(x)}\eta(x, y)$$
$$= \nabla\left(\sum_{j=1}^{p}\frac{f_{ij}(y)}{g_{ij}(y)}\right)\widetilde{\eta}(x, y).$$

Therefore, each sum of fraction $\sum_{j=1}^{p} \dfrac{f_{ij}(x)}{g_{ij}(x)}$ is an invex function with respect to $\widetilde{\eta}(x, y) = \sum_{j=1}^{p} \dfrac{g_{ij}(y)}{g_{ij}(x)}\eta(x, y)$ $\forall\ i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, p$. □

2. Equivalency of sum of fractional multiobjective programming

In this section, we will prove that the following sum of the fractional multiobjective program is equivalent to SOFMP(3).

$$\text{Minimize}\left(\widetilde{\eta}(x,y)\nabla\left(\sum_{j=1}^{p}\frac{f_{1j}(y)}{g_{1j}(y)}\right),\ \widetilde{\eta}(x,y)\nabla\left(\sum_{j=1}^{p}\frac{f_{2j}(y)}{g_{2j}(y)}\right),\ldots,\widetilde{\eta}(x,y)\nabla\left(\sum_{j=1}^{p}\frac{f_{kj}(y)}{g_{kj}(y)}\right)\right) \tag{5}$$

subject to

$h_l(x)\leq 0,\quad \text{and}\quad I(x)=\{l: h_l(x)=0\}\quad \forall\ x\in R^n\quad l\in R^m.$

Each $h_l(x)$ is a constraint function that constructs the feasible region S. $f_{ij}(x)$ and $g_{ij}(x)$ and $h_l(x)$ are continuously differentiable over $S\subseteq R^n$, S denotes the set of all feasible solutions and $(f_{11},\, f_{12},\ldots,f_{kp}) = f : S\to R^k$ $(-g_{11},\, -g_{12},\ldots,-g_{kp}) = -g : S\to R^k$ are invex, and it is assumed that $f_{ij}(x)\geq 0$ and $g_{ij}(x)>0\,\forall\ x,\, y\in S,\, i=1,\,2\ldots,k,\, j=1,\,2,\,\ldots,p$.

Theorem 1 *Let y be any weakly efficient solution of SOFMP(3) and assume that there exists $\alpha\in S$ such that $(\nabla h_l(y),\,\alpha)>0,\, l\in I(y),\, \nabla h_l(y),\, l\in I(y)$ are linearly independent. Further, we assume that h_l is strictly invex with respect to $\widetilde{\eta}$ at y on S and $\widetilde{\eta}(y,\,y)=0$. Then y is a weakly efficient solution of SOFMP(5).*

Proof Since y is any weakly efficient solution of SOFMP(3) and $\nabla h_l(y),\, l\in I(x)$ are linearly independent, KKT conditions are satisfied. Assume that y is not efficient for SOFMP(5). This implies that there exist y^* feasible for SOFMP(5) such that

$$\begin{aligned}&\left[\widetilde{\eta}(y^*,\,y)\nabla\left(\sum_{j=1}^{p}\frac{f_{1j}(y)}{g_{1j}(y)}\right),\ \widetilde{\eta}(y^*,\,y)\nabla\left(\sum_{j=1}^{p}\frac{f_{2j}(y)}{g_{2j}(y)}\right)\ldots,\widetilde{\eta}(y^*,\,y)\nabla\left(\sum_{j=1}^{p}\frac{f_{kj}(y)}{g_{kj}(y)}\right)\right]\\ &\leq\left[\widetilde{\eta}(y,\,y)\nabla\left(\sum_{j=1}^{p}\frac{f_{1j}(y)}{g_{1j}(y)}\right),\ \widetilde{\eta}(y,\,y)\nabla\left(\sum_{j=1}^{p}\frac{f_{2j}(y)}{g_{2j}(y)}\right)\ldots,\widetilde{\eta}(y,\,y)\nabla\left(\sum_{j=1}^{p}\frac{f_{kj}(y)}{g_{kj}(y)}\right)\right]\\ &=(0,\,0,\ldots,0).\end{aligned}$$

Since $\lambda_i\geq 0\quad\forall\ i$,

$$\sum_{i=1}^{k}\lambda_i\nabla\left(\sum_{j=1}^{p}\frac{f_{ij}(y)}{g_{ij}(y)}\right)\widetilde{\eta}(y^*,\,y)\leq 0. \tag{6}$$

It is given that y^* is feasible and $\mu_l\geq 0,\quad \mu_l^T h_l(y^*)\leq 0\quad\forall\quad l$. From this point of view $\quad\mu_l^T h_l(y^*)\leq\mu_l^T h_l(y)$. It is assumed that each $h_l\,\forall\, l$ is strictly invex with $\widetilde{\eta}$ at y.

$$\mu_l^T\nabla h_l(y)\widetilde{\eta}(y^*,\,y)<0. \tag{7}$$

From (6) and (7):

$$\widetilde{\eta}(y^*,y)\left[\sum_{i=1}^{k}\lambda_i\nabla\left(\sum_{j=1}^{p}\frac{f_{ij}(y)}{g_{ij}(y)}\right)\right]+\mu_l^T\nabla h_l(y)<0.$$

This inequality contradicts the fact that

$$\sum_{i=1}^{k} \lambda_i \nabla \left(\sum_{j=1}^{p} \frac{f_{ij}(y)}{g_{ij}(y)} \right) + \mu_l^T \nabla h_l(y) = 0.$$

This proves that y is a weakly efficient solution of SOFMP(5). □

Theorem 2 *If real valued functions $f_{ij}(x)$ and $-g_{ij}(x)$ are invex with respect to the same $\eta(x, y)$ $\forall\ i = 1, 2, \ldots, k,\quad j = 1, 2, \ldots, p,\quad x, y \in S$ then $\sum_{j=1}^{p} \dfrac{f_{ij}(x)}{g_{ij}(x)}$ is an invex function with respect to $\widetilde{\eta}(x, y)$ and $\widetilde{\eta}(y, y) = 0$ according to Kim [18]. If y is any weakly efficient solution of SOFMP(5), then y is also a weakly efficient solution of SOFMP(3).*

Proof Since f_{ij} and $-g_{ij}$ are invex with respect to the same $\eta(x, y)$ $\forall$ $x, y \in S$, $\sum_{j=1}^{p} \dfrac{f_{ij}}{g_{ij}}$, $\forall$ $i = 1, 2, \ldots, k,\quad j = 1, 2, \cdots, p$, is invex with respect to $\widetilde{\eta}(x, y)$ and $\widetilde{\eta}(y, y) = 0$. Assume that y is not efficient for SOFMP(3), and then there exists a feasible solution y^* of SOFMP(3) such that

$$\left(\sum_{j=1}^{p} \frac{f_{1j}(y^*)}{g_{1j}(y^*)}, \sum_{j=1}^{p} \frac{f_{2j}(y^*)}{g_{2j}(y^*)}, \ldots, \sum_{j=1}^{p} \frac{f_{kj}(y^*)}{g_{kj}(y^*)} \right) \leq \left(\sum_{j=1}^{p} \frac{f_{1j}(y)}{g_{1j}(y)}, \sum_{j=1}^{p} \frac{f_{2j}(y)}{g_{2j}(y)}, \ldots, \sum_{j=1}^{p} \frac{f_{kj}(y)}{g_{kj}(y)} \right).$$

As we know, $\sum_{j=1}^{p} \dfrac{f_{ij}}{g_{ij}}$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, p$ is invex with respect to $\widetilde{\eta}$ at y and $\widetilde{\eta}(y, y) = 0$. Then we have

$$\begin{aligned}
&\left(\widetilde{\eta}(y^*, y) \nabla \left(\sum_{j=1}^{p} \frac{f_{1j}(y)}{g_{1j}(y)} \right), \widetilde{\eta}(y^*, y) \nabla \left(\sum_{j=1}^{p} \frac{f_{2j}(y)}{g_{2j}(y)} \right), \ldots, \widetilde{\eta}(y^*, y) \nabla \left(\sum_{j=1}^{p} \frac{f_{kj}(y)}{g_{kj}(y)} \right) \right) \\
&\leq \left(\sum_{j=1}^{p} \frac{f_{1j}(y^*)}{g_{1j}(y^*)} - \sum_{j=1}^{p} \frac{f_{1j}(y)}{g_{1j}(y)}, \sum_{j=1}^{p} \frac{f_{2j}(y^*)}{g_{2j}(y^*)} - \sum_{j=1}^{p} \frac{f_{2j}(y)}{g_{2j}(y)}, \ldots, \sum_{j=1}^{p} \frac{f_{kj}(y^*)}{g_{kj}(y^*)} - \sum_{j=1}^{p} \frac{f_{kj}(y)}{g_{kj}(y)} \right) \\
&\leq (0, 0, \ldots, 0) \\
&= \left(\widetilde{\eta}(y, y) \nabla \left(\sum_{j=1}^{p} \frac{f_{1j}(y)}{g_{1j}(y)} \right), \widetilde{\eta}(y, y) \nabla \left(\sum_{j=1}^{p} \frac{f_{2j}(y)}{g_{2j}(y)} \right), \ldots, \widetilde{\eta}(y, y) \nabla \left(\sum_{j=1}^{p} \frac{f_{kj}(y)}{g_{kj}(y)} \right) \right),
\end{aligned}$$

which contradicts that y is a weakly efficient solution of SOFMP(5). □

3. Optimality conditions for sum of fractional multiobjective programming

In this section, the necessary works of John [16], Kuhn and Tucker [19], and Mishra [20] are introduced and conditions of sufficiency for the weakly efficient solution of the SOFMP with generalized invexity are established. A Lagrangian for the SOFMP is introduced and some theorems based on the saddle point are also established [18].

Theorem 3 *(Fritz John Necessary Conditions) [16]: If y is a weakly efficient solution of MOSFP(3), then there exist λ_i, $i = 1, 2, \ldots, k$ and μ_l, $l = 1, 2, \ldots, m$ such that*

$$\begin{aligned}
&\sum_{i=1}^{k} \lambda_i \nabla \left(\sum_{j=1}^{p} \frac{f_{ij}(y)}{g_{ij}(y)} \right) + \sum_{l=1}^{m} \mu_l \nabla h_l(y) = 0 \\
&\sum_{l=1}^{m} \mu_l h_l(y) = 0 \qquad (8) \\
&(\lambda_1, \lambda_2, \ldots, \lambda_k, \mu_1, \mu_2, \ldots, \mu_m) \geq 0.
\end{aligned}$$

Theorem 4 *(Harush–Kuhn–Tucker Necessary Conditions) [19]: Assume that there exist $\alpha \in S$ such that $(\nabla h_l(y), \alpha) > 0$, $l \in Q(y)$, where $Q(x) = \{l : h_l(x) = 0\}$ for any $x \in S$. If $y \in S$ is a weakly efficient solution of MOSFP(3), then there exists $\lambda_i \geq 0$, $i = 1, 2, \ldots, k$ and $\mu_l \geq 0$, $l = 1, 2, \ldots, m$ such that*

$$\begin{aligned}
&\sum_{i=1}^{k} \lambda_i \nabla \left(\sum_{j=1}^{p} \frac{f_{ij}(y)}{g_{ij}(y)} \right) + \sum_{l=1}^{m} \mu_l \nabla h_l(y) = 0 \\
&\sum_{l=1}^{m} \mu_l h_l(y) = 0 \qquad (9) \\
&(\lambda_1, \lambda_2, \ldots, \lambda_k, \mu_1, \mu_2, \ldots, \mu_m) \geq 0, \, (\lambda_1, \lambda_2, \ldots, \lambda_k) \neq (0, 0, \ldots, 0).
\end{aligned}$$

Theorem 5 *(Harush–Kuhn–Tucker Necessary Conditions) [19]: Assume that $\nabla h_l(y)$, $l \in Q(y)$ are linearly independent. If $y \in S$ is a weakly efficient solution of SOFMP(3), then there exists $\lambda_i \geq 0$, $i = 1, 2, \ldots, k$ and $\mu_l \geq 0$, $l = 1, 2, \ldots, m$ and they satisfy condition (9).*

Theorem 6 *(Harush–Kuhn–Tucker Necessary Conditions) [18]: If $(f_{1j}, f_{2j}, \ldots, f_{kj})$ and $(-g_{1j}, -g_{2j}, \ldots, -g_{kj})$ are invex with respect to the same $\eta(x, y)$ $\forall\ i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, p$, $x, y \in S$ and if $(h_1, h_2, \ldots, h_m)$ are invex with respect to $\widetilde{\eta}$ then suppose that (y, λ, μ) satisfy the KKT conditions (9). Then y is a weakly efficient solution of SOFMP(3).*

Proof Suppose y is not a weakly efficient solution of SOFMP(3). Then for $x \in S$,

$$\sum_{j=1}^{p} \frac{f_{ij}(x)}{g_{ij}(x)} < \sum_{j=1}^{p} \frac{f_{ij}(y)}{g_{ij}(y)} \qquad \forall\, i = 1, 2, \ldots, k, \qquad j = 1, 2, ..., p.$$

As we know, f_{ij} and $-g_{ij}$ $\forall$ $i = 1, 2, \ldots, k$, $j = 1, 2, ..., p$ are invex with respect to the same η. Then by Lemma 2,

$$\sum_{j=1}^{p} \frac{g_{ij}(y)}{g_{ij}(x)} \nabla \left(\frac{f_{ij}(y)}{g_{ij}(y)} \right) \widetilde{\eta}(x, y) < 0.$$

From $(\lambda_1, \lambda_2, \ldots, \lambda_k \geq 0)$,

$$\sum_{i=1}^{k} \lambda_i \nabla \left(\sum_{j=1}^{p} \frac{f_{ij}(y)}{g_{ij}(y)} \right) \widetilde{\eta}(x, y) < 0.$$

Using the KKT conditions,

$$\sum_{l=1}^{m} \mu_l \nabla h_l(y) \widetilde{\eta}(x,y) > 0. \tag{10}$$

Since $(h_1,\, h_2,\, \ldots,\, h_m)$ are invex with respect to $\widetilde{\eta}$,

$$\sum_{l=1}^{m} \mu_l h_l(x) - \sum_{l=1}^{m} \mu_l h_l(y) \geq \sum_{l=1}^{m} \mu_l \nabla h_l(y) \widetilde{\eta}(x,y).$$

Then

$$\sum_{l=1}^{m} \mu_l \nabla h_l(y) \widetilde{\eta}(x,y) \leq 0, \tag{11}$$

which is a contradiction of (10). □

4. Saddle point criteria

In this section, the η-Lagrangian for a multiobjective sum of fractional programming problem SOFMP(3) is defined and a new definition of a weak vector saddle point for the introduced η-Lagrangian function in a sum of fractional multiobjective programming is given.

Definition 5 *An η-Lagrange function is said to be a Lagrange function for the sum of fractional programming multiobjective programming.*

$$L_{\widetilde{\eta}}(x,\mu) = \left(\widetilde{\eta}(x,y) \nabla \left(\sum_{j}^{p} \frac{f_{1j}(y)}{g_{1j}(y)} \right) + \mu_1^T h_1(x), \ldots, \widetilde{\eta}(x,y) \nabla \left(\sum_{j}^{p} \frac{f_{lj}(y)}{g_{lj}(y)} \right) + \mu_l^T h_l(x) \right) \tag{12}$$

Definition 6 *A point $(y, \bar{\mu_l}) \in S \times R_+^m$ is said to be a weak vector saddle point for the $\widetilde{\eta}$-Lagrange function $L_{\widetilde{\eta}}$ if:*

1. $L_{\widetilde{\eta}}(y,\, \mu_l) \ngtr L_{\widetilde{\eta}}(y,\, \bar{\mu_l}) \qquad \forall\; \mu_l \in R_+^m,$

2. $L_{\widetilde{\eta}}(x,\, \bar{\mu_l}) \nless L_{\widetilde{\eta}}(y,\, \bar{\mu_l}) \qquad \forall\; x \in S.$

Theorem 7 *We assume that f_{ij} and g_{ij}, $\forall \quad i = 1, 2, ...k, \quad j = 1, 2, ..., p$, are invex with respect to the same η at y with $\eta(y,\, y) = 0$ and some constraint qualification (CQ) holds at y for SOFMP(5). If $(y, \bar{\mu_l})$ is a weak vector saddle point for $L_{\widetilde{\eta}}$, then y is a weakly efficient solution in SOFMP(3).*

Proof Suppose (y, μ_l) is a saddle point for the Lagrangian $L_{\widetilde{\eta}}$. Then by definition from $6(1)$, we have

$$\begin{aligned} & L_{\widetilde{\eta}}(y,\mu) \ngtr L_{\widetilde{\eta}}(y,\bar{\mu}), \qquad \text{since, } \eta(y,y) = 0, \\ & \mu_l^T h_l(y) \leq \bar{\mu_l}^T h_l(y), \qquad \mu_l \in R_+^m. \end{aligned} \tag{13}$$

Now assume that y is not a weakly efficient solution in MOSFP. Then there exists $y^* \in S$ such that for all $i = 1,\, 2,\, ...k$

$$\sum_{j=1}^{p} \frac{f_{ij}(y^*)}{g_{ij}(y^*)} < \sum_{j=1}^{p} \frac{f_{ij}(y)}{g_{ij}(y)}. \tag{14}$$

Since $y \in S$ and $\bar{\mu_l} \in R^m_+$, $\quad \bar{\mu_l}^T h_l(y) \leq 0$. In equation (13), let $\mu_l = 0 \quad \forall \quad l$

$$\bar{\mu_l}^T h_l(y) \geq 0.$$

Hence,

$$\bar{\mu_l}^T h_l(y) = 0. \tag{15}$$

Since $\sum_{j=1}^{p} \dfrac{f_{ij}}{g_{ij}}, \qquad i = 1,\, 2,\, 3...k,\; j = 1,\, 2,\, 3, ..., p$ is invex with respect to $\widetilde{\eta}$, and then from equation (14)

$$\widetilde{\eta}(y^*,\, y)\nabla \left(\sum_{j=1}^{p} \frac{f_{ij}(y)}{g_{ij}(y)} \right) < 0. \tag{16}$$

Thus, by equations (15) and (16) and using the definition of $L_{\widetilde{\eta}}$, we get

$$\begin{aligned}
L_{\widetilde{\eta}}(y^*,\, \bar{\mu_l}) &= \left(\widetilde{\eta}(y^*,\, y)\nabla \left(\sum_{j=1}^{p} \frac{f_{1j}(y)}{g_{1j}(y)} \right) + \bar{\mu_l}^T h_l(y), ..., \widetilde{\eta}(y^*,\, y)\nabla \left(\sum_{j=1}^{p} \frac{f_{kj}(y)}{g_{kj}(y)} \right) + \bar{\mu_l}^T h_l(y) \right) \\
&< \left(\widetilde{\eta}(y,\, y)\nabla \left(\sum_{j=1}^{p} \frac{f_{1j}(y)}{g_{1j}(y)} \right) + \bar{\mu_l}^T h_l(y), ..., \widetilde{\eta}(y,\, y)\nabla \left(\sum_{j=1}^{p} \frac{f_{kj}(y)}{g_{kj}(y)} \right) + \bar{\mu_l}^T h_l(y) \right) \\
&= L_{\widetilde{\eta}}(y,\, \bar{\mu_l}).
\end{aligned}$$

This is the contradict of definition by $6(2)$. Hence, y is a weakly efficient solution in SOFMP(3). □

Theorem 8 *Let y be a weakly efficient solution in SOFMP(3) at which the constraint qualification is satisfied. Further, we assume that h is invex with respect to $\widetilde{\eta}$ at y and $\widetilde{\eta}(y, \bar{\mu_l})$ is a weak vector saddle point for the $\widetilde{\eta}$-Lagrange function in a sum of fractional multiobjective programming (SOFMP).*

Proof Since y is a weakly efficient solution for SOFMP, by Theorem 4, KKT conditions hold. Assume that $\sum_{i=1}^{k} \bar{\lambda_i} = 1$. Since $h_l(x)$ is invex with respect to $\widetilde{\eta}$ at y and $\bar{\mu_l} \in R^m_+$, it follows that the inequality

$$\bar{\mu_l}^T h(x) - \bar{\mu_l}^T h_l(y) \geq \bar{\mu_l}^T \nabla h_l(x) \widetilde{\eta}(x, y)$$

holds for all $x \in S$. Then, from the KKT conditions,

$$\bar{\mu_l}^T h_l(x) - \bar{\mu_l}^T h_l(y) \geq - \sum_{i=}^{k} \bar{\lambda_i}^T \nabla \left(\sum_{j=1}^{p} \frac{f_{ij}}{g_{ij}} \right) \widetilde{\eta}(x, y).$$

By assumption $\widetilde{\eta}(x,y)=0$, the inequality

$$\sum_{i=1}^{k}\bar{\lambda}_i{}^T\nabla\left(\sum_{j=1}^{p}\frac{f_{ij}(y)}{g_{ij}(y)}\right)\widetilde{\eta}(x,y)+\bar{\mu_l}^T h(x)\geq\sum_{i=1}^{k}\bar{\lambda}_i{}^T\nabla\left(\sum_{j=1}^{p}\frac{f_{ij}(y)}{g_{ij}(y)}\right)\widetilde{\eta}(y,\,y)+\bar{\mu_l}^T h(y)$$

holds for all $x\in S$. Since $\bar{\lambda}_i\geq 0.\forall\ i,\quad \sum_i^k\bar{\lambda}_i=1$ and by the definition of the $\widetilde{\eta}$- Lagrange function, it follows that, for all $x\in S$,

$$\bar{\lambda}^T L_{\eta}(x,\,\bar{\mu_l})\geq\bar{\lambda}^T L_{\widetilde{\eta}}(y,\,\bar{\mu_l}). \tag{17}$$

Assume that $L_{\widetilde{\eta}}(x,\,\bar{\mu_l})<L_{\widetilde{\eta}}(y,\,\bar{\mu_l})$ for all $\mu_l\in R^m_+$. Thus, we obtain

$$\widetilde{\eta}(y,\,y)\nabla\left(\sum_{j=1}^{p}\frac{f_{1j}(y)}{g_{1j}(y)}\right)+\mu_l^T h(y),...,\widetilde{\eta}(y,\,y)\nabla\left(\sum_{j=1}^{p}\frac{f_{kj}(y)}{g_{kj}(y)}\right)+\bar{\mu_l}^T h(y)\leq\widetilde{\eta}(y,\,y)\nabla\left(\sum_{j=1}^{p}\frac{f_{1j}(y)}{g_{1j}(y)}\right)$$

$$+\bar{\mu_l}^T h(y),...,\widetilde{\eta}(y,\,y)\nabla\left(\sum_{j=1}^{p}\frac{f_{kj}(y)}{g_{kj}(y)}\right)+\bar{\mu_l}^T h(y)$$

and

$$\bar{\lambda}^T L_{\widetilde{\eta}}(y,\,\mu_l)\leq\bar{\lambda}^T L_{\widetilde{\eta}}(y,\,\bar{\mu_l})\qquad\forall\quad\mu_l\in R^m_+. \tag{18}$$

Assume that $L_{\widetilde{\eta}}(y,\,\bar{\mu_l})<L_{\widetilde{\eta}}(y,\,\bar{\mu_l})$ for all $\mu_l\in R^m_+$. Then $\bar{\lambda}^T L_{\widetilde{\eta}}(y,\,\bar{\mu_l})<\bar{\lambda}^T L_{\widetilde{\eta}}(y,\,\bar{\mu_l})\qquad\forall\quad\mu_l\in R^m_+$. This contradicts (18), and hence

$$\bar{\lambda}^T L_{\widetilde{\eta}}(y,\,\bar{\mu})\leq\bar{\lambda}^T L_{\widetilde{\eta}}(y,\,\mu)\qquad\forall\quad\mu\in R^m_+. \tag{19}$$

Inequalities (17) and (19) mean than $(y,\,\bar{\mu})$ is a weak vector saddle point for the $\widetilde{\eta}$- Lagrange function in a sum of fractional multiobjective fractional programming (MOSF) (3). □

5. Conclusions

In this paper, optimality conditions for the sum of fractional multiobjective optimization problem are established with generalized invexity. The modified approach is used to establish all the results and the equivalence of the modified problem with the original problem is proved by two theorems. A Lagrangian for the SOFMP is introduced and some saddle point-based results are established.

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