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Magnetic curves on flat para-Kähler manifolds

Mohamed JLELI[1], **Marian Ioan MUNTEANU**[2,*]
[1]Department of Mathematics, King Saud University, Riyadh, Saudi Arabia
[2]Faculty of Mathematics, 'Al. I. Cuza' University of Iaşi Iaşi, Romania

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Abstract: In this paper we prove that spacelike and timelike magnetic trajectories corresponding to the para-Kähler 2-form on a para-Kähler manifold $(M, \mathcal{P}, g)$ are circles on M. We then classify all para-Kähler magnetic curves in pseudo-Euclidean spaces $\mathbb{E}^{2n}_n$.

Key words: Magnetic field, para-Kähler manifold, circle

1. Introduction

An almost para-Hermitian manifold is a manifold M equipped with a pseudo-Riemannian metric g and an almost product structure $\mathcal{P}$ compatible with the metric; namely, $\mathcal{P}$ is a $(1,1)$-type tensor field, $\mathcal{P} \neq \pm I$, such that

$$\mathcal{P}^2 = I,\ g(\mathcal{P}X, \mathcal{P}Y) = -g(X, Y), \tag{1.1}$$

for vector fields X, Y tangent to M, where I is the identity map. Clearly, it follows from (1.1) that the dimension of M is even and the metric g is neutral. An almost para-Hermitian manifold is called *para-Kähler* if it satisfies $\nabla\mathcal{P} = 0$ identically, where ∇ denotes the Levi-Civita connection of M.

Properties of para-Kähler manifolds were first studied in 1948 by Rashevski, who considered a neutral metric of signature (m, m) defined from a potential function on a locally product $2m$-manifold [25]. He called such manifolds stratified spaces. Para-Kähler manifolds were explicitly defined by Rozenfeld in 1949 [26]. Such manifolds were also defined by Ruse in 1949 [27] and studied by Libermann [22] in the context of G-structures. Para-Kähler manifolds have been applied in supersymmetric field theories as well as in string theory in recent years (see, for instance, [10, 11]). An interesting survey on para-Kähler manifolds is given in [17]. See also [12].

In analogy with *holomorphic sectional curvature* of Kähler manifolds, one may define the *para-holomorphic sectional curvature* of para-Kähler manifolds. More precisely, if v and $\mathcal{P}v$ determine a nondegenerate plane at $p \in M$, the sectional curvature $H^{\mathcal{P}}(v) = K(v \wedge \mathcal{P}v)$ is called the para-holomorphic sectional curvature of the $\mathcal{P}$-plane spanned by v. A *para-Kähler space form* is a para-Kähler manifold of constant para-holomorphic sectional curvature. The simplest example of para-Kähler space form is furnished by the flat pseudo-Euclidean $2n$-space described in Section 4. The model of a nonflat para-Kähler space form was constructed in [18]. See also [8].

*Correspondence: marian.ioan.munteanu@gmail.com
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A *magnetic curve* represents a trajectory of a charged particle moving on the manifold under the action of a magnetic field. A *magnetic field* on a pseudo-Riemannian manifold (M, g) is a closed 2-form F. The *Lorentz force* of the magnetic field F is a $(1,1)$-type tensor field Φ given by

$$g(\Phi(X), Y) = F(X, Y), \quad \forall X, Y \in \mathfrak{X}(M). \tag{1.2}$$

The *magnetic trajectories* of F are curves γ on M that satisfy the *Lorentz equation*

$$\nabla_{\gamma'}\gamma' = \Phi(\gamma'), \tag{1.3}$$

where ∇ is the Levi-Civita connection of g. See, e.g., [3, 5, 4]. The Lorentz equation generalizes the equation satisfied by the geodesics of M, namely $\nabla_{\gamma'}\gamma' = 0$. Therefore, from the point of view of the dynamical systems, a geodesic corresponds to a trajectory of a particle without an action of a magnetic field. Hence, magnetic curves generalize geodesics.

Since the Lorentz force is skew symmetric we get that a magnetic curve has constant speed $v(t) = v_0$. When the magnetic curve $\gamma(t)$ is arc-length parametrized $(v_0 = 1)$, it is called a *normal magnetic curve*.

A typical example of magnetic fields is obtained by multiplying the area form on a Riemannian surface by a scalar q (usually called *strength* or *magnitude*). When the surface is of constant Gaussian curvature K, trajectories of such magnetic fields are well known. More precisely, on the sphere $\mathbb{S}^2(K)$, $K > 0$, trajectories are small circles of certain radius, on the Euclidean plane they are circles, and on a hyperbolic plane $\mathbb{H}^2(-K)$, $K > 0$, trajectories can be either closed curves (when $|q| > \sqrt{K}$), or open curves. Moreover, when $|q| = \sqrt{K}$, normal trajectories are horocycles (see, e.g., [9, 28]).

This problem was extended also for different ambient spaces. For example, if the ambient is a complex space form, Kähler magnetic fields are studied (see [2]), and in particular, explicit trajectories for Kähler magnetic fields are found in the complex projective space $\mathbb{CP}^n$ [1].

If the ambient is a contact manifold, the fundamental 2-form defines the so-called *contact magnetic field*. Interesting results are obtained when the manifold is Sasakian; more precisely, it is proved that the angle between the velocity of a normal magnetic curve and the Reeb vector field is constant; that is, they are slant curves. Moreover, explicit description for normal flowlines of the contact magnetic field on a 3-dimensional Sasakian manifold is known [6]. See also [20].

In the case of a 3-dimensional Riemannian manifold (M, g), 2-forms and vector fields may be identified via the Hodge star operator $\star$ and the volume form dv_g of the manifold. Thus, magnetic fields mean divergence-free vector fields (see, e.g., [7]). In particular, Killing vector fields define an important class of magnetic fields, called *Killing magnetic fields*. Recall that a vector field V on M is *Killing* if and only if it satisfies the Killing equation:

$$g(\nabla_Y V, Z) + g(\nabla_Z V, Y) = 0 \tag{1.4}$$

for every vector field Y, Z on M, where ∇ is the Levi-Civita connection on M. See, for example, [7, 15, 16, 23, 24].

In this paper we prove that spacelike and timelike magnetic trajectories corresponding to the para-Kähler 2-form on a para-Kähler manifold $(M, \mathcal{P}, g)$ are circles on M, namely curves of order 2 having constant curvature. Then we classify all para-Kähler magnetic curves in pseudo-Euclidean spaces $\mathbb{E}^{2n}_n$. The main result is Theorem B. *Let $\gamma : I \longrightarrow \mathbb{E}^{2n}_n$ be a magnetic curve corresponding to the standard flat para-Kähler structure* *on* $\mathbb{E}^{2n}_n$ *and with constant strength* $q \neq 0$. *Then, up to a Lorentzian transformation in the ambient space,* γ *belongs to the following list:*

(1a) $\gamma(s) = \frac{1}{q}\big(\mathrm{e}^{qs}w; \mathrm{e}^{qs}w\big)$, $w \in \mathbb{R}^n$, $w \neq 0$;

(1b) $\gamma(s) = \frac{1}{q}\big(-\mathrm{e}^{-qs}w; \mathrm{e}^{-qs}w\big)$, $w \in \mathbb{R}^n$, $w \neq 0$;

(2a) $\gamma(s) = \frac{1}{q}\big(\cosh(qs), 0, \ldots, 0; \sinh(qs), 0, \ldots, 0\big)$;

(2b) $\gamma(s) = \frac{1}{q}\big(\sinh(qs), 0, \ldots, 0; \cosh(qs), 0, \ldots, 0\big)$;

(2c) $\gamma(s) = \frac{1}{q}\big(\sinh(qs), \cosh(qs), 0, \ldots, 0; \cosh(qs), \sinh(qs), 0, \ldots, 0\big)$, *only when* $n \geq 2$.

2. Magnetic trajectories on para-Kähler manifolds

On a Kähler manifold (M, J, g) a closed 2-form $F_q = q\Omega_J$, where Ω_J is the Kähler 2-form on M, is said to be a *Kähler magnetic field* [1, 2]. A smooth curve γ parametrized (usually by its arc-length) is a trajectory of F_q if it satisfies the Lorentz equation $\nabla_{\gamma'}\gamma' = q\, J\gamma'$.

It is a natural problem to study Kähler magnetic fields and their trajectories on Kähler manifolds of constant holomorphic sectional curvature. See, e.g., [21]. On a complex space $\mathbb{C}^n$ the situation is quite trivial. For a complex projective space $\mathbb{C}P^n(c)$, $(c > 0)$, Adachi [1] proved that every trajectory corresponding to a Kähler magnetic field is a small circle on a totally geodesic embedded 2-sphere. In [2], the author gives explicit expressions of magnetic curves in complex hyperbolic spaces $\mathbb{C}H^n(-c)$, $(c > 0)$. While on $\mathbb{C}P^n(c)$ the trajectories are simply closed, on $\mathbb{C}H^n(c)$ the feature of trajectories changes according to the value of $|q|$ is greater or smaller than $\sqrt{c}$.

Consider now a para-Kähler manifold $(M, \mathcal{P}, g)$ and the 2-form $\Omega_{\mathcal{P}}$ defined by $\Omega_{\mathcal{P}}(X, Y) = g(\mathcal{P}X, Y)$, for all $X, Y \in \mathfrak{X}(M)$. Let $\gamma : I \longrightarrow M$ be a smooth curve on M. Then γ is a magnetic trajectory corresponding to the para-Kähler magnetic field $F_q = q\ \Omega_{\mathcal{P}}$, $q \neq 0$, if it satisfies the Lorentz equation

$$\nabla_{\gamma'}\gamma' = q\ \mathcal{P}\gamma'. \tag{2.1}$$

Since $\mathcal{P}$ is skew symmetric, we immediately obtain

$$\frac{d}{dt}g(\gamma', \gamma') = 2g(\nabla_{\gamma'}\gamma', \gamma') = 2qg(\mathcal{P}\gamma', \gamma') = 0,$$

and hence $g(\gamma', \gamma')$ does not depend on the parameter t.

As the metric g is no longer Riemannian, we have to distinguish several cases according to the causality of γ (which is the same at each point). When γ is spacelike or timelike we consider normal magnetic curves, namely those curves γ parametrized by arc-length s.

Let γ be a spacelike magnetic curve on M, i.e. $g(\dot\gamma, \dot\gamma) = 1$. Here by $\dot{}$ we denote the derivative with respect to the parameter s. We have $\nabla_{\dot\gamma}\dot\gamma = \kappa\nu$, where ν is the (first) unit normal to γ and κ is the (first) curvature. Combining this with the Lorentz equation and the fact that $\dot{\gamma}$ is unitary, we get $\kappa = q$ and $\nu = \mathcal{P}\dot{\gamma}$. Then

$$\nabla_{\dot{\gamma}}\nu = \nabla_{\dot{\gamma}}(\mathcal{P}\dot{\gamma}) = \mathcal{P}\nabla_{\dot{\gamma}}\dot{\gamma} = q\mathcal{P}^2\dot{\gamma} = q\dot{\gamma}.$$

It follows that γ has order 2 and its curvature is constant. Hence, γ is a *circle* on the para-Kähler manifold M.

Similar discussion may be done when γ is timelike.

We can state the following.

Theorem A *Let γ be a spacelike or timelike normal magnetic curve with constant strength q on a para-Kähler manifold $(M, g, \mathcal{P})$. Then γ is a circle, i.e. a curve of order 2 with constant curvature $\kappa = q$.*

Remark 1 For lightlike curves the curvature is not defined. Moreover, $\nabla_{\dot{\gamma}}\dot{\gamma}$ is lightlike, too.

3. Magnetic curves on $\mathbb{E}^{2n}_n$

On $\mathbb{R}^{2n}$ consider canonical coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. Define the pseudo-Euclidean metric

$$g = -\sum_{j=1}^{n} dx_j^2 + \sum_{j=1}^{n} dy_j^2, \tag{3.1}$$

and the para-complex structure

$$\mathcal{P}\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}\ ,\ \mathcal{P}\frac{\partial}{\partial y_j} = \frac{\partial}{\partial x_j}\ . \tag{3.2}$$

The manifold $\mathbb{E}^{2n}_n = (\mathbb{R}^{2n}, g, \mathcal{P})$ is a flat para-Kähler manifold. Its fundamental 2-form is given by $\Omega_{\mathcal{P}}(X, Y) = g(\mathcal{P}X, Y)$.

Define the magnetic field $F_q = q\Omega_{\mathcal{P}}$, where $q \neq 0$ is the strength. Let $\gamma : I \subseteq \mathbb{R} \longrightarrow \mathbb{E}^{2n}_n$ be the trajectory corresponding to the magnetic field F_q. Then the Lorentz equation becomes

$$\gamma'' = q\ \mathcal{P}\gamma'. \tag{3.3}$$

As we have already pointed out, due to the skew-symmetry of $\mathcal{P}$, the curve γ has constant "speed"; namely, $g(\gamma', \gamma')$ is constant. As the metric g is pseudo-Riemannian, we distinguish three situations:

1. $g(\gamma', \gamma') = v^2$ (spacelike magnetic curve),
2. $g(\gamma', \gamma') = -v^2$ (timelike magnetic curve),
3. $g(\gamma', \gamma') = 0$ (lightlike magnetic curve).

In the case of spacelike and timelike magnetic curves, we will consider γ parameterized by the arc-length s, i.e. $v = 1$.

We have the following result.

Theorem B *Let $\gamma : I \longrightarrow \mathbb{E}^{2n}_n$ be a magnetic curve corresponding to the standard flat para-Kähler structure on $\mathbb{E}^{2n}_n$ and with strength $q \neq 0$. Then, up to a Lorentzian transformation in the ambient space, γ belongs to the following list:*

(1a) $\gamma(s) = \frac{1}{q}\big(\mathrm{e}^{qs}w; \mathrm{e}^{qs}w\big)\,,\ w \in \mathbb{R}^n\,,\ w \neq 0$*;*

(1b) $\gamma(s) = \frac{1}{q}\big(-\mathrm{e}^{-qs}w; \mathrm{e}^{-qs}w\big)\,,\ w \in \mathbb{R}^n\,,\ w \neq 0$*;*

(2a) $\gamma(s) = \frac{1}{q}\big(\cosh(qs), 0, \ldots, 0; \sinh(qs), 0, \ldots, 0\big)$*;*

(2b) $\gamma(s) = \frac{1}{q}\big(\sinh(qs), 0, \ldots, 0; \cosh(qs), 0, \ldots, 0\big)$*;*

(2c) $\gamma(s) = \frac{1}{q}\big(\sinh(qs), \cosh(qs), 0, \ldots, 0; \cosh(qs), \sinh(qs), 0, \ldots, 0\big)$*, only when* $n \geq 2$*.*

Proof The speed $\dot{\gamma}$ is written as

$$\dot{\gamma} = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial y_j}\ ,$$

where a_j and b_j are smooth functions to be determined. Moreover, they satisfy

$$-\sum_{j=1}^{n} a_j^2 + \sum_{j=1}^{n} b_j^2 = \delta,$$

where $\delta \in \{-1, 0, 1\}$.

The Lorentz equation leads to the following system of ordinary differential equations:

$$\begin{cases} \dot{a}_j = q b_j \\ \dot{b}_j = q a_j, \quad \forall j = 1, \ldots, n. \end{cases}$$

The general solution is given by

$$\begin{cases} a_j = \alpha_j \cosh(qs) + \beta_j \sinh(qs) \\ b_j = \beta_j \cosh(qs) + \alpha_j \sinh(qs), \quad \alpha_j, \beta_j \in \mathbb{R}, \quad j = 1, \ldots, n. \end{cases}$$

Hence, the velocity of γ is given by

$$\dot{\gamma} = \cosh(qs)\ V + \sinh(qs)\mathcal{P}V,$$

where $V = \sum\limits_{j=1}^{n} \alpha_j \frac{\partial}{\partial x_j} + \sum\limits_{j=1}^{n} \beta_j \frac{\partial}{\partial y_j}$. Obviously, $V \neq 0$.

We have to distinguish two cases:

Case 1. V and $\mathcal{P}V$ are linearly dependent. This means V is a constant lightlike vector of the form $V = \sum\limits_{j=1}^{n} \alpha_j \left(\frac{\partial}{\partial x_j} + \varepsilon \frac{\partial}{\partial y_j}\right)$, $\varepsilon = \pm 1$. Thus, the velocity of γ can be expressed as

$$\dot{\gamma} = \big(\cosh(qs) + \varepsilon \sinh(qs)\big)V.$$

It follows that γ is given by

$$\gamma(s) = \gamma_0 + \frac{1}{q}\left(\sinh(qs) + \varepsilon \cosh(qs)\right)V.$$

Denote by $w = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, $w \neq 0$. Then the curve γ is parametrized as:

1a. for $\varepsilon = 1$: $\begin{cases} x(s) = x_0 + \frac{1}{q}\, \mathrm{e}^{qs} w \\ y(s) = y_0 + \frac{1}{q}\, \mathrm{e}^{qs} w, \end{cases}$

1b. for $\varepsilon = -1$: $\begin{cases} x(s) = x_0 - \frac{1}{q}\, \mathrm{e}^{-qs} w \\ y(s) = y_0 + \frac{1}{q}\, \mathrm{e}^{-qs} w. \end{cases}$

Subsequently, γ represents the two bisectrices in a 2-plane in $\mathbb{E}^{2n}_n$, spanned by $(w;0)$ and $(0;w)$.

Case 2. V and $\mathcal{P}V$ are linearly independent, and hence they are orthogonal. We have

$$\delta = g(\dot\gamma, \dot\gamma) = \cosh^2(qs) g(V,V) + \sinh^2(qs) g(\mathcal{P}V, \mathcal{P}V) = g(V,V).$$

2a. $\delta = 1$: Without loss of the generality we may take $V = \bar e_1 = (0, \ldots, 0; 1, 0, \ldots, 0) \in \mathbb{R}^{2n}$. Then $\dot\gamma(s) = \sinh(qs) e_1 + \cosh(qs) \bar e_1$, where $e_1 = (1, 0, \ldots, 0; 0, \ldots, 0)$. Therefore, γ is a spacelike hyperbola in a 2-plane $\mathbb{R}^2_1$ given by

$$\begin{cases} x(s) = x_0 + \frac{1}{q}\, (\cosh(qs), 0, \ldots, 0) \\ y(s) = y_0 + \frac{1}{q}\, (\sinh(qs), 0, \ldots, 0). \end{cases}$$

2b. $\delta = -1$: Take $V = e_1 = (1, 0, \ldots, 0; 0, \ldots, 0) \in \mathbb{R}^{2n}$. The velocity of γ is $\dot\gamma(s) = \cosh(qs) e_1 + \sinh(qs) \bar e_1$. Hence, γ is a timelike hyperbola given by

$$\begin{cases} x(s) = x_0 + \frac{1}{q}\, (\sinh(qs), 0, \ldots, 0) \\ y(s) = y_0 + \frac{1}{q}\, (\cosh(qs), 0, \ldots, 0). \end{cases}$$

2c. $\delta = 0$: Thus, $V = (u, w)$, where $u, w \in \mathbb{R}^n$ are linearly independent vectors in $\mathbb{R}^n$ such that $|u| = |w|$. Here $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^n$. Notice that this situation occurs only when $n \geq 2$. We get the velocity of γ:

$$\dot\gamma(s) = (\cosh(qs) u + \sinh(qs) w, \sinh(qs) u + \cosh(qs) w).$$

Without loss of the generality consider $u = (1, 0, \ldots, 0)$ and $w = (0, 1, 0, \ldots, 0)$. Hence, γ is a hyperbola in a lightlike 2-plane given by

$$\begin{cases} x(s) = x_0 + \frac{1}{q}\, (\sinh(qs), \cosh(qs), 0, \ldots, 0) \\ y(s) = y_0 + \frac{1}{q}\, (\cosh(qs), \sinh(qs), 0, \ldots, 0). \end{cases}$$

After a translation one can take $x_0 = 0$ and $y_0 = 0$. □

Let us conclude this paper with the following remark.

Remark 2 We have obtained that the codimension of a spacelike or timelike magnetic curve γ in the flat para-Kähler manifold may be reduced to 1; namely, there exists a 2-plane invariant by $\mathcal{P}$ such that γ lies on it. See also [1, 13, 14, 19, 21] for results of the same type in other ambient spaces.

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