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Approximation properties of Szász type operators based on Charlier polynomials

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Abstract: In the present paper, we study some approximation properties of the Szász type operators involving Charlier polynomials introduced by Varma and Taşdelen in 2012. First, we establish approximation in a Lipschitz type space and weighted approximation theorems for these operators. Then we obtain the error in the approximation of functions having derivatives of bounded variation.

Key words: Szász operator, Charlier polynomials, modulus of continuity, bounded variation

1. Introduction

In 1950, Szász [20] introduced the following linear positive operators:

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \tag{1.1}$$

where $x \in [0,\infty)$ and $f(x)$ is a continuous function on $[0,\infty)$ whenever the above sum converges uniformly. Many researchers have studied approximation properties of these operators and modified Szász operators by involving different types of polynomials. Jakimovski and Leviatan [13] defined a generalization of Szász operators including the Appell polynomials and gave the approximation properties of these operators. In [21], Varma et al. considered the generalization of Szász operators involving Brenke type polynomials and studied convergence properties by using the Korovkin type theorem and the order of convergence with the help of classical method.

Recently, Altomare et al. [3] introduced a new kind of generalization of Szász–Mirkajan–Kantorovich operators and obtained the rate of convergence by means of suitable moduli of smoothness. Several researchers also defined different types of generalizations of these operators and studied their approximation properties; we refer the reader to those papers (cf. [5, 4, 7, 8, 15, 17]).

In [22], Varma and Taşdelen constituted a link between orthogonal polynomials and the positive linear operators. They considered Szász type operators including Charlier polynomials. These polynomials [11] have the generating functions of the form

$$e^t\left(1-\frac{t}{a}\right)^u = \sum_{k=0}^{\infty}C_k^{(a)}(u)\frac{t^k}{k!}, \quad |t| < a, \tag{1.2}$$

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where $C_k^{(a)}(u) = \sum_{r=0}^{k} \binom{k}{r} (-u)_r \left(\frac{1}{a}\right)^r$ and $(m)_0 = 1, (m)_j = m(m+1)\cdots(m+j-1)$ for $j \geq 1$.

For $C_\gamma[0,\infty) := \{f \in C[0,\infty) : |f(t)| \leq Me^{\gamma t}$ for some $\gamma > 0, M > 0$ and $t \in [0,\infty)\}$, Varma and Taşdelen [22] defined the following Szász type operators involving Charlier polynomials

$$\mathcal{L}_n(f;x,a) = e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)nx)}{k!} f\left(\frac{k}{n}\right), \tag{1.3}$$

where $a > 1$ and $x \in [0,\infty)$. For the special case $a \to \infty$ and $x - \frac{1}{n}$ instead of x, these operators reduce to the operators (1.1). They studied uniform convergence of these operators by applying the Korovkin theorem on compact subsets of $[0,\infty)$ and the order of approximation by using the classical modulus of continuity.

This paper is structured as follows. In Section 2, we present some moment estimates and a result needed to study approximation of functions with derivatives of bounded variation. In Section 3, we discuss the main results of the paper wherein we establish approximation in a Lipschitz type space and weighted approximation theorems for the operators $\mathcal{L}_n$. Lastly, we obtain the rate of convergence for the functions having a derivative of bounded variation on every finite subinterval of $[0,\infty)$, for these operators.

2. Preliminaries

Let $e_i(x) = x^i, i = 0,1,2,\cdots$

Lemma 1 *[22] For the operators $\mathcal{L}_n(f;x,a)$, we get*

(i) $\mathcal{L}_n(e_0(t);x,a) = 1;$

(ii) $\mathcal{L}_n(e_1(t);x,a) = x + \dfrac{1}{n};$

(iii) $\mathcal{L}_n(e_2(t);x,a) = x^2 + \dfrac{x}{n}\left(3 + \dfrac{1}{a-1}\right) + \dfrac{2}{n^2}.$

Lemma 2 *For the operators $\mathcal{L}_n(f;x,a)$, we have*

(i) $\mathcal{L}_n(e_3(t);x,a) = x^3 + \dfrac{x^2}{n}\left(6 + \dfrac{3}{a-1}\right) + \dfrac{2x}{n^2}\left(\dfrac{1}{(a-1)^2} + \dfrac{3}{a-1} + 5\right) + \dfrac{5}{n^3};$

(ii)
$$\mathcal{L}_n(e_4(t);x,a) = x^4 + \frac{x^3}{n}\left(10 + \frac{6}{a-1}\right) + \frac{x^2}{n^2}\left(31 + \frac{30}{a-1} + \frac{11}{(a-1)^2}\right)$$
$$+ \frac{x}{n^3}\left(67 + \frac{31}{a-1} + \frac{20}{(a-1)^2} + \frac{6}{(a-1)^3}\right) + \frac{15}{n^4}.$$

Proof From the generating function of the Charlier polynomials given by (1.2) we have

$$\sum_{k=0}^{\infty}\frac{k^3C_k^{(a)}(-(a-1)nx)}{k!}=e\left(1-\frac{1}{a}\right)^{-(a-1)nx}\left[n^3x^3+x^2n^2\left(6+\frac{3}{a-1}\right)+2xn\left(\frac{1}{(a-1)^2}+\frac{3}{a-1}+5\right)+5\right]$$

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{k^4C_k^{(a)}(-(a-1)nx)}{k!} &= e\left(1-\frac{1}{a}\right)^{-(a-1)nx}\left[n^4x^4+x^3n^3\left(10+\frac{6}{a-1}\right)+x^2n^2\left(31+\frac{30}{a-1}+\frac{11}{(a-1)^2}\right)\right.\\ &+ \left. nx\left(67+\frac{31}{a-1}+\frac{20}{(a-1)^2}+\frac{6}{(a-1)^3}\right)+15\right],\end{aligned}$$

from which the lemma is immediate.

□

Let $e_i^x(t)=(t-x)^i, i=0,1,2\cdots$

Lemma 3 *For the operators* $\mathcal{L}_n(f;x,a)$, *we have*

(i) $\mathcal{L}_n(e_1^x(t);x,a)=\frac{1}{n}$;

(ii) $\mathcal{L}_n(e_2^x(t);x,a)=\frac{ax}{n(a-1)}+\frac{2}{n^2}$;

(iii) $\mathcal{L}_n(e_4^x(t);x,a)=\frac{x}{n^3}\left(17+\frac{49}{(a-1)}-\frac{20}{(a-1)^2}+\frac{6}{(a-1)^3}\right)+\frac{x^2}{n^2}\left(19-\frac{46}{(a-1)}+\frac{3}{(a-1)^2}\right)+\frac{15}{n^4}.$

Proof Using Lemmas 1 and 2, the proof of this lemma easily follows. Hence, the details are omitted. □

To study the rate of convergence of functions having a derivative of bounded variation, let us rewrite the operators (1.3) as

$$\mathcal{L}_n(f;x,a)=\int_0^{\infty}f(w)\frac{\partial}{\partial w}\left\{\mathcal{K}_n(x,w,a)\right\}dw, \tag{2.1}$$

where

$$\mathcal{K}_n(x,w,a)=\begin{cases}\displaystyle\sum_{k\le nw}e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\frac{C_k^{(a)}(-(a-1)nx)}{k!}, & \text{if } 0<w<\infty,\\ 0, & \text{if } w=0.\end{cases}$$

From Lemma 3, for $x\in(0,\infty)$ and sufficiently large n, we have

$$\mathcal{L}_n(|e_1^x(t)|;x,a)\le\left(\mathcal{L}_n(e_2^x(t);x,a)\right)^{1/2}\le\sqrt{\frac{\lambda(a)x}{n}}, \tag{2.2}$$

where $\lambda(a)$ is some positive constant depending on a.
We also get, for $r\ge 2$ and fixed $x\in[0,\infty)$,

$$\mathcal{L}_n(e_{2r}^x(t);x,a)=O(n^{-r});\ \ n\to\infty. \tag{2.3}$$

Lemma 4 *For all* $x\in(0,\infty)$ *and sufficiently large* n, *we have* *(i)* $\vartheta_{n,a}(x,t) = \int_0^t \frac{\partial}{\partial w}\{\mathcal{K}_n(x,w,a)\}\,dw \leq \frac{1}{(x-t)^2}\frac{\lambda(a)x}{n},\ 0 \leq t < x;$

(ii) $1 - \vartheta_{n,a}(x,z) = \int_z^\infty \frac{\partial}{\partial w}\{\mathcal{K}_n(x,w,a)\}\,dw \leq \frac{1}{(z-x)^2}\frac{\lambda(a)x}{n},\ x < z < \infty,$

where $\lambda(a)$ is a constant as described in (2.2).

Proof First we prove (i).

$$\begin{aligned}\vartheta_{n,a}(x,t) &= \int_0^t \frac{\partial}{\partial w}\{\mathcal{K}_n(x,w,a)\}\,dw \leq \int_0^t \left(\frac{x-w}{x-t}\right)^2 \frac{\partial}{\partial w}\{\mathcal{K}_n(x,w,a)\}\,dw \\ &\leq \frac{1}{(x-t)^2}\mathcal{L}_n((w-x)^2;x,a) \\ &\leq \frac{1}{(x-t)^2}\frac{\lambda(a)x}{n}.\end{aligned}$$

The proof of (ii) is similar, and hence it is omitted. □

Theorem 1 *[22] Let $f \in C[0,\infty) \cap E$. Then*

$$\lim_{n\to\infty} \mathcal{L}_n(f;x,a) = f(x),$$

and the operators given by (1.3) converge uniformly in each compact subset of $[0,\infty)$ where

$$E := \left\{ f : [0,\infty) \to \mathbb{R}, |f(x)| \leq Me^{Ax}, A \in \mathbb{R} \quad \text{and} \quad M \in \mathbb{R}^+ \right\}.$$

In what follows, let $\tilde{C_B}[0,\infty)$ be the space of all real valued bounded and uniformly continuous functions f on $[0,\infty)$, endowed with the norm $||f||_{\tilde{C_B}[0,\infty)} = \sup_{x\in[0,\infty)} |f(x)|$.

3. Main results

3.1. Degree of approximation

Let $a_1, a_2 > 0$ be fixed. We consider the following Lipschitz type space (see [18]):

$$Lip_M^{(a_1,a_2)}(r) := \left\{ f \in \tilde{C}_B[0,\infty) : |f(t) - f(x)| \leq M\frac{|t-x|^r}{(t + a_1x^2 + a_2x)^{\frac{r}{2}}}; x,t \in (0,\infty) \right\},$$

where M is a positive constant and $0 < r \leq 1$.

Theorem 2 *Let $f \in Lip_M^{(a_1,a_2)}(r)$ and $r \in (0,1]$. Then, for all $x \in (0,\infty)$, we have*

$$| \mathcal{L}_n(f;x,a) - f(x) | \leq M\left(\frac{\zeta_{n,a}(x)}{a_1x^2 + a_2x}\right)^{\frac{r}{2}},$$

where $\zeta_{n,a}(x) = \mathcal{L}_n(e_2^x(t);x,a)$.

Proof Applying the Hölder's inequality with $p=\dfrac{2}{r}$ and $q=\dfrac{2}{2-r}$, we find that

$$
\begin{aligned}
|\,\mathcal{L}_n(f;x,a)-f(x)\,| &\leq \left\{e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}\left|f\left(\frac{k}{n}\right)-f(x)\right|^{\frac{2}{r}}\right\}^{r/2}\\
&\leq M\left\{e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}\frac{\left(\frac{k}{n}-x\right)^2}{\left(\frac{k}{n}+a_1x^2+a_2x\right)}\right\}^{r/2}.
\end{aligned}
$$

Since $f\in Lip_M^{(a_1,a_2)}(r)$ and $\dfrac{1}{\sqrt{\frac{k}{n}+a_1x^2+a_2x}}<\dfrac{1}{\sqrt{a_1x^2+a_2x}}, \forall x\in(0,\infty)$, we have

$$
\begin{aligned}
|\,\mathcal{L}_n(f;x,a)-f(x)\,| &\leq \frac{M}{(a_1x^2+a_2x)^{\frac{r}{2}}}\left\{e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}\left(\frac{k}{n}-x\right)^2\right\}^{r/2}\\
&\leq M\left(\frac{\zeta_{n,a}(x)}{a_1x^2+a_2x}\right)^{\frac{r}{2}}.
\end{aligned}
$$

This completes the proof of the theorem. □

Next, we obtain a local direct estimate for the operators defined in (1.3) using the Lipschitz-type maximal function of order r introduced by Lenze [16] as

$$\widetilde{\omega}_r(f,x)=\sup_{t\neq x,\ t\in[0,\infty)}\frac{|f(t)-f(x)|}{|t-x|^r},\quad x\in[0,\infty)\ \text{ and }\ r\in(0,1]. \tag{3.1}$$

Theorem 3 *Let $f\in\tilde{C}_B[0,\infty)$ and $0<r\leq 1$. Then, for all $x\in[0,\infty)$, we have*

$$|\mathcal{L}_n(f;x,a)-f(x)|\leq\widetilde{\omega}_r(f,x)(\zeta_{n,a}(x))^{\frac{r}{2}}.$$

Proof From equation (3.1), we have

$$|\mathcal{L}_n(f;x,a)-f(x)|\leq\widetilde{\omega}_r(f,x)\mathcal{L}_n(|t-x|^r;x,a).$$

Applying Hölder's inequality with $p=\dfrac{2}{r}$ and $q=\dfrac{2}{2-r}$, we get

$$|\mathcal{L}_n(f;x,a)-f(x)|\leq\widetilde{\omega}_r(f,x)(\mathcal{L}_n((t-x)^2;x,a))^{\frac{r}{2}}\leq\widetilde{\omega}_r(f,x)(\zeta_{n,a}(x))^{\frac{r}{2}}.$$

Thus, the proof is completed. □

3.2. Weighted approximation

Let $B_\varphi[0,\infty)$ be the space of all real valued functions on $[0,\infty)$ satisfying the condition $|f(x)|\leq M_f\varphi(x)$, where M_f is a positive constant depending only on f and $\varphi(x)=1+x^2$ is a weight function. Let $C_\varphi[0,\infty)$ be the space of all continuous functions in $B_\varphi[0,\infty)$ with the norm $\|f\|_\varphi:=\sup\limits_{x\in[0,\infty)}\dfrac{|f(x)|}{\varphi(x)}$ and

$C^*_\varphi[0,\infty) := \left\{ f \in C_\varphi[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{\varphi(x)} \text{ is finite} \right\}.$

The usual modulus of continuity of f on $[0,b]$ is defined as

$$\omega_b(f,\delta) = \sup_{|t-x|\le\delta} \sup_{x,t\in[0,b]} |f(t)-f(x)|.$$

Theorem 4 *Let $f \in C_\varphi[0,\infty)$. Then we have*

$$\|\mathcal{L}_n(f;\cdot,a) - f\|_{C[0,b]} \le 4M_f(1+b^2)\zeta_{n,a}(b) + 2\omega_{b+1}(f, \sqrt{\zeta_{n,a}(b)}),$$

where $\zeta_{n,a}(b) = \dfrac{ab}{n(a-1)} + \dfrac{2}{n^2}$.

Proof Let $x \in [0,b]$ and $t > b+1$. Then $t - x > 1$, and hence

$$\begin{aligned} |f(t)-f(x)| &\le M_f(2+t^2+x^2) = M_f\left\{2+2x^2+(t-x)^2+2x(t-x)\right\} \\ &\le M_f(t-x)^2\left(3+2x+2x^2\right) \\ &\le 4M_f(1+x^2)(t-x)^2. \end{aligned} \tag{3.2}$$

For $x \in [0,b]$ and $t \in [0,b+1]$ we have

$$|f(t)-f(x)| \le \omega_{b+1}(f;|t-x|) \le \left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f;\delta). \tag{3.3}$$

Thus, from (3.2) and (3.3) for all $x \in [0,b]$ and $t \ge 0$, we have

$$|f(t)-f(x)| \le 4M_f(1+x^2)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta), \delta > 0.$$

Hence, applying Cauchy–Schwarz inequality, we get

$$\begin{aligned} |\mathcal{L}_n(f;x,a) - f(x)| &\le 4M_f(1+x^2)\mathcal{L}_n((t-x)^2;x,a) + \omega_{b+1}(f,\delta)\left(1+\frac{1}{\delta}\mathcal{L}_n(|t-x|;x,a)\right) \\ &\le 4M_f(1+x^2)\zeta_{n,a}(x) + \omega_{b+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\zeta_{n,a}(x)}\right) \\ &\le 4M_f(1+b^2)\zeta_{n,a}(b) + \omega_{b+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\zeta_{n,a}(b)}\right). \end{aligned}$$

Choosing $\delta = \sqrt{\zeta_{n,a}(b)}$, we get the desired result. □

Theorem 5 *Let $f \in C^*_\varphi[0,\infty)$. Then we have*

$$\lim_{n\to\infty} \|\mathcal{L}_n(f;\cdot,a) - f\|_\varphi = 0. \tag{3.4}$$

Proof From [9], we know that it is sufficient to verify the following three equations:

$$\lim_{n\to\infty} \|\mathcal{L}_n(e_m;\cdot,a)-e_m\|_\varphi = 0,\ m=0,1,2. \tag{3.5}$$

Since $\mathcal{L}_n(e_0;x,a)=1$, the condition in (3.5) holds true for $m=0$.

By Lemma 1, we have

$$\|\mathcal{L}_n(e_1;\cdot,a)-e_1\|_\varphi = \sup_{x\geq 0}\frac{1}{1+x^2}\left|x+\frac{1}{n}-x\right| \leq \frac{1}{n}.$$

Thus, $\lim\limits_{n\to\infty}\|\mathcal{L}_n(e_1;\cdot,a)-e_1\|_\varphi=0$. Similarly, we get

$$\begin{aligned}\|\mathcal{L}_n(e_2;\cdot,a)-e_2\|_\varphi &= \sup_{x\geq 0}\frac{1}{1+x^2}\left|x^2+\frac{x}{n}\left(3+\frac{1}{a-1}\right)+\frac{2}{n^2}-x^2\right|\\ &\leq \frac{1}{n}\left(3+\frac{1}{(a-1)}\right)+\frac{2}{n^2},\end{aligned}$$

which implies that $\lim\limits_{n\to\infty}\|\mathcal{L}_n(e_2;\cdot,a)-e_2\|_\varphi=0$. This completes the proof.

□

Next we give a theorem to approximate all functions in $C_\varphi[0,\infty)$. This type of result is discussed in [10] for locally integrable functions.

Theorem 6 *For each* $f\in C_\varphi[0,\infty)$ *and* $\beta>0$, *we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|\mathcal{L}_n(f;x,a)-f(x)|}{(1+x^2)^{1+\beta}}=0.$$

Proof For any fixed $x_0>0$,

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{|\mathcal{L}_n(f;x,a)-f(x)|}{(1+x^2)^{1+\beta}} &\leq \sup_{x\leq x_0}\frac{|\mathcal{L}_n(f;x,a)-f(x)|}{(1+x^2)^{1+\beta}}+\sup_{x\geq x_0}\frac{|\mathcal{L}_n(f;x,a)-f(x)|}{(1+x^2)^{1+\beta}}\\ &\leq \|\ \mathcal{L}_n(f)-f\ \|_{C[0,x_0]}+\|\ f\ \|_\varphi\sup_{x\geq x_0}\frac{|\mathcal{L}_n(1+t^2;x,a)|}{(1+x^2)^{1+\beta}}\\ &\quad +\sup_{x\geq x_0}\frac{|f(x)|}{(1+x^2)^{1+\beta}},\\ &=: I_1+I_2+I_3,\ \text{say.}\end{aligned} \tag{3.6}$$

Since $|f(x)|\leq ||f||_\varphi(1+x^2)$, we have

$$I_3=\sup_{x\geq x_0}\frac{|f(x)|}{(1+x^2)^{1+\beta}}\leq\sup_{x\geq x_0}\frac{||f||_\varphi}{(1+x^2)^\beta}\leq\frac{||f||_\varphi}{(1+x_0^2)^\beta}.$$

Let $\epsilon>0$ be arbitrary. In view of Theorem 1, there exists $n_1\in\mathbb{N}$ such that

$$|\mathcal{L}_n(1+t^2;x,a)-(1+x^2)|<\frac{\epsilon}{3||f||_\varphi},\forall n\geq n_1,$$

or

$$\mathcal{L}_n(1+t^2;x,a) < (1+x^2) + \frac{\epsilon}{3||f||_\varphi}, \forall n \geq n_1.$$

Hence,

$$\begin{aligned} \| f \|_\varphi \frac{|\mathcal{L}_n(1+t^2;x,a)|}{(1+x^2)^{1+\beta}} &< \frac{1}{(1+x^2)^{1+\beta}} \| f \|_\varphi \left((1+x^2) + \frac{\epsilon}{3||f||_\varphi}\right), \forall n \geq n_1 \\ &< \frac{\|f\|_\varphi}{(1+x^2)^\beta} + \frac{\epsilon}{3}, \forall n \geq n_1. \end{aligned} \tag{3.7}$$

Hence, $||f||_\varphi \sup_{x \geq x_0} \frac{|\mathcal{L}_n(1+t^2;x,a)|}{(1+x^2)^{1+\beta}} < \frac{||f||_\varphi}{(1+x_0^2)^\beta} + \frac{\epsilon}{3}, \ \forall n \geq n_1.$

Thus, $I_2 + I_3 < \frac{2||f||_\varphi}{(1+x_0^2)^\beta} + \frac{\epsilon}{3}, \ \forall n \geq n_1.$

Now let us choose x_0 to be so large that $\frac{\|f\|_\varphi}{(1+x^2)^\beta} < \frac{\epsilon}{6}.$

Then,

$$I_2 + I_3 < \frac{2\epsilon}{3}, \ \forall n \geq n_1. \tag{3.8}$$

By Theorem 4, there exists $n_2 \in \mathbb{N}$ such that

$$I_1 = \| \mathcal{L}_n(f) - f \|_{C[0,x_0]} < \frac{\epsilon}{3}, \ \forall n \geq n_2. \tag{3.9}$$

Let $n_0 = \max(n_1, n_2)$. Then, combining (3.6)–(3.9):

$$\sup_{x \in [0,\infty)} \frac{|\mathcal{L}_n(f;x,a) - f(x)|}{(1+x^2)^{1+\beta}} < \epsilon, \quad \forall n \geq n_0.$$

This completes the proof. □

3.3. Rate of approximation

In this section we obtain the estimate of the rate of convergence for the operators $\mathcal{L}_n$ for functions with derivatives of bounded variation. In recent years, several researchers have obtained results in this direction for different sequences of linear positive operators. We refer the reader to some of the related papers (cf. [1, 2, 6, 12, 14, 19], etc.).

Let $f \in DBV_\gamma[0,\infty)$, $\gamma > 0$ be the class of all functions defined on $[0,\infty)$, having a derivative of bounded variation on every finite subinterval of $[0,\infty)$ and $|f(t)| \leq Mt^\gamma$, $\forall \ t > 0$.
It turns out that for $f \in DBV_\gamma[0,\infty)$, we can write

$$f(x) = \int_0^x g(t)dt + f(0),$$

where $g(t)$ is a function of bounded variation on each finite subinterval of $[0,\infty)$.

Theorem 7 *Let $f \in DBV_\gamma[0,\infty)$. Then for every $x \in (0,\infty)$, $r \in \mathbb{N}(2r > \gamma)$ and sufficiently large n, we have*

$$\begin{aligned} |\mathcal{L}_n(f;x,a) - f(x)| \leq & \left|\frac{f'(x+) - f'(x-)}{2}\right| \left\{\frac{\lambda(a)x}{n}\right\}^{1/2} + \frac{1}{n}\left|\frac{f'(x+) + f'(x-)}{2}\right| \\ & + \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} (f'_x) + \frac{\lambda(a)}{n} \sum_{k=1}^{[\sqrt{n}\,]} \bigvee_{x-\frac{x}{k}}^{x+\frac{x}{k}} (f'_x) + |f'(x+)| \left\{\frac{\lambda(a)x}{n}\right\}^{1/2} \\ & + |f(2x) - f(x) - xf'(x+)| \frac{\lambda(a)}{nx} + \frac{C(\gamma,x,r,a)}{n^{\gamma/2}} + |f(x)|\frac{\lambda(a)}{nx}, \end{aligned}$$

where

$$f'_x(t) = \begin{cases} f'(t) - f'(x+), & x < t < \infty, \\ 0, & t = x, \\ f'(t) - f'(x-), & 0 \leq t < x. \end{cases}$$

$\bigvee_c^d(f'_x)$ is the total variation of f'_x on $[c,d]$ and $C(\gamma,x,r,a)$ is a constant depending only on γ, x, r, and a.

Proof By the hypothesis, we may write

$$\begin{aligned} f'(t) = & \frac{1}{2}\Big(f'(x+) + f'(x-)\Big) + f'_x(t) + \frac{1}{2}\Big(f'(x+) - f'(x-)\Big) sgn(t-x) \\ & + \delta_x(t)\Big(f'(t) - \frac{1}{2}\Big(f'(x+) + f'(x-)\Big)\Big), \end{aligned} \tag{3.10}$$

where

$$\delta_x(t) = \begin{cases} 1 \ , & t = x, \\ 0 \ , & t \neq x. \end{cases}$$

From equations (2.1) and (3.10), we have

$$\begin{aligned} \mathcal{L}_n(f;x,a) - f(x) &= \int_0^\infty \frac{\partial}{\partial t}\{\mathcal{K}_n(x,t,a)\} f(t)dt - f(x) = \int_0^\infty (f(t) - f(x)) \frac{\partial}{\partial t}\{\mathcal{K}_n(x,t,a)\}dt \\ &= \int_0^x (f(t) - f(x))\frac{\partial}{\partial t}\{\mathcal{K}_n(x,t,a)\}dt + \int_x^\infty (f(t) - f(x))\frac{\partial}{\partial t}\{\mathcal{K}_n(x,t,a)\}dt \\ &= -\int_0^x \left(\int_t^x f'(u)du\right)\frac{\partial}{\partial t}\{\mathcal{K}_n(x,t,a)\}dt + \int_x^\infty \left(\int_x^t f'(u)du\right)\frac{\partial}{\partial t}\{\mathcal{K}_n(x,t,a)\}dt \\ &=: -F_1(n,x,a) + F_2(n,x,a), \ \text{say}. \end{aligned}$$

By using equation (3.10), we get

$$\begin{aligned} F_1(n,x,a) = & \int_0^x \Bigg\{\int_t^x \Bigg(\frac{1}{2}\Big(f'(x+) + f'(x-)\Big) + f'_x(u) + \frac{1}{2}\Big(f'(x+) - f'(x-)\Big)sgn(u-x) \\ & + \delta_x(u)\Big(f'(u) - \frac{1}{2}\Big(f'(x+) + f'(x-)\Big)\Big)\Bigg)du\Bigg\}\frac{\partial}{\partial t}\{\mathcal{K}_n(x,t,a)\}dt. \end{aligned}$$

Since $\int_x^t (\delta_x(u))du = 0,$ we have

$$\begin{aligned} F_1(n,x,a) &= \frac{1}{2}\Big(f'(x+)+f'(x-)\Big)\int_0^x (x-t)\frac{\partial}{\partial t}\left\{\mathcal{K}_n(x,t,a)\right\}dt + \int_0^x \left(\int_t^x f_x'(u)du\right)\frac{\partial}{\partial t}\left\{\mathcal{K}_n(x,t,a)\right\}dt \\ &\quad -\frac{1}{2}\Big(f'(x+)-f'(x-)\Big)\int_0^x |x-t|\frac{\partial}{\partial t}\left\{\mathcal{K}_n(x,t,a)\right\}dt. \qquad (3.11) \end{aligned}$$

Proceeding similarly, we find that

$$\begin{aligned} F_2(n,x,a) &= \frac{1}{2}\Big(f'(x+)+f'(x-)\Big)\int_x^\infty (t-x)\frac{\partial}{\partial t}\left\{\mathcal{K}_n(x,t,a)\right\}dt + \int_x^\infty \left(\int_x^t f_x'(u)du\right)\frac{\partial}{\partial t}\left\{\mathcal{K}_n(x,t,a)\right\}dt \\ &\quad +\frac{1}{2}\Big(f'(x+)-f'(x-)\Big)\int_x^\infty |t-x|\frac{\partial}{\partial t}\left\{\mathcal{K}_n(x,t,a)\right\}dt. \qquad (3.12) \end{aligned}$$

Combining (3.11) and (3.12), we get

$$\begin{aligned} \mathcal{L}_n(f;x,a)-f(x) &= \frac{1}{2}\Big(f'(x+)+f'(x-)\Big)\int_0^\infty (t-x)\frac{\partial}{\partial t}\left\{\mathcal{K}_n(x,t,a)\right\}dt \\ &\quad +\frac{1}{2}\Big(f'(x+)-f'(x-)\Big)\int_0^\infty |t-x|\frac{\partial}{\partial t}\left\{\mathcal{K}_n(x,t,a)\right\}dt \\ &\quad -\int_0^x \left(\int_t^x f_x'(u)du\right)\frac{\partial}{\partial t}\left\{\mathcal{K}_n(x,t,a)\right\}dt + \int_x^\infty \left(\int_x^t f_x'(u)du\right)\frac{\partial}{\partial t}\left\{\mathcal{K}_n(x,t,a)\right\}dt. \end{aligned}$$

Hence,
$|\mathcal{L}_n(f;x,a)-f(x)|$

$$\begin{aligned} &\leq \left|\frac{f'(x+)+f'(x-)}{2}\right||\mathcal{L}_n(t-x;x,a)| + \left|\frac{f'(x+)-f'(x-)}{2}\right|\mathcal{L}_n(|t-x|;x,a) \\ &\quad +\left|\int_0^x \left(\int_t^x f_x'(u)du\right)\frac{\partial}{\partial t}\left\{\mathcal{K}_n(x,t,a)\right\}dt\right| + \left|\int_x^\infty \left(\int_x^t f_x'(u)du\right)\frac{\partial}{\partial t}\left\{\mathcal{K}_n(x,t,a)\right\}dt\right|. \qquad (3.13) \end{aligned}$$

On an application of Lemma 4 and integration by parts, we obtain

$$\int_0^x \left(\int_t^x f_x'(u)du\right)\frac{\partial}{\partial t}\left\{\mathcal{K}_n(x,t,a)\right\}dt = \int_0^x \left(\int_t^x f_x'(u)du\right)\frac{\partial}{\partial t}\vartheta_{n,a}(x,t)dt = \int_0^x f_x'(t)\vartheta_{n,a}(x,t)dt.$$

Thus,

$$\begin{aligned} \left|\int_0^x \left(\int_t^x f_x'(u)du\right)\frac{\partial}{\partial t}\left\{\mathcal{K}_n(x,t,a)\right\}dt\right| &\leq \int_0^x |f_x'(t)|\vartheta_{n,a}(x,t)dt \\ &\leq \int_0^{x-\frac{x}{\sqrt{n}}} |f_x'(t)|\vartheta_{n,a}(x,t)dt + \int_{x-\frac{x}{\sqrt{n}}}^x |f_x'(t)|\vartheta_{n,a}(x,t)dt. \end{aligned}$$

Since $f'_x(x) = 0$ and $\vartheta_{n,a}(x,t) \leq 1$, we get

$$\begin{aligned}\int_{x-\frac{x}{\sqrt{n}}}^{x} |f'_x(t)|\vartheta_{n,a}(x,t)dt &= \int_{x-\frac{x}{\sqrt{n}}}^{x} |f'_x(t) - f'_x(x)|\vartheta_{n,a}(x,t)dt \leq \int_{x-\frac{x}{\sqrt{n}}}^{x} \bigvee_t^x (f'_x)dt \\ &\leq \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} (f'_x) \int_{x-\frac{x}{\sqrt{n}}}^{x} dt = \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} (f'_x).\end{aligned}$$

Similarly, applying Lemma 4 and putting $t = x - \frac{x}{u}$, we get

$$\begin{aligned}\int_0^{x-\frac{x}{\sqrt{n}}} |f'_x(t)|\vartheta_{n,a}(x,t)dt &\leq \frac{\lambda(a)x}{n} \int_0^{x-\frac{x}{\sqrt{n}}} |f'_x(t)| \frac{dt}{(x-t)^2} \\ &\leq \frac{\lambda(a)x}{n} \int_0^{x-\frac{x}{\sqrt{n}}} \bigvee_t^x (f'_x) \frac{dt}{(x-t)^2} \\ &= \frac{\lambda(a)}{n} \int_1^{\sqrt{n}} \bigvee_{x-\frac{x}{u}}^{x} (f'_x)du \leq \frac{\lambda(a)}{n} \sum_{k=1}^{[\sqrt{n}\,]} \bigvee_{x-\frac{x}{k}}^{x} (f'_x).\end{aligned}$$

Consequently,

$$\left| \int_0^x \left(\int_t^x f'_x(u)du \right) \frac{\partial}{\partial t} \left\{ \mathcal{K}_n(x,t,a) \right\} dt \right| \leq \frac{x}{\sqrt{n}} \bigvee_{x-\frac{x}{\sqrt{n}}}^{x} (f'_x) + \frac{\lambda(a)}{n} \sum_{k=1}^{[\sqrt{n}\,]} \bigvee_{x-\frac{x}{k}}^{x} (f'_x). \tag{3.14}$$

Also, we have

$$\begin{aligned}&\left| \int_x^\infty \left(\int_x^t f'_x(u)du \right) \frac{\partial}{\partial t} \left\{ \mathcal{K}_n(x,t,a) \right\} dt \right| \\ &\leq \left| \int_x^{2x} \left(\int_x^t f'_x(u)du \right) \frac{\partial}{\partial t} (1 - \vartheta_{n,a}(x,t))dt \right| \\ &+ \left| \int_{2x}^\infty \left(\int_x^t f'_x(u)du \right) \frac{\partial}{\partial t} \left\{ \mathcal{K}_n(x,t,a) \right\} dt \right| \\ &\leq \left| \int_{2x}^\infty (f(t) - f(x)) \frac{\partial}{\partial t} \left\{ \mathcal{K}_n(x,t,a) \right\} dt \right| + |f'(x+)| \left| \int_{2x}^\infty (t-x) \frac{\partial}{\partial t} \left\{ \mathcal{K}_n(x,t,a) \right\} dt \right| \\ &+ \left| \int_x^{2x} f'_x(u)du \right| |1 - \vartheta_{n,a}(x,2x)| + \int_x^{2x} |f'_x(t)|(1 - \vartheta_{n,a}(x,t))dt.\end{aligned} \tag{3.15}$$

We may write

$$\begin{aligned}\int_x^{2x} |f'_x(t)|(1 - \vartheta_{n,a}(x,t))dt &= \int_x^{x+\frac{x}{\sqrt{n}}} |f'_x(t)|(1 - \vartheta_{n,a}(x,t))dt + \int_{x+\frac{x}{\sqrt{n}}}^{2x} |f'_x(t)|(1 - \vartheta_{n,a}(x,t))dt \\ &=: I_1 + I_2 \quad \text{(say)}.\end{aligned} \tag{3.16}$$

Since $f'_x(x) = 0$ and $(1 - \vartheta_{n,a}(x,t)) \leq 1$, we have

$$\begin{aligned} I_1 &= \int_x^{x+\frac{x}{\sqrt{n}}} |f'_x(t) - f'_x(x)|(1 - \vartheta_{n,a}(x,t))dt \\ &\leq \int_x^{x+\frac{x}{\sqrt{n}}} \bigvee_x^{x+\frac{x}{\sqrt{n}}} (f'_x)dt \\ &= \frac{x}{\sqrt{n}} \bigvee_x^{x+\frac{x}{\sqrt{n}}} (f'_x). \end{aligned} \tag{3.17}$$

Next, we estimate I_2. Applying Lemma 4 and putting $t = x + \frac{x}{u}$ we have

$$\begin{aligned} I_2 &= \int_{x+\frac{x}{\sqrt{n}}}^{2x} |f'_x(t)|(1 - \vartheta_{n,a}(x,t))dt \\ &\leq \frac{\lambda(a)x}{n} \int_{x+\frac{x}{\sqrt{n}}}^{2x} |f'_x(t) - f'_x(x)| \frac{dt}{(x-t)^2} \\ &\leq \frac{\lambda(a)x}{n} \int_{x+\frac{x}{\sqrt{n}}}^{2x} \bigvee_x^t (f'_x) \frac{dt}{(x-t)^2} \\ &= \frac{\lambda(a)}{n} \int_1^{\sqrt{n}} \bigvee_x^{x+\frac{x}{u}} (f'_x)du \\ &\leq \frac{\lambda(a)}{n} \sum_{k=1}^{[\sqrt{n}\,]} \int_k^{k+1} \bigvee_x^{x+\frac{x}{u}} (f'_x)du \\ &\leq \frac{\lambda(a)}{n} \sum_{k=1}^{[\sqrt{n}\,]} \bigvee_x^{x+\frac{x}{k}} (f'_x). \end{aligned} \tag{3.18}$$

Putting the values of I_1 and I_2 in (3.16), we get

$$\int_x^{2x} |f'_x(t)|(1 - \vartheta_{n,a}(x,t))dt \leq \frac{x}{\sqrt{n}} \bigvee_x^{x+\frac{x}{\sqrt{n}}} (f'_x) + \frac{\lambda(a)}{n} \sum_{k=1}^{[\sqrt{n}\,]} \bigvee_x^{x+\frac{x}{k}} (f'_x). \tag{3.19}$$

In view of (2.2), (3.15), (3.19), and Lemma 4, we get

$$\begin{aligned} \left| \int_x^\infty \left(\int_x^t f'_x(u)du \right) \frac{\partial}{\partial t} \{\mathcal{K}_n(x,t,a)\} dt \right| &\leq M \int_{2x}^\infty t^\gamma \frac{\partial}{\partial t} \{\mathcal{K}_n(x,t,a)\} dt + |f(x)| \int_{2x}^\infty \frac{\partial}{\partial t} \{\mathcal{K}_n(x,t,a)\} dt \\ &\quad + |f'(x+)| \left\{ \frac{\lambda(a)x}{n} \right\}^{1/2} + \frac{\lambda(a)}{nx} |f(2x) - f(x) - xf'(x+)| \\ &\quad + \frac{x}{\sqrt{n}} \bigvee_x^{x+\frac{x}{\sqrt{n}}} (f'_x) + \frac{\lambda(a)}{n} \sum_{k=1}^{[\sqrt{n}\,]} \bigvee_x^{x+\frac{x}{k}} (f'_x). \end{aligned} \tag{3.20}$$

Since $t \leq 2(t-x)$ and $x \leq t-x$ when $t \geq 2x$, applying Hölder's inequality, (2.2), and (2.3), we obtain

$$M\int_{2x}^{\infty} t^{\gamma}\frac{\partial}{\partial t}\{\mathcal{K}_n(x,t,a)\}\,dt + |f(x)|\int_{2x}^{\infty}\frac{\partial}{\partial t}\{\mathcal{K}_n(x,t,a)\}\,dt$$

$$\begin{aligned}
&\leq M2^{\gamma}\int_{2x}^{\infty}(t-x)^{\gamma}\frac{\partial}{\partial t}\{\mathcal{K}_n(x,t,a)\}\,dt + \frac{|f(x)|}{x^2}\int_{0}^{\infty}\frac{\partial}{\partial t}\{\mathcal{K}_n(x,t,a)\}(t-x)^2dt \\
&\leq M2^{\gamma}\left(\int_{0}^{\infty}(t-x)^{2r}\frac{\partial}{\partial t}\{\mathcal{K}_n(x,t,a)\}\,dt\right)^{\gamma/2r} + |f(x)|\frac{\lambda(a)}{nx} \\
&\leq \frac{C(\gamma,x,r,a)}{n^{\gamma/2}} + |f(x)|\frac{\lambda(a)}{nx}. \qquad (3.21)
\end{aligned}$$

Combining (3.13), (3.14), (3.20), and (3.21), we get the required result. □

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