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Some properties of a class of analytic functions defined by generalized Struve functions

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Abstract: The aim of this paper is to define a new operator by using the generalized Struve functions $\sum_{n=0}^{\infty}\frac{(-c/4)^n}{(3/2)_n(k)_n}z^{n+1}$ with $k = p+ (b+2)/2 \neq 0,-1,-2,\dots$ and $b,c,k \in \mathbb{C}$. By using this operator we define a subclass of analytic functions. We discuss some properties of this class such as inclusion problems, radius problems, and some other interesting properties related to this operator.

Key words: Analytic functions, subordination, generalized Struve functions

1. Introduction

Let A be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $E = \{z : |z| < 1\}$. A function f is said to be subordinate to a function g written as $f \prec g$, if there exists a Schwarz function w with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular, if g is univalent in E, then $f(0) = g(0)$ and $f(E) \subset g(E)$.

For any two analytic functions $f(z)$ and $g(z)$ with

$$f(z) = \sum_{n=0}^{\infty} b_n z^{n+1} \text{ and } g(z) = \sum_{n=0}^{\infty} c_n z^{n+1},\ z \in E,$$

the convolution (Hadamard product) is given by

$$(f * g)(z) = \sum_{n=0}^{\infty} b_n c_n z^{n+1},\ z \in E.$$

Consider the following second-order inhomogeneous differential equation and see [16] for more details:

$$z^2 w''(z) + z w'(z) + \left(z^2 - p^2\right) w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\,\Gamma(p+1/2)}. \tag{1.2}$$

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The solution of the homogeneous part is a Bessel function of order p, where p is a real or complex number. The particular solution of the inhomogeneous equation defined in (1.2) is called the Struve function of order p. It is defined as

$$H_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+p+1}}{\Gamma(n+3/2)\Gamma(p+n+3/2)}. \tag{1.3}$$

Now we consider the differential equation

$$z^2 w''(z) + zw'(z) - \left(z^2+p^2\right) w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+1/2)}. \tag{1.4}$$

Equation (1.4) differs from equation (1.2) in the coefficients of $w(z)$. Its particular solution is called the modified Struve function of order p and is given as

$$L_p(z) = -ie^{-ip\pi/2} H_p(iz) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+p+1}}{\Gamma(n+3/2)\Gamma(p+n+3/2)}.$$

Again consider the second-order inhomogenous differential equation

$$z^2 w''(z) + bzw'(z) + \left[cz^2 - p^2 + (1-b)p\right] w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+b/2)}, \tag{1.5}$$

where $b, c, p \in \mathbb{C}$. Equation (1.5) generalizes equations (1.2) and (1.4). In particular for $b=1$, $c=1$, we obtain (1.2) and for $b=1$, $c=-1$, we obtain (1.4). Its particular solution has the series form

$$M_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n (z/2)^{2n+p+1}}{\Gamma(n+3/2)\Gamma(p+n+(b+2)/2)} \tag{1.6}$$

and is called the generalized Struve function of order p. This series is convergent everywhere but not univalent in the open unit disk E. We take the transformation

$$N_{p,b,c}(z) = 2^p\sqrt{\pi}\Gamma(p+(b+2)/2)\, z^{(-p-1)/2} M_{p,b,c}\left(\sqrt{z}\right) = \sum_{n=0}^{\infty} \frac{(-c/4)^n z^n}{(3/2)_n (k)_n}, \tag{1.7}$$

where $k = p+(b+2)/2 \neq 0, -1, -2, \ldots$ and $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \gamma(\gamma+1)\ldots(\gamma+n-1)$. This function is analytic in the whole complex plane and satisfies the differential equation

$$4z^2 w''(z) + 2(2p+b+3) zw'(z) + [cz + 2p + b] w(z) = 2p + b.$$

Some geometric properties such as univalency, starlikeness, convexity, and close-to-convexity of the function $N_{p,b,c}(z)$ were studied recently by Orhan and Yağmur [10] and Yağmur and Orhan [14, 15].

Dziok and Srivastava [3, 4] defined the linear operator H by using the generalized hypergeometric functions and it is given as $H(\alpha_1, \ldots \alpha_s; \beta_1, \ldots \beta_q) : A \to A$ with $\alpha_i \in \mathbb{C}\,(i = 1, 2, \ldots, s)$ and $\beta_i \in \mathbb{C}\backslash\mathbb{Z}_0^-\,(i = 1, 2, \ldots, q)$ such that

$$H(\alpha_1, \ldots \alpha_s; \beta_1, \ldots \beta_q) f(z) = z\, {}_sF_q(\alpha_1, \ldots \alpha_s; \beta_1, \ldots \beta_q; z) * f(z),$$

where

$$ {}_sF_q\left(\alpha_1,\dots\alpha_s;\beta_1,\dots\beta_q;z\right)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\dots(\alpha_s)_n\,z^n}{(\beta_1)_n\dots(\beta_q)_n\,n!},\quad s\leq q+1;s,q\in\mathbb{N}_0=\mathbb{N}\cup\{0\} $$

is the generalized hypergeometric function. Baricz et al. [2] used a similar argument to define a convolution operator $B_k^c : A \to A$ by using generalized Bessel functions and it is given as

$$ B_k^c f(z)=\varphi_{k,c}(z)*f(z)=z+\sum_{n=1}^{\infty}\frac{(-c/4)^n\,a_{n+1}z^{n+1}}{(k)_n\,n!},\quad \left(k=p+\frac{b+1}{2}\notin\mathbb{Z}_0^-,c\in\mathbb{C}\right), $$

where

$$ \varphi_{k,c}(z)=z+\sum_{n=1}^{\infty}\frac{(-c/4)^n\,z^{n+1}}{(k)_n\,n!}. $$

For some refrences for convolution operators see [11, 12, 13].

Now using (1.7), we define the following convolution operator. Let

$$ \varphi_{p,b,c}(z)=2^p\sqrt{\pi}\Gamma\left(p+(b+2)/2\right)z^{(-p+1)/2}M_{p,b,c}\left(\sqrt{z}\right)=z+\sum_{n=1}^{\infty}\frac{(-c/4)^n\,z^{n+1}}{(3/2)_n\,(k)_n}. $$

Then

$$ S_k^c f(z)=\varphi_{p,b,c}(z)*f(z)=z+\sum_{n=1}^{\infty}\frac{(-c/4)^n\,a_{n+1}z^{n+1}}{(3/2)_n\,(k)_n}\quad\left(k=p+\frac{b+2}{2}\notin\mathbb{Z}_0^-,b,c,p\in\mathbb{C}\right). \tag{1.8} $$

It can easily be seen that

$$ z\left(S_{k+1}^c f(z)\right)'=kS_k^c f(z)-(k-1)\,S_{k+1}^c f(z). \tag{1.9} $$

Special cases

(i) For $b=1,\ c=1,$ we have the operator $\mathcal{S}_p : A \to A$ related with the Struve function of order p. It is given as

$$ \begin{aligned} \mathcal{S}_p f(z) &= \varphi_{p,1,1}(z)*f(z)=\left[2^p\sqrt{\pi}\Gamma(p+3/2)\,z^{(-p+1)/2}M_{p,1,1}\left(\sqrt{z}\right)\right]*f(z)\\ &= z+\sum_{n=1}^{\infty}\frac{(-1/4)^n\,a_{n+1}z^{n+1}}{(3/2)_n\,(p+3/2)_n} \end{aligned} $$

and the recursive relation

$$ z\left[\mathcal{S}_{p+1}f(z)\right]'=(p+3/2)\,\mathcal{S}_p f(z)-(p+1/2)\,\mathcal{S}_{p+1}f(z) $$

holds.

(ii) For $b=1,\ c=-1,$ we obtain the operator $\mathfrak{S}_p : A \to A$ related with the modified Struve function of order p. It is given as

$$ \begin{aligned} \mathfrak{S}_p f(z) &= \varphi_{p,1,-1}(z)*f(z)=\left[2^p\sqrt{\pi}\Gamma(p+3/2)\,z^{(-p+1)/2}M_{p,1,-1}\left(\sqrt{z}\right)\right]*f(z)\\ &= z+\sum_{n=1}^{\infty}\frac{(1/4)^n\,a_{n+1}z^{n+1}}{(3/2)_n\,(p+3/2)_n} \end{aligned} $$

and the recursive relation

$$z\left[\mathfrak{S}_{p+1}f\left(z\right)\right]'=\left(p+3/2\right)\mathfrak{S}_{p}f\left(z\right)-\left(p+1/2\right)\mathfrak{S}_{p+1}f\left(z\right)$$

holds.

We define the following class of analytic functions by using the operator $S_k^c f\left(z\right)$.

Definition 1.1 *Let $f\in A$. Then $f\in N_{k,c}^{\alpha}\left(\lambda,\mu,\phi\right)$ for $0<\mu<1,\lambda\in\mathbb{C},k=p+(b+2)/2\neq 0,-1,-2,\ldots,b,c,p\in\mathbb{C}$, and $|\alpha|<\frac{\pi}{2}$, if and only if*

$$e^{i\alpha}\left\{(1+\lambda)\left(\frac{z}{S_{k+1}^c f\left(z\right)}\right)^{\mu}-\lambda\frac{S_k^c f\left(z\right)}{S_{k+1}^c f\left(z\right)}\left(\frac{z}{S_{k+1}^c f\left(z\right)}\right)^{\mu}\right\}\prec\cos\alpha\phi\left(z\right)+i\sin\alpha, \tag{1.10}$$

where $\phi\left(z\right)$ is a convex univalent function with $\phi\left(0\right)=1$.

(i) For $\phi\left(z\right)=\frac{1+Az}{1+Bz},-1\leq B<A\leq 1$, we have the class $N_{k,c}^{\alpha}\left(\lambda,\mu,\frac{1+Az}{1+Bz}\right)$, which consists of functions f such that

$$J\left(\alpha,c,k,f\left(z\right)\right)\prec\frac{1+Az}{1+Bz},$$

where

$$J\left(\alpha,c,k,f\left(z\right)\right)=\frac{1}{\cos\alpha}\left[e^{i\alpha}\left\{(1+\lambda)\left(\frac{z}{S_{k+1}^c f\left(z\right)}\right)^{\mu}-\lambda\frac{S_k^c f\left(z\right)}{S_{k+1}^c f\left(z\right)}\left(\frac{z}{S_{k+1}^c f\left(z\right)}\right)^{\mu}\right\}-i\sin\alpha\right].$$

(ii) For $\phi\left(z\right)=\frac{1+z}{1-z}$, we have the class $N_{k,c}^{\alpha}\left(\lambda,\mu,\frac{1+z}{1-z}\right)$. That is, $f\in N_{k,c}^{\alpha}\left(\lambda,\mu,\frac{1+z}{1-z}\right)$ if

$$J\left(\alpha,c,k,f\left(z\right)\right)\prec\frac{1+z}{1-z}.$$

Since it is well known that for a function $p\left(z\right)\prec\frac{1+z}{1-z}$, then $\mathrm{Re}p\left(z\right)>0$. This implies that $f\in N_{k,c}^{\alpha}\left(\lambda,\mu,\frac{1+z}{1-z}\right)$ if

$$\mathrm{Re}J\left(\alpha,c,k,f\left(z\right)\right)>0.$$

Lemma 1.2 *[8] Let F be analytic and convex in E. If $f,\ g\in A$ and $f,\ g\prec F$, then*

$$\sigma f+\left(1-\sigma\right)g\prec F,\quad 0\leq\sigma\leq 1.$$

Lemma 1.3 *[6] Let h be convex in E with $h(0)=a$ and $\beta\in\mathbb{C}$ such that $Re\beta\geq 0$. If $p\in H\left[a,n\right]$ and*

$$p(z)+\frac{zp'(z)}{\beta}\prec h\left(z\right),$$

then $p(z)\prec q\left(z\right)\prec h\left(z\right)$, where

$$q\left(z\right)=\frac{\beta}{nz^{\beta/n}}\int_0^z h\left(t\right)t^{\beta/n-1}dt$$

and $q\left(z\right)$ is the best dominant.

Lemma 1.4 *[1]. Let a, b, and $c \neq 0, -1, -2\ldots$ be complex numbers. Then, for $Rec > Reb > 0$,*

$$\begin{aligned}
(i) \quad {}_2F_1(a,b,c;z) &= \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt, \\
(ii) \quad {}_2F_1(a,b,c;z) &= {}_2F_1(b,a,c;z), \\
(iii) \quad {}_2F_1(a,b,c;z) &= (1-z)^{-a}\ {}_2F_1\left(a, c-b, c; \frac{z}{z-1}\right).
\end{aligned}$$

Lemma 1.5 *[7] Let $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$. Then*

$$\frac{1+A_2 z}{1+B_2 z} \prec \frac{1+A_1 z}{1+B_1 z}.$$

Lemma 1.6 *[9] Let the function $g(z)$ be analytic and univalent in E and let the functions $\theta(w)$ and $\varphi(w)$ be analytic in a domain D containing $g(E)$, with $\theta(w) \neq 0$ $(w \in g(E))$. Set $Q(z) = zg'(z)\varphi(g(z))$ and $h(z) = \theta(g(z)) + Q(z)$ and suppose that*

(i) $Q(z)$ is univalently starlike in E;

(ii) $Re\frac{zh'(z)}{Q(z)} = Re\left\{\frac{\theta'(g(z))}{\varphi(g(z))} + \frac{zQ'(z)}{Q(z)}\right\} > 0$ $(z \in E)$. If $q(z)$ is analytic in E with $q(0) = g(0), q(E) \subset D$ and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z) \ (z \in E),$$

then $q(z) \prec g(z)$ $(z \in E)$ and $g(z)$ is the best dominant.

2. Main results

Theorem 2.1 *Let $f \in N_{k,c}^{\alpha}(\lambda, \mu, \phi)$. Then for $Re\frac{\mu k}{\lambda} \geq 0$,*

$$e^{i\alpha}\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} \prec \frac{\mu k}{\lambda}\cos\alpha z^{-\frac{\mu k}{\lambda}}\int_0^z \phi(t)\, t^{\frac{\mu k}{\lambda}-1}dt + i\sin\alpha \prec (\cos\alpha)\,\phi(z) + i\sin\alpha.$$

This result is the best possible.

Proof Consider

$$p(z) = \frac{1}{\cos\alpha}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} - i\sin\alpha\right\}. \tag{2.1}$$

Then p is analytic in E with $p(0) = 1$. Therefore, we have

$$e^{i\alpha}\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} = (\cos\alpha)\,p(z) + i\sin\alpha.$$

Differentiating both sides and using (1.9) and simplifying, we obtain

$$\frac{\lambda(\cos\alpha)\,zp'(z)}{\mu k} = \lambda e^{i\alpha}\left\{\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} - \frac{S_k^c f(z)}{S_{k+1}^c f(z)}\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu}\right\}.$$

It follows from the above equation and (2.1) that

$$\begin{aligned} & p(z)+\frac{\lambda}{\mu k} z p'(z) \\ = & \frac{1}{\cos \alpha}\left[e^{i\alpha}\left\{(1+\lambda)\left(\frac{z}{S_{k+1}^{c} f(z)}\right)^{\mu}-\lambda \frac{S_{k}^{c} f(z)}{S_{k+1}^{c} f(z)}\left(\frac{z}{S_{k+1}^{c} f(z)}\right)^{\mu}\right\}-i \sin \alpha\right]. \end{aligned}$$

Since $f \in N_{k,c}^{\alpha}(\lambda, \mu, \phi)$, therefore

$$p(z)+\frac{\lambda}{\mu k} z p'(z) \prec \phi(z).$$

Now using Lemma 1.3 for $\beta=\frac{\mu k}{\lambda}$ with $Re \frac{\mu k}{\lambda} \geq 0,$ we obtain the required result. □

Corollary 2.2 *Let* $f \in N_{k,c}^{\alpha}\left(\lambda, \mu, \frac{1+Az}{1+Bz}\right)$. *Then for* $k, \lambda \in \mathbb{R}$ *and* $\frac{\mu k}{\lambda} \geq 0$,

$$e^{i\alpha}\left(\frac{z}{S_{k+1}^{c} f(z)}\right)^{\mu} \prec h(z) \cos \alpha+i \sin \alpha,$$

where

$$h(z)=\begin{cases} \frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}\ {}_{2}F_{1}\left(1,1, \frac{\mu k}{\lambda}+1 ; \frac{Bz}{1+Bz}\right), & B \neq 0, \\ 1+\frac{\mu k}{\mu k+\lambda} A z, & B=0. \end{cases}$$

Furthermore,

$$Re\left[e^{i\alpha}\left(\frac{z}{S_{k+1}^{c} f(z)}\right)^{\mu}\right]>(\cos \alpha) h(-1).$$

Proof Since $f \in N_{k,c}^{\alpha}\left(\lambda, \mu, \frac{1+Az}{1+Bz}\right)$, therefore from Theorem 2.1, we have

$$e^{i\alpha}\left(\frac{z}{S_{k+1}^{c} f(z)}\right)^{\mu} \prec \frac{\mu k}{\lambda}(\cos \alpha) z^{-\frac{\mu k}{\lambda}} \int_{0}^{z} \frac{1+At}{1+Bt} t^{\frac{\mu k}{\lambda}-1} dt+i \sin \alpha. \tag{2.2}$$

Putting $t=zu$ and after simple calculations, one can get

$$e^{i\alpha}\left(\frac{z}{S_{k+1}^{c} f(z)}\right)^{\mu} \prec\left\{\frac{A}{B}+\frac{\mu k}{\lambda}\left(1-\frac{A}{B}\right) \int_{0}^{1}(1+Buz)^{-1} u^{\frac{\mu k}{\lambda}-1} dt\right\} \cos \alpha+i \sin \alpha.$$

Now using Lemma 1.4 for $a=1,\ b=\frac{\mu k}{\lambda},\ c=b+1$, and $B \neq 0,$ we obtain

$$\begin{aligned} & e^{i\alpha}\left(\frac{z}{S_{k+1}^{c} f(z)}\right)^{\mu} \\ \prec & \left(\frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}\ {}_{2}F_{1}\left(1,1, \frac{\mu k}{\lambda}+1 ; \frac{Bz}{1+Bz}\right)\right) \cos \alpha+i \sin \alpha. \end{aligned}$$

For the case of $B = 0$, it can easily be followed from (2.2) that

$$\begin{aligned} e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} &\prec \left(\frac{\mu k}{\lambda}\int_{0}^{1}(1+Atz)\,t^{\frac{\mu k}{\lambda}-1}dt\right)\cos\alpha+i\sin\alpha. \\ &= \frac{\mu k}{\lambda}\left\{\left(\int_{0}^{1}t^{\frac{\mu k}{\lambda}-1}dt\right)+\int_{0}^{1}Azt^{\frac{\mu k}{\lambda}}dt\right\}\cos\alpha+i\sin\alpha. \\ &= \left\{1+\frac{\mu k}{\mu k+\lambda}Az\right\}\cos\alpha+i\sin\alpha. \end{aligned}$$

Now we have to prove that $\mathrm{Re}\left[e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right] > (\cos\alpha)\,h(-1)$. From (2.2), we can have this relation by using subordination

$$\frac{1}{\cos\alpha}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}-i\sin\alpha\right\}=h(w(z)),$$

where $h(z)=\frac{\mu k}{\lambda}z^{-\frac{\mu k}{\lambda}}\int_{0}^{z}\frac{1+At}{1+Bt}t^{\frac{\mu k}{\lambda}-1}dt$. Therefore,

$$\begin{aligned} \mathrm{Re}\left[\frac{1}{\cos\alpha}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\}\right] &= \mathrm{Re}\frac{\mu k}{\lambda}\int_{0}^{1}\frac{1+Atw(z)}{1+Btw(z)}t^{\frac{\mu k}{\lambda}-1}dt \\ &> \frac{\mu k}{\lambda}\int_{0}^{1}\frac{1-At}{1-Bt}t^{\frac{\mu k}{\lambda}-1}dt \\ &= h(-1). \end{aligned}$$

To show that this result is sharp, we have to prove that $\inf\limits_{|z|<1}\{\mathrm{Re}h(z)\}=h(-1)$. Now

$$\mathrm{Re}h(z)\geq\frac{\mu k}{\lambda}\int_{0}^{1}t^{\frac{\mu k}{\lambda}-1}\frac{1-Atr}{1-Btr}dt=h(-r).$$

Therefore, $h(-r)\to h(-1)$ as $r\to 1^{-}$. □

Theorem 2.3 *Let* $e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\prec\phi(z)\cos\alpha+i\sin\alpha$ *with* $\phi(z)=\frac{1+z}{1-z}$. *Then* $f\in N_{k,c}^{\alpha}(\lambda,\mu,\phi(z))$ *for* $|z|=r<-c+\sqrt{c^{2}+1}$, *where* $c=\left|\frac{\lambda}{\mu k}\right|$.

Proof Let

$$e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}=p(z)\cos\alpha+i\sin\alpha,$$

where $p(z) \prec \frac{1+z}{1-z}$. Then from Theorem 2.1, we have

$$\begin{aligned} & p(z) + \frac{\lambda}{\mu k} z p'(z) \\ = & \frac{1}{\cos\alpha}\left[e^{i\alpha}\left\{(1+\lambda)\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} - \lambda\frac{S_{k}^{c}f(z)}{S_{k+1}^{c}f(z)}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\} - i\sin\alpha\right]. \end{aligned}$$

Since $p(z) \prec \frac{1+z}{1-z}$, then it is well known (see [5]) that:

$$\frac{1-r}{1+r} \leq \mathrm{Re}p(z) \leq |p(z)| \leq \frac{1+r}{1-r} \text{ and } |zp'(z)| \leq \frac{2r\mathrm{Re}p(z)}{1-r^2}. \tag{2.3}$$

Thus, we have

$$\begin{aligned} & \mathrm{Re}\frac{1}{\cos\alpha}\left[e^{i\alpha}\left\{(1+\lambda)\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} - \lambda\frac{S_{k}^{c}f(z)}{S_{k+1}^{c}f(z)}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\} - i\sin\alpha\right] \\ \geq & \mathrm{Re}p(z) - \left|\frac{\lambda}{\mu k}\right| |zp'(z)|. \end{aligned}$$

Using (2.3), we obtain

$$\begin{aligned} & \mathrm{Re}\frac{1}{\cos\alpha}\left[e^{i\alpha}\left\{(1+\lambda)\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} - \lambda\frac{S_{k}^{c}f(z)}{S_{k+1}^{c}f(z)}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\} - i\sin\alpha\right] \\ \geq & \mathrm{Re}p(z) - \frac{2cr\mathrm{Re}p(z)}{1-r^2} \\ = & \mathrm{Re}p(z)\frac{1-r^2-2cr}{1-r^2}. \end{aligned}$$

Since $p(z) \prec \frac{1+z}{1-z}$, therefore $\mathrm{Re}p(z) > 0$. This implies that $f \in N_{k,c}^{\alpha}(\lambda, \mu, \phi(z))$ for $r < -c + \sqrt{c^2+1}$. This result is sharp for the function $p(z) = \frac{1+z}{1-z}$. □

Theorem 2.4 *Let* $0 < \mu < 1$, $k = p + (b+2)/2 \neq 0, -1, -2, \ldots, b, c, p \in \mathbb{C}$. *Then*

$$N_{k,c}^{0}(\lambda_2, \mu, \phi) \subset N_{k,c}^{0}(\lambda_1, \mu, \phi), \; 0 \leq \lambda_1 < \lambda_2.$$

Proof Since $f \in N_{k,c}^{\alpha}(\lambda_2, \mu, \phi)$, therefore we have

$$h_1(z) = (1+\lambda_2)\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} - \lambda_2\frac{S_{k}^{c}f(z)}{S_{k+1}^{c}f(z)}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} \prec \phi(z).$$

From Theorem 2.1 for $\alpha = 0$, we write

$$h_2(z) = \left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} \prec \phi(z), \; z \in E.$$

Now for $\lambda_1 \geq 0$, we obtain

$$\begin{aligned} & (1+\lambda_1)\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} - \lambda_1 \frac{S_k^c f(z)}{S_{k+1}^c f(z)}\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} \\ = & (1-\frac{\lambda_1}{\lambda_2})\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} + \\ & \frac{\lambda_1}{\lambda_2}\left\{(1+\lambda_2)\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} - \lambda_2 \frac{S_k^c f(z)}{S_{k+1}^c f(z)}\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu}\right\} \\ = & \frac{\lambda_1}{\lambda_2} h_1(z) + (1-\frac{\lambda_1}{\lambda_2}) h_2(z). \end{aligned}$$

Using the convexity of the class of the functions $\phi(z)$ and Lemma 1.2, we write

$$\frac{\lambda_1}{\lambda_2} h_1(z) + (1-\frac{\lambda_1}{\lambda_2}) h_2(z) \prec \phi(z), \;\; z \in E,$$

and this implies that $f \in N_{k,c}^0(\lambda_1, \mu, \phi)$. Hence, the proof of the theorem is complete. □

Corollary 2.5 *Let* $0 < \mu < 1,\; k = p + (b+2)/2 \neq 0, -1, -2, \ldots, b, c, p \in \mathbb{C}$. *Then for* $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$,

$$N_{k,c}^0\left(\lambda_2, \mu, \frac{1+A_2 z}{1+B_2 z}\right) \subset N_{k,c}^0\left(\lambda_1, \mu, \frac{1+A_1 z}{1+B1 z}\right), \;\; 0 \leq \lambda_1 < \lambda_2, \; z \in E.$$

Proof Let $f \in N_{k,c}^0\left(\lambda_2, \mu, \frac{1+A_2 z}{1+B_2 z}\right)$. Then

$$h_1(z) = (1+\lambda_2)\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} - \lambda_2 \frac{S_k^c f(z)}{S_{k+1}^c f(z)}\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} \prec \frac{1+A_2 z}{1+B_2 z}.$$

Since $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, therefore by Lemma 1.5, we have

$$h_1(z) = (1+\lambda_2)\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} - \lambda_2 \frac{S_k^c f(z)}{S_{k+1}^c f(z)}\left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} \prec \frac{1+A_1 z}{1+B_1 z}.$$

Theorem 2.1 implies for $\phi(z) = \frac{1+A_1 z}{1+B_1 z}$ that

$$h_2(z) = \left(\frac{z}{S_{k+1}^c f(z)}\right)^{\mu} \prec \frac{1+A_1 z}{1+B_1 z}.$$

Now for $\lambda_2 > \lambda_1 \geq 0,$

$$\begin{aligned} & (1+\lambda_1)\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} - \lambda_1 \frac{S_k^c f(z)}{S_{k+1}^{c}f(z)}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} \\ = \; & (1-\frac{\lambda_1}{\lambda_2})\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} + \\ & \frac{\lambda_1}{\lambda_2}\left\{(1+\lambda_2)\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} - \lambda_2 \frac{S_k^c f(z)}{S_{k+1}^{c}f(z)}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\} \\ = \; & \frac{\lambda_1}{\lambda_2}h_1(z) + (1-\frac{\lambda_1}{\lambda_2})h_2(z). \end{aligned}$$

Using the convexity of the function $\frac{1+A_1 z}{1+B_1 z}$ with Lemma 1.2, we write

$$\frac{\lambda_1}{\lambda_2}h_1(z) + (1-\frac{\lambda_1}{\lambda_2})h_2(z) \prec \frac{1+A_1 z}{1+B_1 z}, \;\; z \in E,$$

and this implies that $f \in N_{k,c}^{0}\left(\lambda_1, \mu, \frac{1+A_1 z}{1+B_1 z}\right).$ □

Theorem 2.6 *Let* $f \in N_{k,c}^{0}(\lambda, \mu, \phi)$, $0 < \mu < 1$, $k = p + (b+2)/2 \neq 0, -1, -2, \ldots, b, c, p \in \mathbb{C}$ *and* $\lambda \leq -1$. *Then*

$$\frac{S_k^c f(z)}{S_{k+1}^{c}f(z)}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} \prec \phi(z).$$

Proof Since $f \in N_{k,c}^{0}(\lambda, \mu, \phi)$, therefore we have

$$(1+\lambda)\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} - \lambda \frac{S_k^c f(z)}{S_{k+1}^{c}f(z)}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} \prec \phi(z).$$

Now consider

$$\begin{aligned} \lambda \frac{S_k^c f(z)}{S_{k+1}^{c}f(z)}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} = \; & (1+\lambda)\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} + \lambda \frac{S_k^c f(z)}{S_{k+1}^{c}f(z)}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} \\ & -(1+\lambda)\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}. \end{aligned}$$

This implies that

$$\begin{aligned} \frac{S_k^c f(z)}{S_{k+1}^{c}f(z)}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} = \; & \left(1+\frac{1}{\lambda}\right)\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} \\ & -\frac{1}{\lambda}\left\{(1+\lambda)\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} + \lambda \frac{S_k^c f(z)}{S_{k+1}^{c}f(z)}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\}. \end{aligned}$$

Using Theorem 2.1, Lemma 1.2, and the convexity of $\phi(z)$ with $\lambda \leq -1$, we have the required result. □

Theorem 2.7 *Let* $f \in N_{k,c}^{\alpha}(\lambda, \mu, h)\,, h(z) = \frac{1+Az}{1+Bz} + \frac{\lambda\mu}{k}\frac{(A-B)z}{(1+Bz)^2}$. *Then for* $Re\frac{\lambda}{\mu k} > 0,$

$$e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} \prec (\cos\alpha)\,\phi(z) + i\sin\alpha,$$

where $\phi(z) = \frac{1+Az}{1+Bz}$. *This result is the best possible.*

Proof Consider

$$p(z) = \frac{1}{\cos\alpha}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} - i\sin\alpha\right\}.$$

Then p is analytic in E with $p(0)=1$. Therefore, we have

$$e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} = (\cos\alpha)\,p(z) + i\sin\alpha.$$

Differentiating both sides, using (1.9), and simplifying, we obtain

$$\frac{\lambda(\cos\alpha)\,zp'(z)}{\mu k} = \lambda e^{i\alpha}\left\{\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} - \frac{S_{k}^{c}f(z)}{S_{k+1}^{c}f(z)}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\}.$$

It follows from the above equation and (2.1) that

$$\begin{aligned} & p(z) + \frac{\lambda}{\mu k}zp'(z) \\ = \; & \frac{1}{\cos\alpha}\left[e^{i\alpha}\left\{(1+\lambda)\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} - \lambda\frac{S_{k}^{c}f(z)}{S_{k+1}^{c}f(z)}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\} - i\sin\alpha\right]. \end{aligned}$$

Since $f \in N_{k,c}^{\alpha}(\lambda, \mu, h)$, therefore

$$p(z) + \frac{\lambda}{\mu k}zp'(z) \prec h(z).$$

Now we choose $g(z) = \frac{1+Az}{1+Bz}$, and then $\theta(w) = w$ and $\varphi(w) = \frac{\mu k}{\lambda}$. It is clear that $g(z)$ is analytic in E with $g(0) = 1$. Also, $\theta(w)$ and $\varphi(w)$ are analytic with $\theta(w) \neq 0$.

We see that

$$Q(z) = zg'(z)\varphi(g(z)) = \frac{\mu k}{\lambda}\frac{(A-B)\,z}{(1+Bz)^2}. \tag{2.4}$$

We have to prove that $Q(z)$ is starlike. In other words, we show that $\mathrm{Re}\frac{zQ'(z)}{Q(z)} > 0$. From (2.4), we have

$$\begin{aligned} \mathrm{Re}\frac{zQ'(z)}{Q(z)} &= \mathrm{Re}\left\{1 - \frac{2Bz}{1+Bz}\right\} \\ &= 1 - 2B\mathrm{Re}\frac{re^{i\psi}}{1+Bre^{i\psi}} \qquad (z = re^{i\psi}) \\ &= \frac{1-B^2r^2}{(1+Br\cos\psi)^2 + B^2r^2\sin^2\psi}. \end{aligned}$$

Since $-1 \le B < 1,\ r < 1$. This implies that $\mathrm{Re}\frac{zQ'(z)}{Q(z)} > 0$. Consider

$$\begin{aligned}\mathrm{Re}\frac{zh'(z)}{Q(z)} &= \mathrm{Re}\left\{\frac{\theta'(g(z))}{\varphi(g(z))} + \frac{zQ'(z)}{Q(z)}\right\} \\ &= \mathrm{Re}\frac{\lambda}{\mu k} + \mathrm{Re}\frac{zQ'(z)}{Q(z)} > 0.\end{aligned}$$

Using Lemma 1.6, we have $e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} \prec (\cos\alpha)\,\phi(z) + i\sin\alpha$. The function $\phi(z) = \frac{1+Az}{1+Bz}$ is the best possible. □

Theorem 2.8 *Let* $f \in N_{k,c}^{\alpha}\left(\lambda,\mu,\frac{1+Az}{1+Bz}\right)$. *Then for* $k,\lambda \in \mathbb{R}$ *and* $\frac{\mu k}{\lambda} \ge 0$,

$$\begin{aligned}&\left.\begin{matrix}\frac{A}{B} + \left(1-\frac{A}{B}\right)\ {}_2F_1\left(1,\frac{\mu k}{\lambda},\frac{\mu k}{\lambda}+1;B\right), & B \ne 0, \\ 1 - \frac{\mu k}{\mu k+\lambda}A, & B = 0.\end{matrix}\right\} \\ &< \frac{1}{\cos\alpha} Re\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\} \\ &< \left\{\begin{matrix}\frac{A}{B} + \left(1-\frac{A}{B}\right)\ {}_2F_1\left(1,\frac{\mu k}{\lambda},\frac{\mu k}{\lambda}+1;-B\right), & B \ne 0, \\ 1 + \frac{\mu k}{\mu k+\lambda}A, & B = 0.\end{matrix}\right.\end{aligned}$$

Proof Since $f \in N_{k,c}^{\alpha}\left(\lambda,\mu,\frac{1+Az}{1+Bz}\right)$, therefore, by using (2.2), we have

$$\frac{1}{\cos\alpha}\mathrm{Re}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\} \prec \mathrm{Re}\frac{\mu k}{\lambda}\int_0^1 \frac{1+Atz}{1+Btz}t^{\frac{\mu k}{\lambda}-1}dt.$$

It follows from the definition of subordination that

$$\begin{aligned}\frac{1}{\cos\alpha}\mathrm{Re}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\} &< \sup_{|z|<1}\mathrm{Re}\left\{\frac{\mu k}{\lambda}\int_0^1 \frac{1+Atz}{1+Btz}t^{\frac{\mu k}{\lambda}-1}dt\right\} \\ &\le \left\{\frac{\mu k}{\lambda}\int_0^1 \sup_{|z|<1}\mathrm{Re}\left\{\frac{1+Atz}{1+Btz}\right\}t^{\frac{\mu k}{\lambda}-1}dt\right\} \\ &< \frac{\mu k}{\lambda}\int_0^1 \frac{1+At}{1+Bt}t^{\frac{\mu k}{\lambda}-1}dt \\ &= \frac{\mu k}{\lambda}\int_0^1 \left\{A/B + \left(\frac{1-A/B}{1+Bt}\right)\right\}t^{\frac{\mu k}{\lambda}-1}dt.\end{aligned}$$

Now using Lemma 1.4 for the case $B \ne 0$, we have

$$\frac{1}{\cos\alpha}\mathrm{Re}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\} < \frac{A}{B} + \left(1-\frac{A}{B}\right)\ {}_2F_1\left(1,\frac{\mu k}{\lambda},\frac{\mu k}{\lambda}+1;-B\right).$$

When $B = 0$, it can be easily seen that

$$\begin{aligned}\frac{1}{\cos\alpha}\text{Re}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\} &< \frac{\mu k}{\lambda}\int_{0}^{1}(1+At)\,t^{\frac{\mu k}{\lambda}-1}dt \\ &= 1+\frac{\mu k}{\mu k+\lambda}A.\end{aligned}$$

We also have

$$\begin{aligned}\frac{1}{\cos\alpha}\text{Re}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\} &> \inf_{|z|<1}\text{Re}\left\{\frac{\mu k}{\lambda}\int_{0}^{1}\frac{1+Atz}{1+Btz}t^{\frac{\mu k}{\lambda}-1}dt\right\} \\ &\geq \left\{\frac{\mu k}{\lambda}\int_{0}^{1}\inf_{|z|<1}\text{Re}\left\{\frac{1+Atz}{1+Btz}\right\}t^{\frac{\mu k}{\lambda}-1}dt\right\} \\ &> \frac{\mu k}{\lambda}\int_{0}^{1}\frac{1-At}{1-Bt}t^{\frac{\mu k}{\lambda}-1}dt \\ &= \frac{\mu k}{\lambda}\int_{0}^{1}\left\{A/B+\left(\frac{1-A/B}{1-Bt}\right)\right\}t^{\frac{\mu k}{\lambda}-1}dt.\end{aligned}$$

Using again Lemma 1.4, we have the required result. □

Theorem 2.9 *Let* $f \in N_{k,c}^{\alpha}\left(\lambda,\mu,\frac{1+Az}{1+Bz}\right)$. *Then for* $k,\lambda \in \mathbb{R}$ *and* $\frac{\mu k}{\lambda} \geq 0$,

$$\begin{aligned}&\left.\begin{array}{ll}\frac{A}{B}+\left(1-\frac{A}{B}\right)\ {}_2F_1\left(1,\frac{\mu k}{\lambda},\frac{\mu k}{\lambda}+1;Br\right), & B\neq 0, \\ 1-\frac{\mu k}{\mu k+\lambda}A, & B=0.\end{array}\right\} \\ &\leq \left|\frac{1}{\cos\alpha}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}\right\}-i\sin\alpha\right| \\ &\leq \left\{\begin{array}{ll}\frac{A}{B}+\left(1-\frac{A}{B}\right)\ {}_2F_1\left(1,\frac{\mu k}{\lambda},\frac{\mu k}{\lambda}+1;-Br\right), & B\neq 0, \\ 1+\frac{\mu k}{\mu k+\lambda}A, & B=0.\end{array}\right.\end{aligned}$$

Proof Since $f \in N_{k,c}^{\alpha}\left(\lambda,\mu,\frac{1+Az}{1+Bz}\right)$, therefore, by using (2.2), we have

$$\frac{1}{\cos\alpha}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}-i\sin\alpha\right\} \prec \frac{\mu k}{\lambda}\int_{0}^{1}\frac{1+Atz}{1+Btz}t^{\frac{\mu k}{\lambda}-1}dt.$$

It follows from the definition of subordination that

$$\frac{1}{\cos\alpha}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu}-i\sin\alpha\right\} = \left\{\frac{\mu k}{\lambda}\int_{0}^{1}\frac{1+Atw(z)}{1+Btw(z)}t^{\frac{\mu k}{\lambda}-1}dt\right\},$$

where $w(z) = c_1 z + c_2 z^2 + \ldots$ is analytic and $|w(z)| \leq |z|$. Therefore,

$$\left|\frac{1}{\cos\alpha}\left\{e^{i\alpha}\left(\frac{z}{S_{k+1}^{c}f(z)}\right)^{\mu} - i\sin\alpha\right\}\right| \leq \left\{\frac{\mu k}{\lambda}\int_{0}^{1}\frac{1+Atr}{1+Btr}t^{\frac{\mu k}{\lambda}-1}dt\right\}.$$

Now using the same process as in the theorem above, we get the required result. □

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