A note on the unit distance problem for planar configurations with $\mathbb{Q}$-independent direction set

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Abstract: Let $T(n)$ denote the maximum number of unit distances that a set of n points in the Euclidean plane $\mathbb{R}^2$ can determine with the additional condition that the distinct unit length directions determined by the configuration must be $\mathbb{Q}$-independent. This is related to the Erdös unit distance problem but with a simplifying additional assumption on the direction set that holds "generically".

We show that $T(n+1) - T(n)$ is the Hamming weight of n, i.e. the number of nonzero binary coefficients in the binary expansion of n, and find a formula for $T(n)$ explicitly. In particular, $T(n)$ is $\Theta(nlog(n))$. Furthermore, we describe a process to construct a set of n points in the plane with $\mathbb{Q}$-independent unit length direction set that achieves exactly $T(n)$ unit distances. In the process of doing this, we show $T(n)$ is also the same as the maximum number of edges a subset of vertices of size n determines in either the countably infinite lattice $\mathbb{Z}^\infty$ or the infinite hypercube graph $\{0,1\}^\infty$.

The problem of determining $T(n)$ can be viewed as either a type of packing or isoperimetric problem.

Key words: Unit distance problem, discrete combinatorics, isoperimetric problems

1. Introduction

Erdös posed the following question [6]: What is the maximum possible number $u(n)$ of unit distances determined by an n-point set in the Euclidean plane? Erdös conjectured that $u(n) = O(n^{1+\epsilon})$ for any $\epsilon > 0$ but the best that is currently known is that $u(n) = O(n^{\frac{4}{3}})$; see for example [1, 5, 12, 13].

Examples are also known (see [4] and in fact some are also described in this paper) that show that $u(n) = \Omega(n \log n)$. Any upper bound for the Erdos unit distance conjecture would give a lower bound on the number of distinct distances determined by n points in the Euclidean plane by simple pigeonholing. The hence related Erdös distinct distance conjecture, recently proven by Guth and Katz [7], says that the number of distinct distances determined by n points in the Euclidean plane is at least $\Omega(\frac{n}{\log(n)})$.

It follows from some recent work of J. Matoušek [10] that if a set of n points, $n \geq 4$, in the plane determines at least $Cn \log(n) \log \log(n)$ unit distances then there must be some integer dependencies within its unit distance set, i.e. some achieved unit difference vector is a $\mathbb{Z}$-linear combination of others. This indicates that configurations that are extremal for the Erdös unit distance problem should contain lots of integral (and hence rational) dependencies in its unit distance set. In particular, it follows from Matoušek's work that if a set of n

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points has rationally independent distinct unit distances, then it determines no more than $O(n \log(n) \log \log(n))$ unit distances.

In this note, we will study these types of sets exclusively. More precisely note that if A is a finite subset of the Euclidean plane $\mathbb{R}^2$ then A determines a unit vector set $U_A = \{x - y | x, y \in A, \|x - y\| = 1\} \subset S^1$, where S^1 is the unit circle. Note that U_A is symmetric i.e. if $u \in U_A$ so is $-u$. Choose a representative of each direction represented in U_A that has principal argument (in radians) in the interval $[0, \pi)$ to form the unit direction set $D_A = \{x - y | x, y \in A, \|x - y\| = 1, Arg(x - y) \in [0, \pi)\} \subset S^1$. Thus U_A is the disjoint union of D_A and $-D_A$.

We say that A has rationally independent unit direction set if the elements of D_A are independent over $\mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers. We show in this paper that most point configurations have rationally independent unit direction set and that this condition is generic in a sense. Note this condition imposes no constraint on differences that are not of unit length nor does it preclude the possibility of a given unit direction vector occurring multiple times in the difference set of A; it only imposes the condition of $\mathbb{Q}$-independence on the **distinct** unit distances as determined in D_A.

We then ask the following question: What is the maximum possible number $T(n)$ of unit distances determined by an n-point set A in the Euclidean plane with rationally independent unit direction set?

In this paper we find an explicit exact formula for $T(n)$ and describe a process of constructing a configuration of n-points in the plane with rationally independent unit direction set that achieves $T(n)$ unit distances.

Theorem 1.1 *Let $n \geq 1$; then $T(n)$, the maximum number of unit distances that a n-point set of the Euclidean plane with rationally independent unit direction set determines, is also equal to:*

(a) The maximum number of edges determined by n points in the standard countably infinite integer lattice $\mathbb{Z}^\infty$.

(b) The maximum number of edges determined by n points in the standard countably infinite hypercube graph $\{0, 1\}^\infty$.

Furthermore, $T(n + 1) - T(n) = H(n)$, where $H(n)$ is the Hamming weight of n i.e. the number of nonzero binary coefficients in the binary expansion of n. If $n = \sum_{j=1}^{t} 2^{k_j}$, where $k_1 > k_2 > \cdots > k_t \geq 0$, then $T(n) = \sum_{j=1}^{t} (k_j 2^{k_j - 1} + (j - 1) 2^{k_j})$ and in particular one always has

$$\frac{n}{4}(\lceil \log n \rceil - 1) < T(n) < n \lceil \log n \rceil,$$

where $\log$ is the base 2 logarithm and $\lceil x \rceil$ is the smallest integer greater or equal to x. Thus $T(n)$ is big-Theta of $n \log n$.

The equivalence of (a) is proven in Theorem 2.7, the equivalence of part (b) is proven in Corollary 3.3, and the formulas for $T(n)$ are proven in Theorem 3.6.

As the first step, we show that any extremal configuration (n-point set in the Euclidean plane with $\mathbb{Q}$-independent unit direction set and that determines the maximum $T(n)$ of unit distances) can be assumed to have a **good** set of unit directions as defined below and furthermore can be assumed to lie in the lattice generated by this good set of directions.

A set of unit directions $u_1, \ldots u_\ell \in S^1$ is **good** if $\|a_1 u_1 + \cdots + a_\ell u_\ell\| = 1$ for $(a_1, \ldots, a_\ell) \in \mathbb{Z}^\ell$ if and only if exactly one a_j is nonzero. We also show that **good** unit direction sets are "generic" (a dense G_δ set) in

the space of collections of unit directions and, in particular, that "most" n-point sets in $\mathbb{R}^2$ have D_A a good unit direction set. Given this, we describe a process of making a n point set with rationally independent unit direction set that achieves the maximum possible number $T(n)$ of unit distances:

Proposition 1.2 *Let $n \geq 1$ be given. Choose d such that $2^{d-1} < n \leq 2^d$ and any collection of unit directions $u_0, \ldots, u_{d-1} \in S^1$ that is* **good***. Then let*

$$A = \{a_0 u_0 + \cdots + a_{d-1} u_{d-1} | a_j \in \{0, 1\}, 0 \leq a_0 + a_1 2 + \cdots + a_{d-1} 2^{d-1} < n\}$$

Then A is an n-point set in the Euclidean plane with rationally independent unit distance set that determines exactly the maximum possible $T(n)$ unit distances.

Proposition 1.2 follows from the reductions in section 2 and the results in section 3. Using the notion of good directions sets, we reduce the problem to a packing problem in $\mathbb{Z}^\infty$, the countably infinite integer lattice, and then to a problem in $\{0, 1\}^\infty$ the countably infinite hypercube graph. These lattice reductions are similar to what is discussed in work of Brass (see [4] for a reference), but with a particular lattice arising specific to our particular situation. We reduce to this canonical lattice by reducing the problem to configurations with good unit direction sets.

This final packing/graph theoretic result for hypercube graphs is interesting in its own right and had been previously studied. The result says that the best way to choose n points, $n \leq 2^d$, in the d-dimensional hypercube $\{0, 1\}^d$ graph to maximize the number of edges determined (an edge is determined every time two vectors have Hamming distance one, i.e. differ in exactly one coordinate) is to choose the n points as the binary representations of the numbers 0 through $n - 1$.

Theorem 1.3 *Fix $n \leq 2^d$. Let V be an n-point subset of the hypercube graph on vertex set $\{0, 1\}^d$ that determines $E(V)$ edges. Let $V_n \subseteq \{0, 1\}^d$ be the vertex set on n vertices given by the binary expansions of the numbers 0 through $n - 1$; then $E(V_n) \geq E(V)$.*

We provide an alternate proof of this result in this paper for completeness and because we feel it is somewhat shorter and cleaner than those that have previously appeared in the literature [2, 8, 9]. These "packing problems" are also related to isoperimetric problems [3].

2. Good direction sets

Let A be an n-point set in the Euclidean plane ($\mathbb{R}^2$ equipped with the Euclidean metric). We define its unit difference set

$$U_A = \{x - y | x, y \in A, \|x - y\| = 1\} \subset S^1$$

where S^1 is the unit circle. Note that U_A is symmetric i.e. if $u \in U_A$ so is $-u$. Choose a representative of each direction represented in U_A that has principal argument (in radians) in the interval $[0, \pi)$ to form the unit direction set

$$D_A = \{x - y | x, y \in A, \|x - y\| = 1, Arg(x - y) \in [0, \pi)\} \subset S^1.$$

We will say that A has rationally independent unit direction set if the elements of D_A are independent over $\mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers.

The unit direction graph determined by A is the graph whose vertex set is the set A and where there is an edge between $a_1, a_2 \in A$ if and only if $\|a_1 - a_2\| = 1$.

Definition 2.1 *An **extremal configuration** will denote an n-point set A in the Euclidean plane with rationally independent unit direction set that achieves the maximum possible number $T(n)$ of unit distances subject to these conditions.*

Note that trivially $T(1) = 0$ and so we might as well restrict our attention to $n \geq 2$. Our first lemma shows that we may always assume certain properties about our extremal configurations:

Lemma 2.2 *For any fixed $n \geq 2$, there exists an extremal configuration A such that $(0,0) \in A, (1,0) \in D_A$ and the unit direction graph determined by A is path connected. Thus $A \subset \mathbb{Z}(D_A) \cong \mathbb{Z}^d$, where $d = |D_A|$. Here $\mathbb{Z}(D_A)$ denotes the set of all $\mathbb{Z}$-linear combinations of elements in D_A. We also have $1 \leq d \leq n-1$.*

Proof It is clear that as $n \geq 2$, D_A is nonempty and thus there is a pair of points $a_1, a_2 \in A$ such that $\|a_1 - a_2\| = 1$. After applying a translation and a rotation to all the points of A (which changes D_A by a rotation and so disturbs neither $\mathbb{Q}$-independence nor $|D_A|$), we may assume $a_1 = (0,0)$ and $a_2 = (1,0)$ lie in A. Thus WLOG $(0,0) \in A$ and $(1,0) \in A$ and hence $(1,0) \in D_A$ also.

Now suppose that the unit direction graph of A was disconnected. Let C_1 be a component and C_2 the rest of the graph. Then C_1 contains a right-most vector (vector with maximal x-coordinate) $\hat{u}$ and C_2 contains a left-most vector (vector with minimal x-coordinate) $\hat{v}$. We may then translate C_2 so that the image of $\hat{v}$ lies one unit to the right of $\hat{u}$. This changes no internal connections in either C_1 (which was left alone) or C_2 but raises the number of connections between them. It does not change the direction set D_A either as $(1,0) \in D_A$ already and no other unit directions were created in the process as the translated C_2 as a set lies one unit to the right of the rightmost point(s) of C_1. Thus we achieve a new extremal configuration with at least one more unit distance than the one we started with, which contradicts the extremality of the original configuration. Thus an extremal configuration always has a (path) connected unit direction graph.

Now as the unit direction graph of A is path connected and $(0,0) \in A$, we know for any $x \in A$ it is possible to go from $(0,0)$ to x with a path of edges consisting of vectors in $U_A = D_A \cup -D_A$. Thus A is contained in the $\mathbb{Z}$-span of D_A as claimed. Finally as the elements of D_A are $\mathbb{Q}$-independent, they are also $\mathbb{Z}$-independent and so the $\mathbb{Z}$-span of D_A is a free abelian group of rank d where $d = |D_A|$.

List A as $A = \{a_1, \ldots, a_n\}$. Now note that the $\mathbb{Z}$-span of D_A is itself contained in the $\mathbb{Z}$-span of $\{a_i - a_j | a_i, a_j \in A\}$. This later set is contained in the $\mathbb{Z}$-span of the set $\{a_2 - a_1, a_3 - a_1, \ldots, a_n - a_1\}$, a free abelian group of rank $\leq n-1$. Thus $d \leq n-1$ as the rank d of a subgroup of free abelian group of rank $n-1$ must have $d \leq n-1$ as $\mathbb{Z}$ is a PID. $\square$

Note that while the $\mathbb{Z}$-span of D_A is a subgroup of $\mathbb{R}^2$ isomorphic to $\mathbb{Z}^d$, it cannot be a closed subgroup unless $d \leq 2$. This is because by Lie theory the only closed subgroups of $\mathbb{R}^2$ are isomorphic to $\mathbb{R}^m, \mathbb{Z}^m$ with $m \leq 2$ or $\mathbb{R} \times \mathbb{Z}$, and thus the closure of the $\mathbb{Z}$-span of D_A must be all of $\mathbb{R}^2$ whenever $d > 2$, i.e. it must generate a dense "lattice".

By Lemma 2.2, an extremal configuration A of any size $n \geq 2$ may be constructed by choosing a "lattice" $\mathbb{Z}^d \subseteq \mathbb{R}^2$ and distributing n points in it to maximize the number of unit distances achieved by the set. Furthermore, the basis vectors of this lattice have unit length and can be assumed to be achieved by the set as differences. Note that any other pair of points of unit distance in the lattice would have difference vector

in the $\mathbb{Z}$-span of these basis vectors and so such a pair would have to be avoided by the set A to maintain the condition of $\mathbb{Q}$-independence of unit directions.

This indicates that a "good lattice" to work with would be one where the only pairs of lattice points with unit distance apart are those that differ by plus or minus a basis vector. This motivates the definition of a "good direction set", which we make next:

Definition 2.3 *Let S^1 be the unit circle in the Euclidean plane and let $E \subset S^1$ be the set of points on the unit circle whose principal argument lies in the interval $[0, \pi)$. Let $F(E, d) = \{(u_1, \ldots, u_d) | u_j \in E, u_i \neq u_j, 1 \leq i < j \leq d\}$ be the configuration space of d-tuples of distinct directions in E, topologized as a subspace of E^d with the product topology.*

*$(u_1, \ldots, u_d) \in F(E, d)$ is a **good set of directions** if whenever $\sum_{k=1}^{d} a_k u_k \in S^1$ with $(a_1, \ldots, a_d) \in \mathbb{Z}^d$, we necessarily have all but one a_j is zero. (Note that this nonzero a_j would then have to be ± 1.)*

Note that E^d is homeomorphic to $[0, \pi)^d$ and is hence locally compact, Hausdorff. $F(E, d)$ is an open subset of E^d and so is also locally compact, Hausdorff and hence in particular a Baire space. Further note that a good set of directions has $\mathbb{Z}$-span equal to a "lattice" where the only pairs of vertices that have unit distance apart occur when the difference is plus or minus a basis vector. We record some properties of good direction sets in the next proposition.

Proposition 2.4 *Let $\{u_1, \ldots, u_d\} \subset E \subset S^1$ be a good direction set; then $\{u_1, \ldots, u_d\}$ is $\mathbb{Q}$-independent and thus the $\mathbb{Z}$-span of $\{u_1, \ldots, u_d\}$ is isomorphic to $\mathbb{Z}^d \subset \mathbb{R}^2$. We can then express any $\hat{v}$ in the $\mathbb{Z}$-span of $\{u_1, \ldots, u_d\}$ uniquely as an integer d-tuple $(a_1, \ldots, a_d)$, where $\hat{v} = \sum_{j=1}^{d} a_j u_j$. The a_j's are referred to as the lattice coordinates of $\hat{v}$. Then in this lattice two vectors $\hat{v}_1, \hat{v}_2$ have Euclidean distance one if and only if they have ℓ_1 distance one, i.e. if and only if they differ in exactly one lattice coordinate by an amount ± 1.*

Proof Let $\{u_1, \ldots, u_d\}$ be a good direction set. If $d = 1$ the set is $\mathbb{Q}$-independent as $u_1 \in S^1$ is nonzero and so WLOG $d \geq 2$. Suppose that this set were rationally dependent. Then $q_1 u_1 + \cdots + q_d u_d = 0$ for rational numbers q_j, not all zero. We can then clear denominators and conclude that $a_1 u_1 + \cdots + a_d u_d = 0$ for integers a_j, not all zero. WLOG suppose $a_1 \neq 0$; then we can write $(a_1 - 1)u_1 + \cdots + a_d u_d = -u_1$ and so the left-hand side of the equation lies on S^1. Since the set of directions is good, this implies either $a_2 = \cdots = a_d = 0$, which implies $a_1 u_1 = 0$ and so $a_1 = 0$, also a contradiction, or that $a_1 = 1$, and all but one of the other a_j's are equal to 0. Without loss of generality let a_2 be the other a_j that is nonzero and then we get $u_1 + a_2 u_2 = 0$, which implies $u_1 = -a_2 u_2$, implying $a_2 = \pm 1$. However, both $u_1, u_2 \in E$ and so we conclude $u_1 = u_2$. This is a contradiction as good direction sets are defined to consist of distinct elements or by noticing this would mean $1u_1 - 2u_2 = -u_1 \in S^1$, contradicting the good direction condition. Thus we conclude good direction sets are $\mathbb{Q}$-independent and so the abelian group they generate is free abelian of rank d as claimed. Finally if $v_1 = a_1 u_1 + \cdots + a_d u_d$ and $v_2 = b_1 u_1 + \cdots + b_d u_d$ for $(a_1, \ldots, a_d), (b_1, \ldots, b_d) \in \mathbb{Z}^d$, it is easy to check that $\|v_1 - v_2\| = 1$ if and only if $\|(a_1, \ldots, a_d) - (b_1, \ldots, b_d)\|_1 = 1$ using the definition of good direction set. (Here $\| \cdot \|$ is the Euclidean norm on $\mathbb{R}^2$ while $\| \cdot \|_1$ is the ℓ_1-norm on $\mathbb{Z}^d$.) □

Example 2.5 *Identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. The set $\{1, \xi = e^{\frac{2\pi i}{3}}\} \subset E \subset S^1$ is $\mathbb{Q}$-independent but is not a good direction set as $1(1) + 1(\xi) = -\xi^2 \in S^1$.*

We now show that the set $G \subset F(E,d)$ of good sets of unit directions is a dense G_δ set. Recall a G_δ set is a countable intersection of open sets.

Theorem 2.6 *Let $G \subseteq F(E,d)$ be the set of good direction sets $(u_1, \ldots, u_d)$; then G is a dense G_δ set in $F(E,d)$.*

Proof   First note that when $d = 1$, $G = F(E,1)$ and so there is nothing to show. Thus WLOG assume $d \geq 2$. For any $(a_1, \ldots, a_d) \in \mathbb{Z}^d$ with two or more a_j nonzero, define $A_{(a_1,\ldots,a_d)} = \{(u_1, \ldots, u_d) \in F(E,d)| \sum_{j=1}^{d} a_j u_j \in S^1\}$. Note that the function $f_{(a_1,\ldots,a_d)}(u_1, \ldots, u_d) = a_1 u_1 + \cdots + a_d u_d : F(E,d) \to \mathbb{R}^2$ is continuous and so $A_{(a_1,\ldots,a_d)} = f_{(a_1,\ldots,a_d)}^{-1}(S^1)$ is hence a closed subset of $F(E,d)$. We now show that $A_{(a_1,\ldots,a_d)}$ is nowhere dense in $F(E,d)$. Toward this let $U \subseteq F(E,d)$ where U is nonempty and open. We have to show that there exists $(v_1, \ldots, v_d) \in U$ such that $(v_1, \ldots, v_d) \notin A_{(a_1,\ldots,a_d)}$.

Setting θ_{jk} to be the angle between v_j and v_k, we have $(v_1, \ldots, v_d) \in A_{(a_1,\ldots,a_d)}$ if and only if $\sum_{j<k} 2a_j a_k cos(\theta_{jk}) = 1 - \sum_{j=1}^{d} a_j^2$.

Now note if we parametrize $E \subset S^1$ by the principal argument and $t_j \in [0, \pi)$ is the argument of v_j, we have $\theta_{ij} = t_i - t_j$ and so the last equation can be written as:

$$2 \sum_{i<j}^{d} a_i a_j cos(t_i - t_j) = 1 - a_1^2 - \cdots - a_d^2.$$

Suppose contrary to what we want to show that $U \subseteq A_{(a_1,\ldots,a_d)}$ then the equation above holds for all choices of $t_1, \ldots, t_d$ in an nonempty open subset of $F([0,\pi),d) \subseteq [0,\pi)^d$ and so we may differentiate it with respect to any of the variables t_j on this open set. Differentiating this last identity with respect to t_1 twice would kill all terms not involving t_1 and replicate those involving t_1 yielding:

$$\sum_{1<j}^{d} a_1 a_j cos(t_1 - t_j) = 0$$

and thus imply that the net contribution of all terms involving t_1 in the original identity is zero! Continuing in this way with t_2 and then t_3 etc., we see that the identity can only hold for a nonempty open set of $(t_1, \ldots, t_d)$ if and only if its left-hand side is identically zero. This yields

$$0 = 1 - a_1^2 - \cdots - a_d^2$$

which is a contradiction as at least two of the a_j's are nonzero integers and so the right-hand side is negative.

Thus $U \not\subset A_{(a_1,\ldots,a_d)}$ and we can conclude that $A_{(a_1,\ldots,a_d)}$ is a nowhere dense closed set.

As $G = F(E,d) - \cup_{(a_1,\ldots,a_d) \in \mathbb{Z}^d} A_{(a_1,\ldots,a_d)}$, where the union is taken only over d-tuples in $\mathbb{Z}^d$ with at least two coordinates nonzero, we conclude G is a dense G_δ set as its complement is a countable union of nowhere dense closed sets in the Baire space $F(E,d)$ (see [11]). $\square$

Let $\mathbb{Z}^d$ denote the integer lattice graph whose vertices are given by the set $\mathbb{Z}^d$ and where there is an edge joining $(u_1, \ldots, u_d)$ with $(v_1, \ldots, v_d)$ if and only if they differ in exactly one coordinate and in this coordinate

the entries differ by ± 1. Alternatively $\hat{u}$ and $\hat{v}$ are joined by an edge exactly when $\|\hat{u} - \hat{v}\|_1 = 1$, where $\| \cdot \|_1$ is the ℓ^1-norm.

Note that we may regard these graphs as nested $\mathbb{Z}^1 \subset \mathbb{Z}^2 \subset \mathbb{Z}^3 \subset \ldots$ and we define $\mathbb{Z}^\infty$ as their union graph, the integer lattice graph on countably infinitely many coordinates. The vertices of this graph are eventually zero integer sequences and two such sequences $\hat{u}, \hat{v}$ are joined by an edge if $\|\hat{u} - \hat{v}\|_1 = 1$.

The next theorem relates $T(n)$, the maximum number of unit distances determined by n points in the Euclidean plane with $\mathbb{Q}$-independent unit distance set, with the maximum number of edges that n points in $\mathbb{Z}^\infty$ can determine.

Theorem 2.7 $T(n)$ *is equal to the maximum number of edges that n points in $\mathbb{Z}^\infty$ can determine. It is also the maximum number of edges that n points in $\mathbb{Z}^{n-1}$ can determine.*

Proof When $n = 1$, $T(1) = 0$ and there is nothing to show and so assume $n \geq 2$. By Lemma 2.2, we may take A an extremal set of n points such that A is a subset of the $\mathbb{Z}$-span of D_A and hence is a subset of $\mathbb{Z}^d \subseteq \mathbb{Z}^{n-1}$. However, if D_A is not a good set of directions, there can be some pairs $\hat{u}, \hat{v}$ in this "lattice" that have Euclidean unit distance apart but that have $\|\hat{u} - \hat{v}\|_1 \neq 1$. For these pairs, we cannot have both members of the pair in A as that would generate a unit difference vector (which is not one of the unit basis vectors of the lattice) that is an integral combination of the basis unit vectors indicating a $\mathbb{Z}$-linear dependency in D_A. This contradicts that A has rationally independent unit direction vectors. Thus the placement of the n points of A in $\mathbb{Z}[D_A] \cong \mathbb{Z}^d$ must avoid placing two points of A in these sorts of pair locations. Subject to these restrictions, A is a placement of n points in the standard $\mathbb{Z}^d$ graph that maximizes the number of edge connections.

On the other hand, we know by Theorem 2.6 that we can find $(u_1, \ldots, u_d)$ a set of good directions that generate a "good lattice" $\mathbb{Z}^d$ and an arbitrary placement of n points in this lattice will yield a set $B \subset \mathbb{R}^2$ with $D_B \subseteq \{u_1, \ldots, u_d\}$ and hence rationally independent unit directions.

Comparing the situation for sets A and B we see that they are both subsets of the standard $\mathbb{Z}^d$ integer lattice graph but A is constrained while B is not, in the sense we may choose B to correspond to A under the isomorphism of these lattices or we could choose B to be something else. Thus we see that we never lose if we choose B inside a lattice generated by a good set of directions so as to maximize the number of edges determined by B within the graph $\mathbb{Z}^d$. Furthermore, because the set of directions is good, the edge count within the standard $\mathbb{Z}^d$-lattice is the same as the unit distance count determined by the set B, i.e. two points in the lattice are a unit distance apart in the Euclidean metric if and only if they are unit distance apart in the ℓ^1-metric.

Thus extremal configurations for the Euclidean unit distance problem correspond to extremal configurations within the standard lattice $\mathbb{Z}^\infty$ or in fact $\mathbb{Z}^d$ for some $d \leq n - 1$.

Note also: given a set C of n points in the standard integer lattice $\mathbb{Z}^\infty$ that maximize edge count, we may translate one to the origin and argue once again by extremality for path connectedness of the unit distance graph. From this it follows easily that there are at most $n - 1$ coordinates for which the elements of C can have nonzero entries in these coordinates, i.e. C can be viewed as a subset of $\mathbb{Z}^{n-1}$. $\square$

3. Edge maximizing configurations in $\mathbb{Z}^d$ and $\{0,1\}^d$

In the following, an induced subgraph on a set of vertices V refers to the subgraph within a given graph whose vertex set is V and whose edge set is obtained by taking all the edges joining the vertices in V in the ambient

graph. Throughout this section let $\mathbb{Z}^d$ denote the standard d-dimensional integer lattice graph whose vertex set is $\mathbb{Z}^d$ and where two integer vectors are adjacent if and only if they differ in exactly one coordinate by ± 1. Furthermore, let $\{0,1\}^l \times \mathbb{Z}^d \subset \mathbb{Z}^{d+l}$ and $\{0,1\}^d \subset \mathbb{Z}^d$ be given the induced subgraph structures also.

We seek to find edge maximizing configurations in the sense that we would like to place n points in **some** lattice $\mathbb{Z}^m$ so that the number of edge connections amongst those points is maximized. The first lemma and its corollary shows that any such edge maximizing configuration can be found inside the hypercube graph $\{0,1\}^d$ for some d.

Let $V \subseteq \mathbb{Z}^d$ be a finite set of vertices; we can and will always translate V so that the number of edges it determines is unchanged and such that $0 \in V$ and all $v = (x_1, \ldots, x_d) \in V$ have $x_i \geq 0$ for all $i = 1, \ldots, d$. We then define $M_j(V) = \max_{v \in V}\{x_j\}$ and $M(V) = \max_{j \in \{1,\ldots,d\}} M_j$. Note that $M_j(V)$ is the maximum jth coordinate achieved in the set of vectors V and $M(V)$ is a nonnegative integer.

Lemma 3.1 *Let $V \subset \mathbb{Z}^d$ be a finite set and $G = (V, E)$ be the induced subgraph of $\mathbb{Z}^d$ and $M(V) \geq 2$, where $M(V)$ is the quantity defined in the previous paragraph. Then there exists $V' \subset \{0,1\}^l \times \mathbb{Z}^d$ and induced subgraph $G' = (V', E')$ of $\{0,1\}^l \times \mathbb{Z}^d$ with $|V| = |V'|$ and $|E| = |E'|$. Furthermore, $M(V') = M(V) - 1$.*

Proof Let M denote $M(V)$ throughout. Without loss of generality we assume $M_1(V) = M \geq 2$. Let V_1 be the set of vertices in V with $x_1 = M$ and let $V_0 = V \setminus V_1$. Note that $v = (M, a_2, \cdots, a_d) \in V_1$ shares an edge with $w = (b_1, b_2, \cdots, b_d) \in V_0$ if and only if $b_1 = M - 1$ and $b_j = a_j$ for $j = 2, \cdots, d$.

We map the elements of V into $\{0,1\} \times \mathbb{Z}^d$ as follows: if $v = (M, a_2, \cdots, a_d) \in V_1$, then $v \rightarrow (1, M - 1, a_2, \cdots, a_d)$ and if $w = (b_1, \cdots, b_d) \in V_0$, then $w \rightarrow (0, b_1, \cdots, b_d)$. Let $G' = (V', E')$ be the graph resulting from this mapping. By construction $|V'| = |V|$. We see that $v \in V_1$ and $w \in V_0$ share in edge in G if and only if their images share an edge in G'. Moreover, we also see that $v_1, v_2 \in V_1$ share an edge in G if and only if their images share an edge in G' and likewise $w_1, w_2 \in V_0$ share an edge in G if and only if their images share an edge in G'. Thus $|E'| = |E|$. We have $M_2(V') = M_1(V) - 1$ and $M(V) - 1 \leq M(V') \leq M(V)$. At this point, if $M(V') = M(V) - 1$ we are done. Otherwise, there is another coordinate x_j with $M_j(V') = M(V)$ and we repeat the process for this coordinate. Continuing in this fashion eventually results in a graph with $M(V') = M(V) - 1$ since there are at most d possible (original) coordinates to consider. $\square$

Corollary 3.2 *Let $G = (V, E)$ be a finite induced subgraph of the standard integer lattice graph $\mathbb{Z}^d$. Then there exists an induced subgraph $G' = (V', E')$ on $\{0,1\}^{d'}$ for some $d' \in \mathbb{N}$, with $|V| = |V'|$ and $|E| = |E'|$.*

Proof After $M(V) - 1$ iterations of lemma 3.1 we have $M(V') = 1$, which implies the resulting graph is now an induced subgraph of $\{0,1\}^{d'}$. $\square$

In particular, suppose K is an extremal graph on $\mathbb{Z}^\infty$, i.e. $|V(K)| = n$ and $|E(K)|$ equals the maximum number of edges determined by n points in $\mathbb{Z}^\infty$. The corollary tells us that there is a graph K' on $\{0,1\}^\infty$ with the same number of vertices and edges; hence

$$\max_{\substack{G \in \mathbb{Z}^\infty \\ |V(G)|=n}} |E(G)| = |E(K)| = |E(K')| \leq \max_{\substack{G \in \{0,1\}^\infty \\ |V(G)|=n}} |E(G)|$$

Since the inequality in the other direction is obvious we have established the following.

Corollary 3.3 *The maximum number of edges determined by n points in the standard countably infinite integer lattice $\mathbb{Z}^\infty$ is equal to the maximum number of edges determined by n points in the standard countably infinite hypercube graph $\{0,1\}^\infty$.*

This reduces the problem of finding $T(n)$ to the edge maximizing problem on $\{0,1\}^\infty$, which we now focus on. The solution that we give in theorem 3.4 is known, and was proved in [2, 8, 9]. We provide an alternate proof of this result for completeness and because we feel it is somewhat shorter and cleaner than those that have previously appeared. See also [3] for an overview of related isoperimetric problems.

We label the elements of the d dimensional unit hypercube $\{0,1\}^d$ via binary representation. That is, let $v_j = (a_{d-1}, a_{d-2}, \ldots, a_0)$, where $j = \sum_{k=0}^{d-1} a_k 2^k$, each $a_k \in \{0,1\}$. In the following, for any graph embedded in $\{0,1\}^d$ it is implied that vertices v_i and v_j are adjacent if and only if the Hamming distance between v_i and v_j is 1, i.e. if and only if v_i and v_j differ in exactly one coordinate.

For the sake of clarity, we use the following example to introduce some definitions we will use. Although they are introduced in this example, it should be clear how they are defined in general.

3.1. Example: $d = 5$

Consider the array for $(x_4, x_3, \cdots, x_0) \in \{0,1\}^5$:

$$
\begin{array}{c|ll|c}
 & v_0 = (0,0,0,0,0) & v_8 = (0,1,0,0,0) & \\
 & v_1 = (0,0,0,0,1) & v_9 = (0,1,0,0,1) & \\
\text{Block} & v_2 = (0,0,0,1,0) & v_{10} = (0,1,0,1,0) & \text{Block} \\
\text{A} & v_3 = (0,0,0,1,1) & v_{11} = (0,1,0,1,1) & \text{B} \\
 & \vdots & \vdots & \\
 & v_7 = (0,0,1,1,1) & v_{15} = (0,1,1,1,1) & \\
\hline
 & v_{16} = (1,0,0,0,0) & v_{24} = (1,1,0,0,0) & \\
 & v_{17} = (1,0,0,0,1) & v_{25} = (1,1,0,0,1) & \\
\text{Block} & v_{18} = (1,0,0,1,0) & v_{26} = (1,1,0,1,0) & \text{Block} \\
\text{C} & & & \text{D} \\
 & \vdots & \vdots & \\
 & v_{23} = (1,0,1,1,1) & v_{31} = (1,1,1,1,1) & \\
\end{array}
$$

The blocks are determined by using the leftmost two coordinates: Block A: $(0,0,*)$, Block B: $(0,1,*)$, Block C: $(1,0,*)$, Block D: $(1,1,*)$. Note that induced subgraphs of $\{0,1\}^5$ never contain "diagonal edges" between a vertex in block A and a vertex in block D or between a vertex in block B and a vertex in block C.

Arrays of this form will also be used to represent induced subgraphs of $\{0,1\}^5$ by putting a dark dot next to vertices that occur in the subgraph. For instance, the induced subgraph T with vertex set

$$V = \{v_0, v_1, v_9, v_{10}, v_{11}, v_{19}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}, v_{28}, v_{29}, v_{30}\}$$

is represented by

$$
\begin{array}{ll}
v_0 = (0,0,0,0,0) \bullet & v_8 = (0,1,0,0,0) \\
v_1 = (0,0,0,0,1) \bullet & v_9 = (0,1,0,0,1) \bullet \\
v_2 = (0,0,0,1,0) & v_{10} = (0,1,0,1,0) \bullet \\
v_3 = (0,0,0,1,1) & v_{11} = (0,1,0,1,1) \bullet \\
v_4 = (0,0,1,0,0) & v_{12} = (0,1,1,0,0) \\
v_5 = (0,0,1,0,1) & v_{13} = (0,1,1,0,1) \\
v_6 = (0,0,1,1,0) & v_{14} = (0,1,1,1,0) \\
v_7 = (0,0,1,1,1) & v_{15} = (0,1,1,1,1)
\end{array}
$$

Block A ... Block B

$$
\begin{array}{ll}
v_{16} = (1,0,0,0,0) & v_{24} = (1,1,0,0,0) \bullet \\
v_{17} = (1,0,0,0,1) & v_{25} = (1,1,0,0,1) \bullet \\
v_{18} = (1,0,0,1,0) & v_{26} = (1,1,0,1,0) \bullet \\
v_{19} = (1,0,0,1,1) \bullet & v_{27} = (1,1,0,1,1) \bullet \\
v_{20} = (1,0,1,0,0) & v_{28} = (1,1,1,0,0) \bullet \\
v_{21} = (1,0,1,0,1) & v_{29} = (1,1,1,0,1) \bullet \\
v_{22} = (1,0,1,1,0) & v_{30} = (1,1,1,1,0) \bullet \\
v_{23} = (1,0,1,1,1) \bullet & v_{31} = (1,1,1,1,1)
\end{array}
$$

Block C ... Block D

In a graph G, let $V_A(G)$ denote the vertices of G in block A ($V_B(G), V_C(G), V_D(G)$ defined analogously). Note that each vertex in $V_D(G)$ shares an edge with at most one vertex in $V_C(G)$ (the element directly to the left). For instance, in the graph above v_{19} and v_{27} share an edge but there are no other vertices in V_D sharing an edge with a vertex in V_C. We call these "horizontal" edges, as with edges connecting a vertex of V_B with the vertex to the left in V_A. The horizontal edges in a graph G will be denoted $E^{hor}(G)$. In the graph above, $|E^{hor}| = 2$, the horizontal edges are $[v_{19}, v_{27}]$ and $[v_1, v_9]$. Similarly, each vertex of $V_D(G)$ shares an edge with at most one vertex of $V_B(G)$. For instance, in the graph above v_{25} shares an edge with v_9 but no other vertices in V_B. We call these "vertical" edges as with edges connecting a vertex of $V_C(G)$ with a vertex of $V_A(G)$. The vertical edges in a graph G will be denoted $E^{vert}(G)$. In the graph above, $|E^{vert}| = 3$, the vertical edges are $[v_9, v_{25}], [v_{10}, v_{26}], [v_{11}, v_{27}]$. Let $V_{AB}(G) = V_A(G) \cup V_B(G)$ and define $V_{AC}(G), V_{BD}(G)$ and $V_{CD}(G)$ analogously. One can observe that the following inequalities hold in general:

$$
|E^{hor}(G)| \leq \min\{|V_{AC}(G)|, |V_{BD}(G)|\} \tag{1}
$$

$$
|E^{vert}(G)| \leq \min\{|V_{AB}(G)|, |V_{CD}(G)|\} \tag{2}
$$

Let $E_A(G)$ be the edges in the subgraph of G induced by $V_A(G)$, and $E_B(G), E_C(G), E_D(G)$ defined analogously. In particular, an edge is in $E_A(G)$ if and only if it is an edge of G that connects a vertex in $V_A(G)$ with another vertex in $V_A(G)$. Similarly, let $E_{AB}(G)$ be the edges in the subgraph of G induced by $V_{AB}(G)$, and $E_{AC}(G), E_{BD}(G), E_{CD}(G)$ defined analogously. In particular, $E_{AB}(G)$ contains precisely all the edges in $E_A(G)$, all the edges in $E_B(G)$, and those edges of G that connect a vertex in $V_A(G)$ and a vertex in $V_B(G)$.

For example, in the graph T above $E_A = \{[v_0, v_1]\}, E_B = \{[v_9, v_{11}], [v_{10}, v_{11}]\}$ and with the horizontal edge $[v_1, v_9]$ we have $|E_{AB}| = 4$. The reader can verify that $|E_C| = 1, |E_D| = 9$ and with the horizontal edge $[v_{19}, v_{27}]$ we have $|E_{CD}| = 11$. With the vertical edges $[v_9, v_{25}], [v_{10}, v_{26}], [v_{11}, v_{27}]$ and the counts for $|E_A|, |E_B|, |E_C|, |E_D|$ we obtain $|E_{AC}| = 2$ and $|E_{BD}| = 14$.

Note that the edge set $E(G)$ of a graph G can be written as a disjoint union as

$$
E(G) = E_{AB}(G) \cup E_{CD}(G) \cup E^{vert}(G) \tag{3}
$$

$$
E(G) = E_{AC}(G) \cup E_{BD}(G) \cup E^{hor}(G) \tag{4}
$$

We also make the following definitions. We say a graph is "completely arranged" if $V = \{v_0, v_1, \cdots, v_I\}$ for some I. We say a graph is "horizontally arranged" if $V_{AB} = \{v_0, v_1, \cdots, v_J\}$ and $V_{CD} = \{v_{16}, v_{17}, \cdots, v_K\}$ for some J and K. We say a graph is "vertically arranged" if V_{AC} is filled consecutively downward starting from v_0, and V_{BD} is filled consecutively downward starting from v_8. Completely arranging a graph is replacing it by the completely arranged graph with the same number of vertices. Horizontall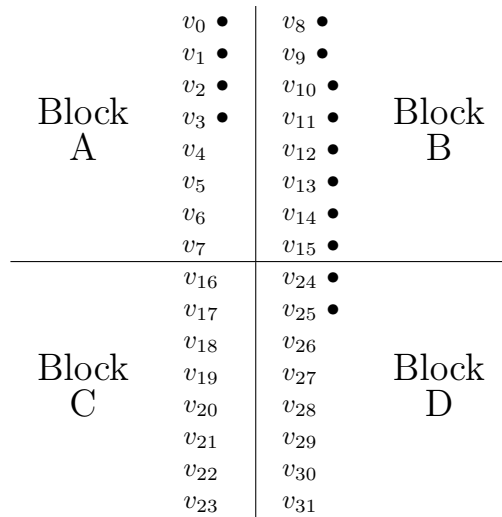y arranging a graph is replacing it by the horizontally arranged graph with the same $|V_{AB}|$ and $|V_{CD}|$. Vertically arranging a graph is replacing it by the vertically arranged graph with the same $|V_{AC}|$ and $|V_{BD}|$. For example, the graph T above is neither vertically nor horizontally arranged. Horizontally arranging T gives

$$
\begin{array}{c|c}
\begin{array}{l}
v_0\ \bullet \\
v_1\ \bullet \\
v_2\ \bullet \\
\text{Block}\quad v_3\ \bullet \\
\text{A}\qquad v_4\ \bullet \\
v_5 \\
v_6 \\
v_7 \\
\end{array}
&
\begin{array}{l}
v_8 \\
v_9 \\
v_{10} \\
v_{11}\qquad \text{Block} \\
v_{12}\qquad\ \text{B} \\
v_{13} \\
v_{14} \\
v_{15} \\
\end{array}
\\ \hline
\begin{array}{l}
v_{16}\ \bullet \\
v_{17}\ \bullet \\
v_{18}\ \bullet \\
\text{Block}\quad v_{19}\ \bullet \\
\text{C}\qquad v_{20}\ \bullet \\
v_{21}\ \bullet \\
v_{22}\ \bullet \\
v_{23}\ \bullet \\
\end{array}
&
\begin{array}{l}
v_{24}\ \bullet \\
v_{25} \\
v_{26} \\
v_{27}\qquad \text{Block} \\
v_{28}\qquad\ \text{D} \\
v_{29} \\
v_{30} \\
v_{31} \\
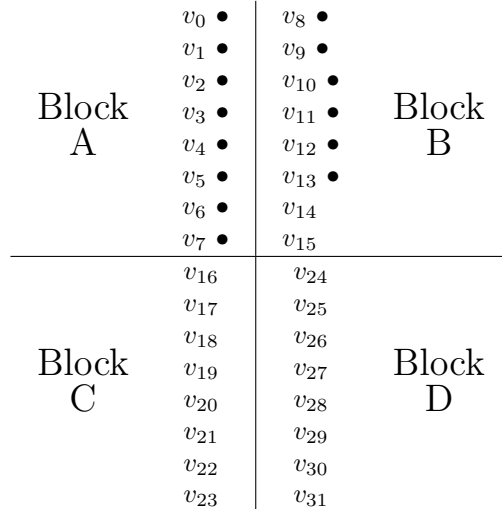\end{array}
\end{array}
$$
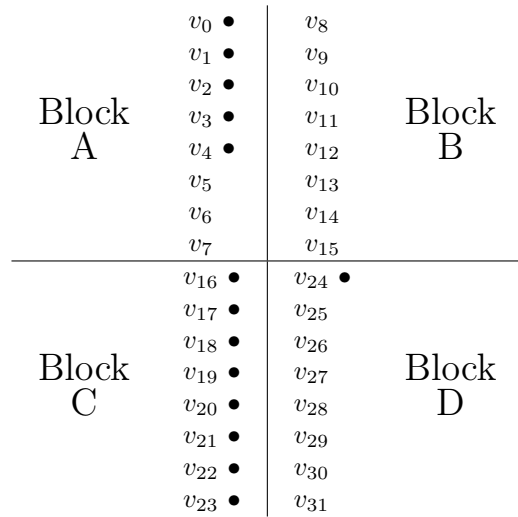
The reader can verify that horizontally arranging T increases $|E_{AB}|$ from 4 to 5, $|E_{CD}|$ from 11 to 13, and $|E^{vert}|$ from 3 to 5, and hence from equation (3) the total number of edges has increased from 18 to 23.

Vertically arranging T gives

$$
\begin{array}{c|c}
\begin{array}{l}
v_0\ \bullet \\
v_1\ \bullet \\
v_2\ \bullet \\
\text{Block}\quad v_3\ \bullet \\
\text{A}\qquad v_4 \\
v_5 \\
v_6 \\
v_7 \\
\end{array}
&
\begin{array}{l}
v_8\ \bullet \\
v_9\ \bullet \\
v_{10}\ \bullet \\
v_{11}\ \bullet\quad \text{Block} \\
v_{12}\ \bullet\quad\ \text{B} \\
v_{13}\ \bullet \\
v_{14}\ \bullet \\
v_{15}\ \bullet \\
\end{array}
\\ \hline
\begin{array}{l}
v_{16} \\
v_{17} \\
v_{18} \\
\text{Block}\quad v_{19} \\
\text{C}\qquad v_{20} \\
v_{21} \\
v_{22} \\
v_{23} \\
\end{array}
&
\begin{array}{l}
v_{24}\ \bullet \\
v_{25}\ \bullet \\
v_{26} \\
v_{27}\qquad \text{Block} \\
v_{28}\qquad\ \text{D} \\
v_{29} \\
v_{30} \\
v_{31} \\
\end{array}
\end{array}
$$

We see that vertically arranging T increases $|E_{AC}|$ from 2 to 4, $|E_{BD}|$ from 14 to 15, and $|E^{hor}|$ from 2 to 4, and hence from equation (4) the total number of edges has increased from 18 to 23.

Completely arranging T gives

$$
\begin{array}{c|c}
v_0 \bullet & v_8 \bullet \\
v_1 \bullet & v_9 \bullet \\
v_2 \bullet & v_{10} \bullet \\
\text{Block} \quad v_3 \bullet & v_{11} \bullet \quad \text{Block} \\
\text{A} \quad v_4 \bullet & v_{12} \bullet \quad \text{B} \\
v_5 \bullet & v_{13} \bullet \\
v_6 \bullet & v_{14} \\
v_7 \bullet & v_{15} \\
\hline
v_{16} & v_{24} \\
v_{17} & v_{25} \\
v_{18} & v_{26} \\
\text{Block} \quad v_{19} & v_{27} \quad \text{Block} \\
\text{C} \quad v_{20} & v_{28} \quad \text{D} \\
v_{21} & v_{29} \\
v_{22} & v_{30} \\
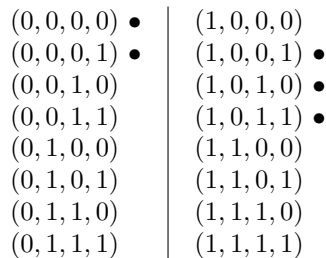v_{23} & v_{31}
\end{array}
$$

We see that completely arranging T increases the total number of edges from 18 to 25.

Notice all vertices in blocks A and B have the form $(0, x_{d-2}, \cdots, x_0)$. Let τ_{AB} be the graph isomorphism from block A $\cup$ block B onto the hypercube graph $\{0,1\}^{d-1}$ defined by the vertex mapping $(0, x_{d-2}, \cdots, x_0) \to (x_{d-2}, \cdots, x_0)$. This is equivalent to ignoring the leading coordinate of the binary expansion associated with a given vertex, and hence preserves the ordering of vertices as given by binary expansion. Similarly, define the graph isomorphism τ_{CD} from block C $\cup$ block D onto the hypercube graph $\{0,1\}^{d-1}$ by the vertex mapping $(1, x_{d-2}, \cdots, x_0) \to (x_{d-2}, \cdots, x_0)$. Notice that a given graph G is horizontally arranged by the following steps:

(i) apply τ_{AB} to the subgraph of G induced by $V_{AB}(G)$

(ii) completely arrange the resulting subgraph of $\{0,1\}^{d-1}$

(iii) apply τ_{AB}^{-1} to obtain the resulting A and B blocks

(iv) using τ_{CD} repeat the steps analogous to (i)–(iii) for the subgraph of G induced by $V_{CD}(G)$ to obtain the resulting C and D blocks

For example, to obtain the A and B blocks after horizontally arranging T, we first apply step (i) to the subgraph of T induced by $V_{AB}(T)$:

$$
\begin{array}{c|c}
(0,0,0,0) \bullet & (1,0,0,0) \\
(0,0,0,1) \bullet & (1,0,0,1) \bullet \\
(0,0,1,0) & (1,0,1,0) \bullet \\
(0,0,1,1) & (1,0,1,1) \bullet \\
(0,1,0,0) & (1,1,0,0) \\
(0,1,0,1) & (1,1,0,1) \\
(0,1,1,0) & (1,1,1,0) \\
(0,1,1,1) & (1,1,1,1)
\end{array}
$$

Then we completely arrange this graph by step (ii):

$$
\begin{array}{ll}
(0,0,0,0) \bullet & (1,0,0,0) \\
(0,0,0,1) \bullet & (1,0,0,1) \\
(0,0,1,0) \bullet & (1,0,1,0) \\
(0,0,1,1) \bullet & (1,0,1,1) \\
(0,1,0,0) \bullet & (1,1,0,0) \\
(0,1,0,1) & (1,1,0,1) \\
(0,1,1,0) & (1,1,1,0) \\
(0,1,1,1) & (1,1,1,1)
\end{array}
$$

Applying step (iii):

$$
\text{Block A} \quad
\begin{array}{ll}
(0,0,0,0,0) \bullet & (0,1,0,0,0) \\
(0,0,0,0,1) \bullet & (0,1,0,0,1) \\
(0,0,0,1,0) \bullet & (0,1,0,1,0) \\
(0,0,0,1,1) \bullet & (0,1,0,1,1) \\
(0,0,1,0,0) \bullet & (0,1,1,0,0) \\
(0,0,1,0,1) & (0,1,1,0,1) \\
(0,0,1,1,0) & (0,1,1,1,0) \\
(0,0,1,1,1) & (0,1,1,1,1)
\end{array}
\quad \text{Block B}
$$

We see these are the A and B blocks after horizontally arranging T.

We can vertically arrange a graph by a similar sequence of steps. Notice that all vertices in blocks A and C have 0 for their second coordinate. Let τ_{AC} be the graph isomorphism from block A $\cup$ block C onto the hypercube graph $\{0,1\}^{d-1}$ defined by the vertex mapping $(x_{d-1}, 0, x_{d-3}, \cdots, x_0) \to (x_{d-1}, x_{d-3}, \cdots, x_0)$. This is equivalent to ignoring the second coordinate of the binary expansion associated with a given vertex, and hence preserves the ordering of vertices as given by binary expansion. Similarly, define the graph isomorphism τ_{BD} from block B $\cup$ block D onto the hypercube graph $\{0,1\}^{d-1}$ by the vertex mapping $(x_{d-1}, 1, x_{d-3}, \cdots, x_0) \to (x_{d-1}, x_{d-3}, \cdots, x_0)$. Notice that a given graph G is vertically arranged by the following steps:

(i) apply τ_{AC} to the subgraph of G induced by $V_{AC}(G)$

(ii) completely arrange the resulting subgraph of $\{0,1\}^{d-1}$

(iii) apply τ_{AC}^{-1} to obtain the resulting A and C blocks

(iv) using τ_{BD} repeat the steps analogous to (i)–(iii) for the subgraph of G induced by $V_{BD}(G)$ to obtain the resulting B and D blocks

For example, for the graph T above, applying steps (i)–(iii) for the subgraph of T induced by $V_{AC}(T)$

we obtain the same blocks A and C from vertically arranging T:

$$
\begin{array}{lll}
(0,0,0,0)\ \bullet & (0,0,0,0)\ \bullet & (0,0,0,0,0)\ \bullet \\
(0,0,0,1)\ \bullet & (0,0,0,1)\ \bullet & (0,0,0,0,1)\ \bullet \\
(0,0,1,0) & (0,0,1,0)\ \bullet & (0,0,0,1,0)\ \bullet \\
(0,0,1,1) & (0,0,1,1)\ \bullet & \quad\text{Block}\quad (0,0,0,1,1)\ \bullet \\
(0,1,0,0) & (0,1,0,0) & \quad\ \ \text{A}\quad (0,0,1,0,0) \\
(0,1,0,1) & (0,1,0,1) & (0,0,1,0,1) \\
(0,1,1,0) & (0,1,1,0) & (0,0,1,1,0) \\
(0,1,1,1) & (0,1,1,1) & (0,0,1,1,1) \\
(1,0,0,0) & (1,0,0,0) & (1,0,0,0,0) \\
(1,0,0,1) & (1,0,0,1) & (1,0,0,0,1) \\
(1,0,1,0) & (1,0,1,0) & (1,0,0,1,0) \\
(1,0,1,1)\ \bullet & (1,0,1,1) & \quad\text{Block}\quad (1,0,0,1,1) \\
(1,1,0,0) & (1,1,0,0) & \quad\ \ \text{C}\quad (1,0,1,0,0) \\
(1,1,0,1) & (1,1,0,1) & (1,0,1,0,1) \\
(1,1,1,0) & (1,1,1,0) & (1,0,1,1,0) \\
(1,1,1,1)\ \bullet & (1,1,1,1) & (1,0,1,1,1)
\end{array}
$$

with $\xrightarrow{\text{Step(i)}}$, $\xrightarrow{\text{Step(ii)}}$, $\xrightarrow{\text{Step(iii)}}$ between columns.

Let λ_i be the graph automorphism of the hypercube graph $\{0,1\}^d$ that changes the x_i coordinate of each vertex from 0 to 1 or 1 to 0 (for instance $(x_{d-1}, x_{d-2}, \cdots, x_i = 1, \cdots, x_0)$ maps to $(x_{d-1}, x_{d-2}, \cdots, x_i = 0, \cdots, x_0)$ and vice versa). Observe that applying λ_4 in this example interchanges block A with C and block B with D. In particular, $\lambda_4(T)$ (which changes the leftmost coordinate in the graph T) is given by

$$
\begin{array}{ll}
\text{Block} & v_0 = (0,0,0,0,0) \quad\ v_8 = (0,1,0,0,0)\ \bullet \\
\text{A} & v_1 = (0,0,0,0,1) \quad\ v_9 = (0,1,0,0,1)\ \bullet \\
 & v_2 = (0,0,0,1,0) \quad\ v_{10} = (0,1,0,1,0)\ \bullet \\
 & v_3 = (0,0,0,1,1)\ \bullet \ \ v_{11} = (0,1,0,1,1)\ \bullet \quad \text{Block B} \\
 & v_4 = (0,0,1,0,0) \quad\ v_{12} = (0,1,1,0,0)\ \bullet \\
 & v_5 = (0,0,1,0,1) \quad\ v_{13} = (0,1,1,0,1)\ \bullet \\
 & v_6 = (0,0,1,1,0) \quad\ v_{14} = (0,1,1,1,0)\ \bullet \\
 & v_7 = (0,0,1,1,1)\ \bullet \ \ v_{15} = (0,1,1,1,1) \\
\text{Block} & v_{16} = (1,0,0,0,0)\ \bullet \ \ v_{24} = (1,1,0,0,0) \\
\text{C} & v_{17} = (1,0,0,0,1)\ \bullet \ \ v_{25} = (1,1,0,0,1)\ \bullet \\
 & v_{18} = (1,0,0,1,0) \quad\ v_{26} = (1,1,0,1,0)\ \bullet \\
 & v_{19} = (1,0,0,1,1) \quad\ v_{27} = (1,1,0,1,1)\ \bullet \quad \text{Block D} \\
 & v_{20} = (1,0,1,0,0) \quad\ v_{28} = (1,1,1,0,0) \\
 & v_{21} = (1,0,1,0,1) \quad\ v_{29} = (1,1,1,0,1) \\
 & v_{22} = (1,0,1,1,0) \quad\ v_{30} = (1,1,1,1,0) \\
 & v_{23} = (1,0,1,1,1) \quad\ v_{31} = (1,1,1,1,1)
\end{array}
$$

Similarly, $\lambda_3(T)$ interchanges block A with B and C with D.

Let $\sigma_{i,j}$ be the graph automorphism of $\{0,1\}^d$ that interchanges the x_i and x_j coordinates of each vertex. Note that $\sigma_{3,4}(T)$ results in interchanging blocks B and C while blocks A and D are left alone.

We will utilize the facts that for any induced subgraph G, $|E(\lambda_i(G))| = |E(G)|$ and $|E(\sigma_{i,j}(G))| = |E(G)|$ in general as $\sigma_{i,j}$ and λ_i are graph automorphisms of the hypercube graph.

We are now ready to state and prove our result.

Theorem 3.4 *Let G be an induced subgraph of $\{0,1\}^d$ with vertices adjacent if and only if they are separated by a Hamming distance 1. Then $|E(G)| \leq |E(H)|$, where H is the completely arranged graph with $|V(H)| = |V(G)|$, i.e. $V(H) = \{v_0, v_1, \cdots, v_{|V(G)|-1}\}$, where v_j is defined by binary representation as above.*
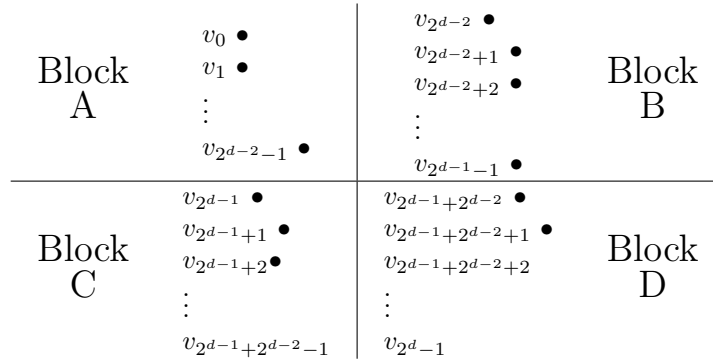
Proof We proceed by induction on d. For the base step $d = 1$, let G be an induced subgraph of $\{0,1\}^1$. Then G has 0, 1, or 2 vertices. If $|V(G)| = 0$ or 1, then $|E(G)| = 0$. If $|V(G)| = 2$, then $|E(G)| = 1$. In all cases $|E(G)| = |E(H)|$, where H is the completely arranged graph with $|V(H)| = |V(G)|$.

For the inductive step, fix $d \geq 2$, let G be an induced subgraph of $\{0,1\}^d$, and assume the conclusion holds for $d - 1$. Consider the array for $(x_{d-1}, x_{d-2}, \cdots, x_0) \in \{0,1\}^d$; there are 2^{d-2} vertices in each block:

Block A	v_0 v_1 $\vdots$ $v_{2^{d-2}-1}$	$v_{2^{d-2}}$ $v_{2^{d-2}+1}$ $\vdots$ $v_{2^{d-1}-1}$	Block B
Block C	$v_{2^{d-1}}$ $v_{2^{d-1}+1}$ $\vdots$ $v_{2^{d-1}+2^{d-2}-1}$	$v_{2^{d-1}+2^{d-2}}$ $v_{2^{d-1}+2^{d-2}+1}$ $\vdots$ v_{2^d-1}	Block D

We make the following observations using our inductive hypothesis. Recall that horizontally arranging G can be accomplished by first applying τ_{AB} to $V_{AB}(G)$, completely arranging the resulting subgraph of $\{0,1\}^{d-1}$, applying τ_{AB}^{-1} to the result, and then doing the analogous sequence of moves applied to $V_{CD}(G)$. Since τ_{AB} is a graph isomorphism and the inductive hypothesis tells us that subgraphs of $\{0,1\}^{d-1}$ can be completely arranged without decreasing the number of edges, we see that the number of edges on the subgraph of G induced by $V_{AB}(G)$ does not decrease from these moves. In other words, horizontally arranging G does not decrease $|E_{AB}|$, and by the same reasoning $|E_{CD}|$ does not decrease either. Since we have equality in equation (2) in a horizontally arranged graph, it follows from equation (3) that horizontally arranging G does not decrease the total number of edges. Since vertically arranging G can similarly be accomplished by applying graph isomorphisms on $V_{AC}(G)$ and $V_{BD}(G)$, and completely arranging subgraphs of $\{0,1\}^{d-1}$, we see that vertically arranging G does not decrease $|E_{AC}|$ or $|E_{BD}|$, and from equations (1) and (4) we see that vertically arranging G does not decrease the total number of edges.

We proceed by applying a sequence of operations that does not decrease the number of edges in the graph and results in the completely arranged graph with the same number of vertices. We first vertically arrange G. If $|V_{AC}| < |V_{BD}|$, apply λ_{d-2} (interchanging block A with B and block C with D). This results in a vertically arranged graph G' with $|V_{AC}| \geq |V_{BD}|$, i.e. of the form:

Block A	v_0 • v_1 • $\vdots$ $v_{2^{d-2}-1}$ •	$v_{2^{d-2}}$ • $v_{2^{d-2}+1}$ • $v_{2^{d-2}+2}$ • $\vdots$ $v_{2^{d-1}-1}$ •	Block B
Block C	$v_{2^{d-1}}$ • $v_{2^{d-1}+1}$ • $v_{2^{d-1}+2}$ • $\vdots$ $v_{2^{d-1}+2^{d-2}-1}$	$v_{2^{d-1}+2^{d-2}}$ • $v_{2^{d-1}+2^{d-2}+1}$ • $v_{2^{d-1}+2^{d-2}+2}$ $\vdots$ v_{2^d-1}	Block D

We now proceed by cases on the form of G'.

Case 1: Block B is full

By the form of G', we know block A is full. We can then horizontally arrange, which results in a completely arranged graph.
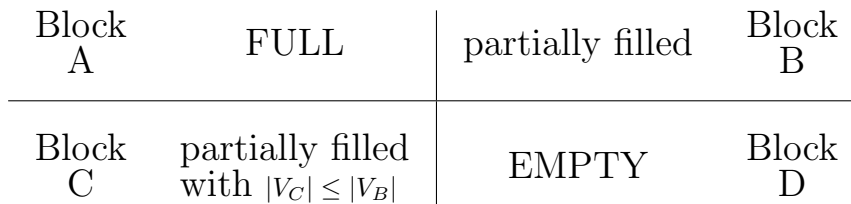
Case 2: Block B is not full

By the form of G', we know block D is empty.

2(a): Block C is empty

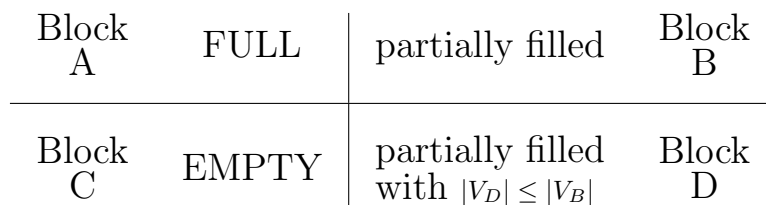Horizontally arranging results in a completely arranged graph.

2(b): Block C is nonempty

By the form of G' we know block A is full. If $|V_B| < |V_C|$, we apply $\sigma_{d-2,d-1}$ (interchanging blocks B and C). If this results in B full or C empty the graph is completely arranged (since A is full and D is empty here). Otherwise, we now have the vertically arranged form:

Block A	FULL	partially filled	Block B
Block C	partially filled with $|V_C| \le |V_B|$	EMPTY	Block D

From equation (3) we see that $|E| = |E_{AB}| + |E_{CD}| + |E^{vert}| = |E_{AB}| + |E_{CD}| + |V_C|$ in this form. We now interchange blocks C and D giving us a graph of the form:

Block A	FULL	partially filled	Block B
Block C	EMPTY	partially filled with $|V_D| \le |V_B|$	Block D

Clearly this operation did not change $|E_{AB}|$ and $|E_{CD}|$ and observe that we now have $|E| = |E_{AB}| + |E_{CD}| + |E^{vert}| = |E_{AB}| + |E_{CD}| + |V_D|$ since $|V_D| \le |V_B|$. Thus we see that the number of edges was preserved by this operation.

We now vertically arrange the graph. If this does not completely fill block B, we have both blocks C and D empty and the graph is completely arranged. If this does completely fill block B we can horizontally arrange, which results in a completely arranged graph.

We see that in all cases the graph can be completely arranged without reducing the number of edges. □

Corollary 3.5 *Let $G = (V, E)$ be a finite induced subgraph of some standard integer lattice $\mathbb{Z}^m$ with $|V| = n$. Let $2^{d-1} < n \le 2^d$. Then $|E(G)| \le |E(H)|$, where H is the completely arranged graph on $\{0, 1\}^d$ with $|V(H)| = n$, i.e. $V(H) = \{v_0, v_1, \cdots, v_{n-1}\}$, where v_j is defined by binary representation as above.*

Proof This follows from corollary 3.2 and theorem 3.4. □

Therefore, the completely arranged graphs are extremal configurations that maximize edge count in $\{0, 1\}^\infty$ and $\mathbb{Z}^\infty$ for a given number of vertices. Thus counting their edges gives us $T(n)$. For a natural number j, let $H(j)$ be the number of nonzero digits in the binary expansion of j (Hamming weight) and make the following observation: starting with the graph with vertex set $V = \{v_0, v_1, \ldots, v_{k-1}\}$, if the single vertex v_k is added the number of edges increases by $H(k)$, i.e. $\nabla T(k) = T(k+1) - T(k) = H(k)$. To see this, note that if we replace a single 1 in v_k by 0 we obtain $v_j \in V$, whereas if we replace a single 0 in v_k by 1 we obtain $v_j \notin V$. We then have $T(n) = T(n) - T(1) = \sum_{k=1}^{n-1} \nabla T(k) = \sum_{k=0}^{n-1} H(k)$. We will use this to obtain an exact expression for $T(n)$.

Theorem 3.6 *(a) $T(2^d) = d2^{d-1}$ for $d \in \mathbb{N}$.*

(b) More generally, if $n = \sum_{j=1}^{t} 2^{k_j}$, where $k_1 > k_2 > \cdots > k_t \ge 0$, then $T(n) = \sum_{j=1}^{t} (k_j 2^{k_j - 1} + (j-1)2^{k_j})$.

(c) For all $n \in \mathbb{N}$, $n(\lceil \log n \rceil - 1)/4 < T(n) < n \lceil \log n \rceil$, where $\log$ is the base 2 logarithm.

Proof

(a) If $n = 2^d$, we note that each vertex in the $d-$dimensional unit hypercube has degree d, and so the sum of degrees over all vertices is $d2^d$; then $T(n) = d2^{d-1}$ follows from the handshaking theorem.

(b) For $2^{d-1} < n \le 2^d$ and $n = \sum_{j=1}^{t} 2^{k_j}$, where $k_1 > k_2 > \cdots > k_t \ge 0$, define $s_r = \sum_{j=1}^{r} 2^{k_j}$. We have:

$$T(n) = \sum_{j=0}^{n-1} H(j) = \sum_{j=0}^{s_1 - 1} H(j) + \sum_{j=s_1}^{s_2 - 1} H(j) + \cdots + \sum_{j=s_{t-1}}^{s_t - 1} H(j)$$

We now notice that for $s_{r-1} \le j \le s_r - 1$, $H(j) = r - 1 + H(j - s_{r-1})$ and so

$$\sum_{j=s_{r-1}}^{s_r - 1} H(j) = \sum_{j=s_{r-1}}^{s_r - 1} (r - 1 + H(j - s_{r-1})) = \sum_{m=0}^{2^{k_r} - 1} (r - 1 + H(m)) = (r-1)2^{k_r} + k_r 2^{k_r - 1}$$

Here we used that $T(2^k) = \sum_{m=0}^{2^k - 1} H(m) = k2^{k-1}$ by part (a).

Then

$$
\begin{aligned}
T(n) &= \sum_{j=0}^{s_1-1} H(j) + \sum_{j=s_1}^{s_2-1} H(j) + \cdots + \sum_{j=s_{t-1}}^{s_t-1} H(j) \\
&= k_1 2^{k_1-1} + 2^{k_2} + k_2 2^{k_2-1} + 22^{k_3} + k_3 2^{k_3-1} + \cdots + (t-1)2^{k_t} + k_t 2^{k_t-1} \\
&= \sum_{j=1}^{t} (k_j 2^{k_j-1} + (j-1)2^{k_j}).
\end{aligned}
$$

(c) Let d be an integer. Note that $d = \lceil \log n \rceil \Leftrightarrow d - 1 < \log n \leq d \Leftrightarrow 2^{d-1} < n \leq 2^d$. Then since $T(n)$ is clearly strictly monotonically increasing we have $(\lceil \log n \rceil - 1)n/4 \leq (d-1)2^{d-2} = T(2^{d-1}) < T(n) \leq T(2^d) = d2^{d-1} < n \lceil \log n \rceil$. $\qquad \square$

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