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On metallic Riemannian structures

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Abstract: The paper is devoted to the study of metallic Riemannian structures. An integrability condition and curvature properties for these structures by means of a Φ-operator applied to pure tensor fields are presented. Examples of these structures are also given.

Key words: Conformal metric, metallic structure, pure tensor, Riemannian manifold, tensor bundle, twin metric

1. Introduction

Let M be an n-dimensional manifold. We point out here and once that all geometric objects considered in this paper are supposed to be of class C^∞.

The number $\eta = \frac{1+\sqrt{5}}{2} \approx 1,61803398874989...$, which is the positive root of the equation $x^2 - x - 1 = 0$, represents the golden mean. There are two most important generalizations of the golden mean. The first of them is the golden p-proportions being a positive root of the equation $x^{p+1} - x^p - 1 = 0$, $(p = 0, 1, 2, 3, ...)$ in [13]. The other, called the metallic means family or metallic proportions, was introduced by de Spinadel in [2, 3, 5, 4]. For two positive integers p and q, the positive solution of the equation $x^2 - px - q = 0$ is named members of the metallic means family. All the members of the metallic means family are positive quadratic irrational numbers $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$. These numbers $\sigma_{p,q}$ are also called (p, q)-metallic numbers. Inspired by the metallic means family, Hretcanu and Crasmareanu [8] constructed a new structure on a Riemannian manifold and named it a metallic structure. Indeed, a metallic structure is a polynomial structure with the structural polynomial $Q(J) = J^2 - pJ - qI$. Polynomial structures on a manifold were defined in [7]. A polynomial structure F of degree d on a connected manifold M means that a (1, 1)-tensor field F satisfies the following algebraic polynomial equation:

$$Q(F) = F^d + a_1F^{d-1} + ... + a_{d-1}F + a_dI = 0,$$

where $a_1, a_2, ..., a_d$ are real numbers and I is the identity tensor of type $(1, 1)$.

Given a Riemannian manifold (M, g) endowed with the metallic structure J, then the triple (M, J, g) is named a metallic Riemannian manifold if

$$g(JX, Y) = g(X, JY) \tag{1}$$

or equivalently

$$g(JX, JY) = g(J^2X, Y) = g((pJ + qI)X, Y) = pg(JX, Y) + qg(X, Y)$$

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for all vector fields X and Y on M [8]. The Riemannian metric (1) is referred to as J-compatible or pure metric [11, 14, 17].

In general, a $(0,s)$-tensor field t is pure with respect to a $(1,1)$-tensor field ψ if and only if the following condition holds:

$$t(\psi Y_1, Y_2, ..., Y_s) = t(Y_1, \psi Y_2, ..., Y_s) = ... = t(Y_1, Y_2, ..., \psi Y_s)$$

for any vector fields $Y_1, Y_2, ..., Y_s$ on M. The Tachibana operator Φ_ψ applied to the $(0,s)$-tensor field t is defined by

$$\begin{aligned} (\Phi_\psi t)(X, Y_1, ..., Y_s) &= (\psi X)\, t(Y_1, ..., Y_s) - Xt(\psi Y_1, ..., Y_s) \\ &+ \sum_{\lambda=1}^{s} t(Y_1, ..., (L_{Y_\lambda}\psi)X, ..., Y_s), \end{aligned} \tag{2}$$

where L_Y denotes the Lie differentiation with respect to Y [11, 14, 17]. If the pure tensor t satisfies $\Phi_\psi t = 0$, then it is called a Φ-tensor. If ψ is a product structure, then a Φ-tensor is a decomposable tensor.

2. Locally decomposable metallic Riemannian structures

Let (M, g, F) be a locally decomposable Riemannian manifold. This means that the Riemannian manifold (M, g) is equipped with an almost product structure F, $F^2 = I$, such that

$$g(FX, Y) = g(X, FY)$$

and

$$\nabla F = 0$$

for all vector fields X and Y on M, where ∇ is the operator of the Riemannian covariant derivation. The theory of Riemannian almost product structures was initiated by Yano in [16]. The classification of Riemannian almost product structure with respect to their covariant derivatives was described by Naveira in [9]. In [10], it was shown that the condition $\nabla F = 0$ is equivalent to decomposability of the pure metric g, i.e. $\Phi_F g = 0$, where Φ_F is the Tachibana operator [11, 14, 17]:

$$(\Phi_F g)(X, Y, Z) = (FX)(g(Y, Z)) - X(g(FY, Z)) + g((L_Y F)X, Z) + g(Y, (L_Z F)X).$$

As is known, a polynomial structure F is integrable if and only if it is possible to introduce a torsion-free linear connection ∇ with respect to which the structure tensor F is covariantly constant [15]. By using the Tachibana operator, we can give another condition of integrability for a metallic Riemannian structure.

Theorem 2.1 *Let M be a metallic Riemannian manifold equipped with a metallic structure J and a Riemannian metric g. Then:*

a) J is integrable if $\Phi_J g = 0$,

b) the condition $\Phi_J g = 0$ is equivalent to $\nabla J = 0$, where ∇ is the Levi-Civita connection of g.

Proof The proof is similar to that of Theorem 2.1 in [6], so we omit it. □

Next, we are going to give relationships between the almost product structures and metallic structures on M.

Proposition 2.2 *[8] If J is a metallic structure on M, then*

$$F_{\pm} = \pm(\frac{2}{2\sigma_{p,q} - p}J - \frac{p}{2\sigma_{p,q} - p}I) \tag{3}$$

are two almost product structures on M. Conversely, every almost product structure F on M induces two metallic structures on M, given as follows:

$$J_{\pm} = \frac{p}{2}I \pm (\frac{2\sigma_{p,q} - p}{2})F.$$

Due to (3), it follows that:

i) A Riemannian metric g is pure with respect to a metallic structure J if and only if the Riemannian metric g is pure with respect to the almost product structures $F_{\pm}$ associated with J.

ii) The dependence between $\Phi_{F_{\pm}}g$ and $\Phi_J g$ is as follows:

$$\Phi_{F_{\pm}}g = \pm\frac{2}{2\sigma_{p,q} - p}\Phi_J g, \tag{4}$$

from which, in view of Theorem 2.1, we can say that the metallic Riemannian structure J is integrable if $\Phi_{F_{\pm}}g = 0$, i.e. the Riemannian metric g is decomposable. If (M, J, g) is a metallic Riemannian manifold with a decomposable pure metric, then we call it a locally decomposable metallic Riemannian manifold. Hence, we have the following.

Proposition 2.3 *Let M be a metallic Riemannian manifold equipped with a metallic structure J and a Riemannian metric g. The manifold M is a locally decomposable metallic Riemannian manifold if and only if $\Phi_{F_{\pm}}g = 0$, where $F_{\pm}$ are the almost product structures associated with J.*

The twin metallic Riemannian metric is defined by

$$G(X, Y) = g(JX, Y)$$

for all vector fields X and Y on M. One can easily prove that G is pure with respect to J. If we apply the Φ_J-operator to the metric G, standard calculations give

$$(\Phi_J G)(X, Y, Z) = (\Phi_J g)(X, JY, Z) + g(N_J(X, Y), Z). \tag{5}$$

Thus, (5) implies the following result.

Proposition 2.4 *Let M be a metallic Riemannian manifold equipped with a metallic structure J and a Riemannian metric g. Then $\Phi_J g = 0$ is equivalent to $\Phi_J G = 0$ if $N_J = 0$, where N_J is Nijenhuis tensor constructed from J.*

We now turn our attention to the Riemannian curvature tensor field R of the locally decomposable metallic Riemannian manifold (M, J, g).

Theorem 2.5 *Let M be a metallic Riemannian manifold equipped with a metallic structure J and a Riemannian metric g. The Riemannian curvature tensor field is a J-tensor field.*

Proof The Riemannian curvature tensor field R of the metallic Riemannian metric g is pure with respect to the metallic structure J, i.e.

$$R(JY_1, Y_2, Y_3, Y_4) = R(Y_1, JY_2, Y_3, Y_4) = R(Y_1, Y_2, JY_3, Y_4) = R(Y_1, Y_2, Y_3, JY_4).$$

From (2), the Tachibana operator Φ_J applied to the Riemannian curvature tensor field R of type $(0,4)$ can be written as follows:

$$(\Phi_J R)(X, Y_1, Y_2, Y_3, Y_4) = (\nabla_{JX} R)(Y_1, Y_2, Y_3, Y_4) - (\nabla_X R)(JY_1, Y_2, Y_3, Y_4). \quad (6)$$

We can also say that in the locally decomposable metallic Riemannian manifold, the covariant derivative of the Riemannian curvature tensor R with respect to the Levi-Civita connection of g is pure. Using Bianchi's second identity and purity conditions, simple calculations give

$$(\Phi_J R)(X, Y_1, Y_2, Y_3, Y_4) = 0.$$

We omit standard calculations (see also [6]). □

By (3) and (6), we can find, in a similar way as for (4), the following:

$$\Phi_{F_\pm} R = \pm \frac{2}{2\sigma_{p,q} - p} \Phi_J R. \quad (7)$$

In view of Theorem 2.5 and (7), we have the result below.

Proposition 2.6 *Let M be a metallic Riemannian manifold equipped with a metallic structure J and a Riemannian metric g. The Riemannian curvature tensor field is a decomposable tensor field.*

3. Metallic structures with conformal metrics

Given a Riemannian metric g, we can easily define a new Riemannian metric $\widetilde{g}$ in terms of g by multiplying g by a smooth function f, or for vector fields X and Y on M,

$$\widetilde{g}(X, Y) = e^{2f} g(X, Y).$$

The metric $\widetilde{g}$ is called conformal to the Riemannian metric g.

Let us consider that (M, J, g) is a metallic Riemannian manifold. Immediately, we can say that $(M, J, \widetilde{g})$ is also a metallic Riemannian manifold. If we apply the Φ_J-operator to the conformal metric $\widetilde{g}$, we get

$$\begin{aligned}(\Phi_J \widetilde{g})(X, Y, Z) &= (JX)(e^{2f} g(Y, Z)) - X(e^{2f} g(JY, Z)) + e^{2f} g((L_Y J)X, Z) \\ &\quad + e^{2f} g(Y, (L_Z J)X) \\ &= (JX)(e^{2f}) g(Y, Z) - X(e^{2f}) g(JY, Z) + e^{2f} (\Phi_J g)(X, Y, Z).\end{aligned}$$

By using (3) and (4), we have

$$\begin{aligned}(\Phi_J \widetilde{g})(X, Y, Z) &= \pm \frac{2\sigma_{p,q} - p}{2} \{(F_\pm X)(e^{2f}) g(Y, Z) - X(e^{2f}) g(F_\pm Y, Z) \\ &\quad + e^{2f} (\Phi_{F_\pm} g)(X, Y, Z)\},\end{aligned}$$

where $F_\pm$ are the almost product structures associated with J. Therefore, we have following theorem.

Theorem 3.1 *Let M be a metallic Riemannian manifold equipped with a metallic structure J and a Riemannian metric g. Then $(M, J, \widetilde{g} = e^{2f}g)$ is a locally decomposable metallic Riemannian manifold if and only if the function f is constant.*

4. Examples

Example 1. Let M be an n-dimensional Riemannian manifold with a Riemannian metric g and denote by $\pi : T_1^1(M) \to M$ its $(1,1)$-tensor bundle with fibers the $(1,1)$-tensor spaces to M. Then $T_1^1(M)$ is an $n+n^2$-dimensional smooth manifold and some local charts induced naturally from local charts on M may be used. Namely, a system of local coordinates $(U; x^j)$ in M induces on $T_1^1(M)$ a system of local coordinates $(\pi^{-1}(U); x^j, x^{\bar{j}} = t_j^i)$ $j = 1, ..., n$, $\bar{j} = n+1, ..., n+n^2$, $J = 1, ..., n+n^2$, where (t_j^i) are the Cartesian coordinates in each $(1,1)$-tensor space $T^1_{1(P)}M$ at $P \in M$ with respect to the natural base.

Let $X = X^i \frac{\partial}{\partial x^i}$ and $A = A_j^i \frac{\partial}{\partial x^i} \otimes dx^j$ be the local expressions in U of a vector field X and a $(1,1)$-tensor field A on M, respectively. Then the vertical lift ${}^VA$ of A and the horizontal lift ${}^HX$ of X are given, with respect to the induced coordinates, by

$$ {}^VA = \begin{pmatrix} {}^VA^j \\ {}^VA^{\bar{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ A_j^i \end{pmatrix} \tag{8} $$

and

$$ {}^HX = \begin{pmatrix} {}^HX^j \\ {}^HX^{\bar{j}} \end{pmatrix} = \begin{pmatrix} X^j \\ X^s(\Gamma_{sj}^m t_m^i - \Gamma_{sm}^i t_j^m) \end{pmatrix}, \tag{9} $$

where Γ_{ij}^h are the coefficients of the Levi-Civita connection ∇ of g. The vector fields γA and $\tilde{\gamma} A$ on $T_1^1(M)$ are respectively defined by

$$ \gamma A = \begin{pmatrix} 0 \\ t_j^m A_m^i \end{pmatrix}, $$

$$ \tilde{\gamma} A = \begin{pmatrix} 0 \\ (t_m^i A_j^m) \end{pmatrix}. $$

From (8) we easily see that the vector fields γA and $\tilde{\gamma} A$ determine respectively the global vector fields on $T_1^1(M)$ [1].

The Sasaki type metric ${}^Sg$ on $T_1^1(M)$ is defined by the following three equations:

$$ {}^Sg\left({}^VA, {}^VB\right) = g(A, B), \tag{10} $$

$$ {}^Sg\left({}^VA, {}^HY\right) = 0, \tag{11} $$

$$ {}^Sg\left({}^HX, {}^HY\right) = g\left(X, Y\right), \tag{12} $$

for any vector fields X and Y and $(1,1)$-tensor fields A, B on M, where $g(A, B) = g_{it} g^{jl} A_j^i B_l^t$ (see [12]). From equations (10)–(12) we easily see that the horizontal distribution H, induced by ∇_g and determined by the horizontal lifts, is orthogonal to the fibers of $T_1^1(M)$. Let now E be a nowhere zero vector field on M. For any vector field X and covector field $\tilde{E} = g \circ E$ on M, we define the vertical lift ${}^V(X \otimes \tilde{E})$ of X with respect to E. The map $X \to {}^V(X \otimes \tilde{E})$ is a monomorphism. Hence, an n-dimensional C^∞ vertical distribution V^E is defined on $T_1^1(M)$. Let $V^\perp$ be the distribution on $T_1^1(M)$, which is orthogonal to H and V^E. Then H, V^E, and $V^\perp$ are mutually orthogonal distributions with respect to the Sasaki type metric ${}^Sg$. We define a $(1,1)$-tensor field $\widetilde{J}$ on $T_1^1(M)$ by

$$\begin{cases} \widetilde{J}^H X = \frac{p}{2}{}^H X + (\frac{2\sigma_{p,q} - p}{2}){}^V(X \otimes \tilde{E}), \\ \widetilde{J}^V(X \otimes \tilde{E}) = \frac{p}{2}{}^V(X \otimes \tilde{E}) + (\frac{2\sigma_{p,q} - p}{2}){}^H X, \\ \widetilde{J}({}^V A) = \sigma_{p,q}{}^V A, \end{cases} \tag{13}$$

for any vector field X and $(1,1)$-tensor field A on M, where $\tilde{E} = g \circ E$ is a covector field on M. The restrictions of $\widetilde{J}$ to $H + V^E$ and $V^\perp$ are endomorphisms, and hence $\widetilde{J}$ is a $(1,1)$-tensor field on $T_1^1(M)$. It is easily see that $\widetilde{J}^2 - p\widetilde{J} - qI = 0$, i.e. $\widetilde{J}$ is a metallic structure on $T_1^1(M)$.

Theorem 4.1 *Let (M, g) be a Riemannian manifold and $T_1^1(M)$ be its tensor bundle equipped with the Sasaki type metric ${}^Sg$ and the metallic structure $\tilde{J}$ defined by (13). The triple $\left(T_1^1(M), \tilde{J}, {}^Sg\right)$ is a metallic Riemannian manifold if and only if $g(E, E) = 1$.*

Proof We calculate

$$A\left(\tilde{X}, \tilde{Y}\right) = {}^Sg\left(\tilde{J}\tilde{X}, \tilde{Y}\right) - {}^Sg\left(\tilde{X}, \tilde{J}\tilde{Y}\right)$$

for any vector fields $\tilde{X}$ and $\tilde{Y}$ on $T_1^1(M)$. From (10)–(12) and (13), we obtain

$$\begin{aligned} A\left({}^H X, {}^H Y\right) &= {}^Sg\left(\tilde{J}^H X, {}^H Y\right) - {}^Sg\left({}^H X, \tilde{J}^H Y\right) \\ &= {}^Sg\left(\frac{p}{2}{}^H X + (\frac{2\sigma_{p,q} - p}{2}){}^V(X \otimes \tilde{E}), {}^H Y\right) \\ &\quad - {}^Sg\left({}^H X, \frac{p}{2}{}^H Y + (\frac{2\sigma_{p,q} - p}{2}){}^V(Y \otimes \tilde{E})\right) \\ &= 0, \\ A\left({}^V(X \otimes \tilde{E}), {}^V(Y \otimes \tilde{E})\right) &= {}^Sg\left(\tilde{J}^V(X \otimes \tilde{E}), {}^V(Y \otimes \tilde{E})\right) \\ &\quad - {}^Sg\left({}^V(X \otimes \tilde{E}), \tilde{J}^V(Y \otimes \tilde{E})\right) \\ &= {}^Sg\left(\frac{p}{2}{}^V(X \otimes \tilde{E}) + (\frac{2\sigma_{p,q} - p}{2}){}^H X, {}^V(Y \otimes \tilde{E})\right) \\ &\quad - {}^Sg\left({}^V(X \otimes \tilde{E}), \frac{p}{2}{}^V(Y \otimes \tilde{E}) + (\frac{2\sigma_{p,q} - p}{2}){}^H Y\right) \\ &= 0, \end{aligned}$$

$$\begin{aligned}
A\left({}^{V}(X\otimes\tilde{E}),{}^{H}Y\right) &= {}^{S}g\left(\tilde{J}^{V}(X\otimes\tilde{E}),{}^{H}Y\right)-{}^{S}g\left({}^{V}(X\otimes\tilde{E}),\tilde{J}^{H}Y\right)\\
&= {}^{S}g\left(\frac{p}{2}^{V}(X\otimes\tilde{E})+(\frac{2\sigma_{p,q}-p}{2})^{H}X,{}^{H}Y\right)\\
&\quad -{}^{S}g\left({}^{V}(X\otimes\tilde{E}),\frac{p}{2}^{H}Y+(\frac{2\sigma_{p,q}-p}{2})^{V}(Y\otimes\tilde{E})\right)\\
&= (\frac{2\sigma_{p,q}-p}{2})\{{}^{S}g({}^{H}X,{}^{H}Y)-{}^{S}g({}^{V}(X\otimes\tilde{E}),{}^{V}(Y\otimes\tilde{E}))\\
&= (\frac{2\sigma_{p,q}-p}{2})\{g(X,Y)-g(X,Y)g(E,E)\}\\
A\left({}^{V}A,{}^{V}B\right) &= {}^{S}g\left(\tilde{J}^{V}A,{}^{V}B\right)-{}^{S}g\left({}^{V}A,\tilde{J}^{V}B\right)\\
&= \sigma_{p,q}\{{}^{S}g\left({}^{V}A,{}^{V}B\right)-{}^{S}g\left({}^{V}A,{}^{V}B\right)\}\\
&= 0,\\
A\left({}^{V}A,{}^{V}(Y\otimes\tilde{E})\right) &= {}^{S}g\left(\tilde{J}^{V}A,{}^{V}(Y\otimes\tilde{E})\right)-{}^{S}g\left({}^{V}A,\tilde{J}^{V}(Y\otimes\tilde{E})\right)\\
&= \sigma_{p,q}{}^{S}g\left({}^{V}A,{}^{V}(Y\otimes\tilde{E})\right)\\
&\quad -{}^{S}g\left({}^{V}A,\frac{p}{2}^{V}(Y\otimes\tilde{E})+(\frac{2\sigma_{p,q}-p}{2})^{H}Y\right)\\
&= 0,\\
A\left({}^{V}A,{}^{H}Y\right) &= {}^{S}g\left(\tilde{J}^{V}A,{}^{H}Y\right)-{}^{S}g\left({}^{V}A,\tilde{J}^{H}Y\right)\\
&= \sigma_{p,q}{}^{S}g\left({}^{V}A,{}^{H}Y\right)\\
&\quad -{}^{S}g\left({}^{V}A,\frac{p}{2}^{H}X+(\frac{2\sigma_{p,q}-p}{2})^{V}(X\otimes\tilde{E})\right)\\
&= 0.
\end{aligned}$$

From the equations above, we say that ${}^{S}g$ is pure with respect to $\tilde{J}$ if and only if $g(E,E)=1$. This completes the proof. □

Now we consider the covariant derivative of $\tilde{J}$ with respect to the Levi-Civita connection of ${}^{S}g$. For this, first we state the following proposition.

Proposition 4.2 *[12] Let (M,g) be a Riemannian manifold and $T_1^1(M)$ be its tensor bundle equipped with the Sasaki type metric ${}^{S}g$. Then the corresponding Levi-Civita connection satisfies the following relations:*

i) ${}^{S}\nabla_{{}^{H}X}{}^{H}Y={}^{H}(\nabla_X Y)+\frac{1}{2}(\tilde{\gamma}-\gamma)R(X,Y)$,

ii) ${}^{S}\nabla_{{}^{V}A}{}^{H}Y=\frac{1}{2}{}^{H}\left(g^{bl}R(t_b,A_l)Y+g_{at}(t^a(g^{-1}\circ R(\quad,Y)\tilde{A}^t))\right)$,

iii) ${}^{S}\nabla_{{}^{H}X}{}^{V}B={}^{V}(\nabla_X B)+\frac{1}{2}{}^{H}\left(g^{bj}R(t_b,B_j)X+g_{ai}(t^a(g^{-1}\circ R(\quad,X)\tilde{B}^i))\right)$,

iv) ${}^{S}\nabla_{{}^{V}A}{}^{V}B=0$,

for all vector fields X, Y and $(1,1)$-tensor fields A, B on M, where $A_l=(A_l{}^i)$, $\tilde{A}^t=(g^{bl}A_l{}^{\;t})=(A^{b\,t}_{\cdot})$,

$t_l = (t_l{}^a)$, $t^a = (t_b{}^a)$, $R(\quad, X)Y$ *is a* $(1,1)$*-tensor field and* $g^{-1} \circ R(\quad, X)Y$ *is a vector field.*

By using Proposition 4.2 we calculate

$$({}^S\nabla_{\widetilde{X}} \widetilde{J})\widetilde{Y} = {}^S\nabla_{\widetilde{X}}(\widetilde{J}\widetilde{Y}) - \widetilde{J}({}^S\nabla_{\widetilde{X}}\widetilde{Y})$$

for all vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}$ on $T_1^1(M)$. Then we get

$$\begin{aligned}
({}^S\nabla_{{}^H X}\widetilde{J})^H Y &= \frac{2\sigma_{p,q} - p}{2}\, {}^V(Y \otimes (g \circ \nabla_X E)) + \frac{p - 2\sigma_{p,q}}{4}(\tilde{\gamma} - \gamma)R(X,Y) \\
&\quad + \frac{2\sigma_{p,q} - p}{4}\, {}^H\{g^{bj}\, R(t_b, (Y\tilde{\otimes}\tilde{E})_j)X + g_{ai}\,(t^a(g^{-1} \circ R(\quad, X)(\widetilde{Y\tilde{\otimes}\tilde{E}})^i)\}, \\
&\quad ({}^S\nabla_{{}^H X}\widetilde{J})^V B \\
&= \frac{2\sigma_{p,q} - p}{4}\, {}^H\{g^{bj}\, R(t_b, B_j)X + g_{ai}\,(t^a(g^{-1} \circ R(\quad, X)\tilde{B}^i)\} \\
&\quad + \frac{p - 2\sigma_{p,q}}{4}\, {}^V\{[g^{bj}\, R(t_b, B_j)X + g_{ai}\,(t^a(g^{-1} \circ R(\quad, X)\tilde{B}^i)] \otimes \tilde{E}\}, \\
&\quad ({}^S\nabla_{{}^H X}\widetilde{J})^V (Y\tilde{\otimes}\tilde{E}) \\
&= \frac{p}{2}\, {}^V(Y \otimes (g \circ \nabla_X E)) + \frac{2\sigma_{p,q} - p}{4}(\tilde{\gamma} - \gamma)R(X,Y) \\
&\quad + \frac{p - 2\sigma_{p,q}}{4}\, {}^V\{[g^{bj}\, R(t_b, (Y\tilde{\otimes}\tilde{E})_j)X + g_{ai}\,(t^a(g^{-1} \circ R(\quad, X)(\widetilde{Y\tilde{\otimes}\tilde{E}})^i)] \otimes \tilde{E}\}, \\
&\quad ({}^S\nabla_{{}^V A}\widetilde{J})^H Y \\
&= \frac{p - 2\sigma_{p,q}}{4}\, {}^V\{[g^{bj}\, R(t_b, A_j)Y + g_{ai}(t^a\,(g^{-1} \circ R(\quad, Y)\tilde{A}^i))] \otimes \tilde{E}\}, \\
&\quad ({}^S\nabla_{{}^V A}\widetilde{J})^V (Y \otimes \tilde{E}) \\
&= \frac{2\sigma_{p,q} - p}{4}\, {}^H\{g^{bj}\, R(t_b, A_j)Y + g_{ai}(t^a\,(g^{-1} \circ R(\quad, Y)\tilde{A}^i))\}, \\
&\quad ({}^S\nabla_{{}^V(X\otimes\tilde{E})}\widetilde{J})^V (Y \otimes \tilde{E}) \\
&= \frac{2\sigma_{p,q} - p}{4}\, {}^H\{g^{bj}\, R(t_b, (X \otimes \tilde{E})_j)Y + g_{ai}(t^a\,(g^{-1} \circ R(\quad, Y)(\widetilde{X\tilde{\otimes}\tilde{E}})^i))\}, \\
&\quad ({}^S\nabla_{{}^V(X\otimes\tilde{E})}\widetilde{J})^H Y \\
&= \frac{p - 2\sigma_{p,q}}{4}\, {}^V\{[g^{bj}\, R(t_b, (X \otimes \tilde{E})_j)Y + g_{ai}(t^a\,(g^{-1} \circ R(\quad, Y)(\widetilde{X\tilde{\otimes}\tilde{E}})^i))] \otimes \tilde{E}\}, \\
&\quad ({}^S\nabla_{{}^V A}\widetilde{J})^V B = 0,\ ({}^S\nabla_{{}^V(X\otimes\tilde{E})}\widetilde{J})^V (Y \otimes \tilde{E}) = 0.
\end{aligned}$$

Therefore, from the last equations, we have the following result.

Theorem 4.3 *Let* (M,g) *be a Riemannian manifold and* $T_1^1(M)$ *be its tensor bundle equipped with the Sasaki type metric* ${}^S g$ *and the metallic structure* $\tilde{J}$ *defined by (13). The triple* $\left(T_1^1(M), \tilde{J}, {}^S g\right)$ *is a locally decomposable* *metallic Riemannian manifold if and only if* M *is locally flat and* $g(E,E)=1$*,* $\nabla E=0$*, where* ∇ *is the Levi-Civita connection of* g*.*

Example 2. Let us consider the $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ endowed with the Euclidean metric g , i.e.

$$g = \begin{pmatrix} \delta_j^i & 0 \\ 0 & \delta_{\bar{j}}^{\bar{i}} \end{pmatrix}, \; i,j = 1,...,k, \quad \bar{i},\bar{j} = k+1,...,n.$$

The canonical product structure on $\mathbb{R}^n$ is given by

$$F = \begin{pmatrix} 0 & \delta_{\bar{j}}^i \\ \delta_j^{\bar{i}} & 0 \end{pmatrix}, \; i,j = 1,...,k, \quad \bar{i},\bar{j} = k+1,...,n.$$

The triple $(\mathbb{R}^n, F, g)$ is a locally decomposable Euclidean space. Metallic structures $J_\pm$ on $\mathbb{R}^n$ obtained from F are as follows:

$$J_\pm = \begin{pmatrix} \frac{p}{2}\delta_j^i & \pm(\frac{2\sigma_{p,q}-p}{2})\delta_{\bar{j}}^i \\ \pm(\frac{2\sigma_{p,q}-p}{2})\delta_j^{\bar{i}} & \frac{p}{2}\delta_{\bar{j}}^{\bar{i}} \end{pmatrix}.$$

The triple $(\mathbb{R}^n, J_\pm, g)$ is a locally decomposable metallic Euclidean space.

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