

Dynamic behavior of a second-order nonlinear rational difference equation

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Abstract: This paper deals with the global attractivity of positive solutions of the second-order nonlinear difference equation

$$x_{n+1} = \frac{ax_n^k + b \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + cx_{n-1}^k}{Ax_n^k + B \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + Cx_{n-1}^k}, \quad k = 3, 4, \dots, n = 0, 1, \dots,$$

where the parameters a, b, c, A, B, C and the initial values x_0, x_{-1} are arbitrary positive real numbers.

Key words: Global stability, difference equations, local asymptotic stability, periodicity

1. Introduction and preliminaries

The study of difference equations is a very rich research field, and difference equations have been applied in several mathematical models in biology, economics, genetics, population dynamics, medicine, and so forth. Solving difference equations and studying the asymptotic behavior of their solutions has attracted the attention of many authors, see for example [1, 3, 4, 6, 7, 8, 9, 10, 12, 14, 15, 16, 17, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31].

In [18, 19] the global stability of positive solutions of the difference equations

$$x_{n+1} = \frac{ax_n^3 + bx_n^2 x_{n-1} + cx_n x_{n-1}^2 + dx_{n-1}^3}{Ax_n^3 + Bx_n^2 x_{n-1} + Cx_n x_{n-1}^2 + Dx_{n-1}^3}, \quad (1)$$

and

$$x_{n+1} = \frac{ax_n^4 + bx_n^3 x_{n-1} + cx_n^2 x_{n-1}^2 + dx_n x_{n-1}^3 + ex_{n-1}^4}{Ax_n^4 + Bx_n^3 x_{n-1} + Cx_n^2 x_{n-1}^2 + Dx_n x_{n-1}^3 + Ex_{n-1}^4}, \quad (2)$$

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was investigated. These equations are special cases of the difference equation

$$x_{n+1} = \frac{ax_n^k + \sum_{j=1}^{k-1} b_j x_n^j x_{n-1}^{k-j} + cx_{n-1}^k}{Ax_n^k + \sum_{j=1}^{k-1} B_j x_n^j x_{n-1}^{k-j} + Cx_{n-1}^k}. \tag{3}$$

Here and motivated by the above-mentioned papers, we study the global character of positive solutions of the difference equation Eq. (3) with $b_i = b$, $B_i = B$, $i = 1, 2, \dots, k - 1$. That is the equation

$$x_{n+1} = \frac{ax_n^k + b \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + cx_{n-1}^k}{Ax_n^k + B \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + Cx_{n-1}^k}, \quad k = 3, 4, \dots, n = 0, 1, \dots, \tag{4}$$

where the parameters a, b, c, A, B, C and the initial values x_0, x_{-1} are arbitrary positive real numbers. Now we recall some definitions and results that will be useful in our investigation; for more details see [11]. Let I be an interval of real numbers and let

$$F : I \times I \longrightarrow I$$

be a continuous function. Consider the difference equation

$$x_{n+1} = F(x_n, x_{n-1}), \quad n = 0, 1, \dots, \tag{5}$$

with initial values $x_{-1}, x_0 \in I$.

Definition 1 A point $\bar{x} \in I$ is called an equilibrium point of Eq.(5) if

$$\bar{x} = F(\bar{x}, \bar{x}).$$

Definition 2 Let \bar{x} be an equilibrium point of Eq.(5).

i) The equilibrium \bar{x} is called locally stable if for every $\epsilon > 0$, there exist $\delta > 0$ such that for all $x_{-1}, x_0 \in I$ with $|x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \delta$, we have $|x_n - \bar{x}| < \epsilon$, for all $n \geq -1$.

ii) The equilibrium \bar{x} is called locally asymptotically stable if it is locally stable, and if there exists $\gamma > 0$ such that if $x_{-1}, x_0 \in I$ and $|x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma$ then

$$\lim_{n \rightarrow +\infty} x_n = \bar{x}.$$

iii) The equilibrium \bar{x} is called a global attractor if for all $x_{-1}, x_0 \in I$, we have

$$\lim_{n \rightarrow +\infty} x_n = \bar{x}.$$

iv) The equilibrium \bar{x} is called global asymptotically stable if it is locally stable and a global attractor.

v) The equilibrium \bar{x} is called unstable if it is not locally stable.

Suppose F is continuously differentiable in some open neighborhood of \bar{x} , and let

$$p = \frac{\partial F}{\partial x}(\bar{x}, \bar{x}), \quad q = \frac{\partial F}{\partial y}(\bar{x}, \bar{x}).$$

Then the equation

$$y_{n+1} = py_n + qy_{n-1}, \quad n = 0, 1, \dots \tag{6}$$

is called the linearized equation of Eq. (5) about the equilibrium point \bar{x} and

$$\lambda^2 - p\lambda - q = 0 \tag{7}$$

is called the characteristic equation of Eq. (6) about \bar{x} .

The following theorem is very useful in establishing local stability.

Theorem 1 (Linearized stability)

1. If all the roots of Eq. (7) lie in the open unit disk $|\lambda| < 1$, then the equilibrium point \bar{x} of Eq. (5) is locally asymptotically stable.
2. If at least one root of Eq. (7) has absolute value greater than one, then the equilibrium point \bar{x} of Eq. (5) is unstable.

To establish convergence results, we need the following two theorems from [2].

Theorem 2 Let $[a, b]$ be a closed and bounded interval of real numbers and let $F \in C([a, b]^{k+1}, [a, b])$ satisfy the following conditions:

1. The function $F(z_1, \dots, z_{k+1})$ is monotonic in each of its arguments.
2. For each $m, M \in [a, b]$ and for each $i \in \{1, \dots, k + 1\}$, we define

$$M_i(m, M) = \begin{cases} M, & \text{if } F \text{ is increasing in } z_i \\ m, & \text{if } F \text{ is decreasing in } z_i \end{cases}$$

and

$$m_i(m, M) = M_i(M, m)$$

and assume that if (m, M) is a solution of the system:

$$\begin{cases} M = F(M_1(m, M), \dots, M_{k+1}(m, M)) \\ m = F(m_1(m, M), \dots, m_{k+1}(m, M)) \end{cases}$$

then $M = m$.

Then there exists exactly one equilibrium \bar{x} of the equation

$$x_{n+1} = F(x_n, x_{n-1}, \dots, x_{n-k}), n = 0, 1, \dots \tag{8}$$

and every solution of Eq. (8) converges to \bar{x} .

Theorem 3 Let I be a set of real numbers and let

$$F : I \times I \longrightarrow I$$

be a function $F(u, v)$, which decreases in u and increases in v . Then for every solution $\{x_n\}_{n=-1}^\infty$ of the equation

$$x_{n+1} = F(x_n, x_{n-1}), n = 0, 1, \dots,$$

the subsequences $\{x_{2n}\}_{n=0}^\infty$ and $\{x_{2n+1}\}_{n=-1}^\infty$ of even and odd terms of the solution do exactly one of the following:

- i) They are both monotonically increasing.
- ii) They are both monotonically decreasing.
- ii) Eventually, one of them is monotonically increasing and the other is monotonically decreasing.

The following results [5, 13] give the rate of convergence for solutions of a system of difference equations. Let us consider the system of difference equations

$$X_{n+1} = (A + B_n) X_n, n \in \mathbb{N}_0, \tag{9}$$

where X_n is an m -dimensional vector, $A \in C^{m \times m}$ is a constant matrix, and $B : Z^+ \rightarrow C^{m \times m}$ is a matrix function satisfying

$$\|B_n\| \rightarrow 0 \tag{10}$$

as $n \rightarrow \infty$.

Theorem 4 (Perron’s First Theorem) Suppose that condition (10) holds. If X_n is a solution of (9), then either $X_n = 0$ for all large n or

$$\rho = \lim_{n \rightarrow \infty} \frac{\|X_{n+1}\|}{\|X_n\|} \tag{11}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A .

Theorem 5 (Perron’s Second Theorem) Suppose that condition (10) holds. If X_n is a solution of (9), then either $X_n = 0$ for all large n or

$$\rho = \lim_{n \rightarrow \infty} (\|X_n\|)^{1/n} \tag{12}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A .

2. Stability of the positive solutions

Let $f : (0, +\infty)^2 \rightarrow (0, +\infty)$ be the function defined by

$$f(x, y) = \frac{ax^k + b \sum_{j=1}^{k-1} x^j y^{k-j} + cy^k}{Ax^k + B \sum_{j=1}^{k-1} x^j y^{k-j} + Cy^k}.$$

In the following theorem we study the periodicity of the positive solutions.

Theorem 6 *Let*

$$\begin{aligned} \check{p}_1 &= aC - cA + aB + bC, \\ \check{p}_2 &= aC - cA + cC + aA - (k - 1)(bA + cB) + k(aB + bC), \\ \check{q}_i &= aC - cA - i(bA + cB) + (i + 1)(aB + bC), \quad i = 1, 2, \dots, k - 2. \end{aligned}$$

Assume that $\check{p}_1, \check{p}_2, \check{q}_1, \check{q}_2, \dots, \check{q}_{k-2} \geq 0$. Then Eq. (4) has no positive prime period-two solution.

Proof For the sake of contradiction, assume that there exist distinct positive real number α and β , such that

$$\dots, \alpha, \beta, \alpha, \beta, \dots$$

is a period-two solution of Eq. (4). Then α, β satisfy the system

$$\alpha = f(\beta, \alpha), \quad \beta = f(\alpha, \beta).$$

Hence

$$\beta \cdot f(\beta, \alpha) - \alpha \cdot f(\alpha, \beta) = \beta \cdot \frac{(a\beta^k + b \sum_{j=1}^{k-1} \beta^j \alpha^{k-j} + c\alpha^k)}{A\beta^k + B \sum_{j=1}^{k-1} \beta^j \alpha^{k-j} + C\alpha^k} - \alpha \cdot \frac{(a\alpha^k + b \sum_{j=1}^{k-1} \alpha^j \beta^{k-j} + c\beta^k)}{A\alpha^k + B \sum_{j=1}^{k-1} \alpha^j \beta^{k-j} + C\beta^k} = 0,$$

which gives

$$(\beta - \alpha) \cdot \frac{F(\beta, \alpha)}{K(\beta, \alpha)} = 0,$$

where

$$\begin{aligned} F(\beta, \alpha) &= aC(\alpha^{2k} + \beta^{2k}) + \check{p}_1\beta\alpha(\alpha^{2k-2} + \beta^{2k-2}) + \check{q}_1\beta^2\alpha^2(\alpha^{2k-4} + \beta^{2k-4}) \\ &+ \check{q}_2\beta^3\alpha^3(\alpha^{2k-6} + \beta^{2k-6}) + \dots + \check{q}_{k-2}\beta^{k-1}\alpha^{k-1}(\alpha^2 + \beta^2) + \check{p}_2\beta^k\alpha^k \\ &+ bB\left(\sum_{j=1}^{k-1} \beta^j \alpha^{k-j}\right)^2, \\ K(\beta, \alpha) &= (A\beta^k + B \sum_{j=1}^{k-1} \beta^j \alpha^{k-j} + C\alpha^k)(A\alpha^k + B \sum_{j=1}^{k-1} \alpha^j \beta^{k-j} + C\beta^k). \end{aligned}$$

Since $\frac{F(\beta, \alpha)}{K(\beta, \alpha)} > 0$, we get $\beta = \alpha$, which is a contradiction. □

In the sequel we need the following real numbers: $r_1 = aB - bA$, $r_2 = aC - cA$, $r_3 = bC - Bc$.

Lemma 1 1) Assume that $\frac{a}{A} \geq \max\{\frac{b}{B}, \frac{c}{C}\}$ and $r_3 \geq 0$. Then f is increasing in x for each y and it is decreasing in y for each x .

2) Assume that $\frac{a}{A} \leq \min\{\frac{b}{B}, \frac{c}{C}\}$ and $r_3 \leq 0$. Then f is decreasing in x for each y and it is increasing in y for each x .

Proof 1. We have $r_3 \geq 0$ and it is easy to see that $\frac{a}{A} \geq \max\{\frac{b}{B}, \frac{c}{C}\}$ implies

$$r_1, r_2 \geq 0.$$

Therefore, the result follows from the two formulas

$$\frac{\partial f}{\partial x}(x, y) = \frac{r_1 \sum_{j=1}^{k-1} (k-j)x^{j+k-1}y^{k-j} + r_2 kx^{k-1}y^k + r_3 \sum_{j=1}^{k-1} jx^{j-1}y^{2k-j}}{(Ax^k + B \sum_{j=1}^{k-1} x^j y^{k-j} + Cy^k)^2},$$

$$\frac{\partial f}{\partial y}(x, y) = -\frac{r_1 \sum_{j=1}^{k-1} (k-j)x^{j+k}y^{k-j-1} + r_2 kx^k y^{k-1} + r_3 \sum_{j=1}^{k-1} jx^j y^{2k-j-1}}{(Ax^k + B \sum_{j=1}^{k-1} x^j y^{k-j} + Cy^k)^2}.$$

2. The proof of 2) is similar and it will be omitted. □

In the following result, we show that every positive solution of Eq. (4) is bounded.

Theorem 7 Let $\{x_n\}_{n=-1}^{+\infty}$ be a positive solution of Eq. (4).

1) Assume that $\frac{a}{A} \geq \max\{\frac{b}{B}, \frac{c}{C}\}$ and $r_3 \geq 0$. Then

$$\frac{c}{C} \leq x_n \leq \frac{a}{A}$$

for all $n \geq 1$.

2) Assume that $\frac{a}{A} \leq \min\{\frac{b}{B}, \frac{c}{C}\}$ and $r_3 \leq 0$. Then

$$\frac{a}{A} \leq x_n \leq \frac{c}{C}$$

for all $n \geq 1$.

Proof 1. We have

$$x_{n+1} - \frac{a}{A} = \frac{-r_1 \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} - r_2 x_n^k}{A(Ax_n^k + B \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + x_{n-1}^k)},$$

$$x_{n+1} - \frac{c}{C} = \frac{r_3 \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + r_2 x_n^k}{C(Ax_n^k + B \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + x_{n-1}^k)}.$$

Now using the fact that $\frac{a}{A} \geq \max\{\frac{b}{B}, \frac{c}{C}\}$, we get $r_1, r_2 \geq 0$. Therefore, it follows that

$$\frac{c}{C} \leq x_n \leq \frac{a}{A},$$

for all $n \geq 1$.

2. The proof of 2) is similar and will be omitted. □

The locally stability of the unique positive equilibrium point $\bar{x} = \frac{a+(k-1)b+c}{A+(k-1)B+C}$ of Eq. (4) is described in the following theorem.

Theorem 8 Assume that

$$\frac{k|(k-1)(r_1+r_3)+2r_2|}{(a+(k-1)b+c)(A+(k-1)B+C)} < 1.$$

Then the positive equilibrium point $\bar{x} = \frac{a+(k-1)b+c}{A+(k-1)B+C}$ of Eq. (4) is locally asymptotically stable.

Proof The linearized equation of Eq. (4) about $\bar{x} = \frac{a+(k-1)b+c}{A+(k-1)B+C}$ is

$$y_{n+1} = py_n + qy_{n-1},$$

where

$$p = \frac{\partial f}{\partial x}(\bar{x}, \bar{x}) = \frac{r_1 \sum_{j=1}^{k-1} (k-j)\bar{x}^{2k-1} + r_2 k \bar{x}^{2k-1} + r_3 \sum_{j=1}^{k-1} j \bar{x}^{2k-1}}{\left(A\bar{x}^k + B \sum_{j=1}^{k-1} \bar{x}^k + C\bar{x}^k\right)^2},$$

$$= \frac{\bar{x}^{2k-1} \left[r_1 \sum_{j=1}^{k-1} k - r_1 \sum_{j=1}^{k-1} j + r_2 k + r_3 \sum_{j=1}^{k-1} j \right]}{\bar{x}^{2k} \left(A + B \sum_{j=1}^{k-1} 1 + C \right)^2},$$

$$\begin{aligned}
 &= \frac{k(k-1)(r_1+r_3)+2r_2k}{2\bar{x}\left(A+B\sum_{j=1}^{k-1}1+C\right)^2}, \\
 &= \frac{k(k-1)(r_1+r_3)+2kr_2}{2(a+(k-1)b+c)(A+(k-1)B+C)}.
 \end{aligned}$$

Similarly, we have

$$q = \frac{\partial f}{\partial y}(\bar{x}, \bar{x}) = -\frac{k(k-1)(r_1+r_3)+2kr_2}{2(a+(k-1)b+c)(A+(k-1)B+C)}.$$

The associated characteristic equation is

$$\lambda^2 - p\lambda - q = 0.$$

Let h and g be the two functions defined by

$$h(\lambda) = \lambda^2, \quad g(\lambda) = p\lambda + q.$$

We have

$$|g(\lambda)| \leq |p| + |q| = \frac{k|(k-1)(r_1+r_3)+2r_2|}{(a+(k-1)b+c)(A+(k-1)B+C)} < 1 = |h(\lambda)|, \quad \forall \lambda \in \mathbb{C} : |\lambda| = 1.$$

Thus, by Rouché’s theorem, all zeros of $\lambda^2 - p\lambda - q = 0$ lie in $|\lambda| < 1$. Therefore, by Theorem (1), \bar{x} is locally asymptotically stable. \square

The following two theorems are devoted to the global stability of the positive equilibrium point $\bar{x} = \frac{a+(k-1)b+c}{A+(k-1)B+C}$ of Eq. (4).

Theorem 9 *Let*

$$\begin{aligned}
 \check{p}_1 &= aC - cA + aB + bC, \\
 \check{p}_2 &= aC - cA + cC + aA - (k-1)(bA + cB) + k(aB + bC), \\
 \check{q}_i &= aC - cA - i(bA + cB) + (i+1)(aB + bC), \quad i = 1, 2, \dots, k-2.
 \end{aligned}$$

1. Assume that

- $\frac{a}{A} \leq \min\{\frac{b}{B}, \frac{c}{C}\}$ and $r_3 \leq 0$.
- $\frac{k|(k-1)(r_1+r_3)+2r_2|}{(a+(k-1)b+c)(A+(k-1)B+C)} < 1$.
- $\check{p}_1, \check{p}_2, \check{q}_i \geq 0$.

Then the positive equilibrium point $\bar{x} = \frac{a+(k-1)b+c}{A+(k-1)B+C}$ of Eq.(4) is globally asymptotically stable.

Proof Let $\{x_n\}_{n=-1}^{+\infty}$ be a positive solution of Eq. (4) with $x_{-1}, x_0 \in (0, +\infty)$. By Theorem (8) we need only to prove that \bar{x} is a global attractor, that is $\lim_{n \rightarrow \infty} x_n = \bar{x}$. By Lemma (1) part 2, we see that the function

$$f(x, y) = \frac{ax^k + b \sum_{j=1}^{k-1} x^j y^{k-j} + cy^k}{Ax^k + B \sum_{j=1}^{k-1} x^j y^{k-j} + Cy^k}$$

satisfies the Hypotheses of Theorem (3); also by Theorem (7) part 2 the solution is bounded. Hence, we get $\lim_{n \rightarrow \infty} x_{2n} = l_1, \lim_{n \rightarrow \infty} x_{2n+1} = l_2$, with $l_1 = f(l_2, l_1), l_2 = f(l_1, l_2)$. Now, in view of Theorem (6) (and its proof), Eq. (4) has no prime period-two solutions and we have

$$l_1 = l_2 = \bar{x} = \frac{a + (k - 1)b + c}{A + (k - 1)B + C}.$$

□

Theorem 10 *Let*

$$\begin{aligned} p_1 &= cA - aC + bA + cB, \\ p_2 &= cA - aC + aA + cC - (k - 1)(bC + aB) + k(bA + cB), \\ q_i &= cA - aC - i(bC + aB) + (i + 1)(bA + cB), i = 1, 2, \dots, k - 2. \end{aligned}$$

1. *Assume that*

- $\frac{a}{A} \geq \max\{\frac{b}{B}, \frac{c}{C}\}$ and $r_3 \geq 0$.
- $\frac{k|(k - 1)(r_1 + r_3) + 2r_2|}{(a + (k - 1)b + c)(A + (k - 1)B + C)} < 1$.
- $p_1, p_2, q_i \geq 0$.

Then the positive equilibrium point $\bar{x} = \frac{a+(k-1)b+c}{A+(k-1)B+C}$ of Eq. (4) is globally asymptotically stable.

Proof Let $\{x_n\}_{n=-1}^{+\infty}$ be a positive solution of Eq. (4) with $x_{-1}, x_0 \in [\frac{c}{C}, \frac{a}{A}]$. By Theorem (8) we need only to prove that \bar{x} is a global attractor, that is $\lim_{n \rightarrow \infty} x_n = \bar{x}$. Consider again the function

$$f(x, y) = \frac{ax^k + b \sum_{j=1}^{k-1} x^j y^{k-j} + cy^k}{Ax^k + B \sum_{j=1}^{k-1} x^j y^{k-j} + Cy^k}.$$

Suppose that $(m, M) \in [\frac{c}{C}, \frac{a}{A}] \times [\frac{c}{C}, \frac{a}{A}]$ is a solution of the system

$$M = f(M_1(m, M), M_2(m, M)), m = f(m_1(m, M), m_2(m, M)). \tag{13}$$

By Lemma (1) part 1, we have

$$M_1(m, M) = M, M_2(m, M) = m, m_1(m, M) = m, m_2(m, M) = M. \tag{14}$$

From (13) and (14), we get

$$M = \frac{aM^k + b \sum_{j=1}^{k-1} M^j m^{k-j} + cm^k}{AM^k + B \sum_{j=1}^{k-1} M^j m^{k-j} + Cm^k}, m = \frac{am^k + b \sum_{j=1}^{k-1} m^j M^{k-j} + cM^k}{Am^k + B \sum_{j=1}^{k-1} m^j M^{k-j} + CM^k}.$$

Hence,

$$M \cdot \frac{am^k + b \sum_{j=1}^{k-1} m^j M^{k-j} + cM^k}{Am^k + B \sum_{j=1}^{k-1} m^j M^{k-j} + CM^k} - m \cdot \frac{aM^k + b \sum_{j=1}^{k-1} M^j m^{k-j} + cm^k}{AM^k + B \sum_{j=1}^{k-1} M^j m^{k-j} + Cm^k} = 0,$$

which can be written as

$$(M - m) \cdot \frac{L(m, M)}{K(m, M)} = 0,$$

where

$$L(m, M) = cA(M^{2k} + m^{2k}) + p_1mM(M^{2k-2} + m^{2k-2}) + q_1m^2M^2(M^{2k-4} + m^{2k-4}) + q_2m^3M^3(M^{2k-6} + m^{2k-6}) + \dots + q_{k-2}m^{k-1}M^{k-1}(M^2 + m^2) + p_2m^kM^k + bB(\sum_{j=1}^{k-1} m^j M^{k-j})^2, \text{ and}$$

$$K(m, M) = (Am^k + B \sum_{j=1}^{k-1} m^j M^{k-j} + CM^k)(AM^k + B \sum_{j=1}^{k-1} M^j m^{k-j} + Cm^k).$$

Since $\frac{L(m, M)}{K(m, M)} > 0$, we get $M = m$. From this fact and by Lemma (1) part 1, all Hypotheses of Theorem (2) are satisfied and we have $\lim_{x \rightarrow \infty} x_n = \bar{x}$. □

3. Rate of convergence and numerical examples

Now we estimate the rate of convergence of a solution that converges to the positive equilibrium point $\bar{x} = \frac{a+(k-1)b+c}{A+(k-1)B+C}$ of Eq. (4). First, we will find a system that satisfies the error terms. Hence, we consider the quantity

$$x_{n+1} - \bar{x} = \frac{ax_n^k + b \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + cx_{n-1}^k}{Ax_n^k + B \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + Cx_{n-1}^k} - \frac{a + (k - 1)b + c}{A + (k - 1)B + C}$$

$$\begin{aligned}
 &= \frac{\left(r_1 (k-1) x_n^k + r_2 x_n^k + r_3 \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} \right) - \left(r_1 \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + r_2 x_{n-1}^k + r_3 (k-1) x_{n-1}^k \right)}{\left(Ax_n^k + B \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + Cx_{n-1}^k \right) (A + (k-1)B + C)} \\
 &= \frac{r_1 \left((k-1) x_n^k - \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} \right) + r_2 (x_n^k - x_{n-1}^k) + r_3 \left(\sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} - (k-1) x_{n-1}^k \right)}{\left(Ax_n^k + B \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + Cx_{n-1}^k \right) (A + (k-1)B + C)}
 \end{aligned}$$

for $n \in \mathbb{N}_0$. The last equality can be written as follows:

$$\begin{aligned}
 x_{n+1} - \bar{x} &= \frac{r_1 [(x_n^k - x_n x_{n-1}^{k-1}) + (x_n^k - x_n^2 x_{n-1}^{k-2}) + \dots + (x_n^k - x_n^{k-1} x_{n-1})]}{\left(Ax_n^k + B \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + Cx_{n-1}^k \right) (A + (k-1)B + C)} \\
 &+ \frac{r_2 (x_n - x_{n-1}) (x_n^{k-1} + x_n^{k-2} x_{n-1} + \dots + x_{n-1}^{k-1})}{\left(Ax_n^k + B \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + Cx_{n-1}^k \right) (A + (k-1)B + C)} \\
 &+ \frac{r_3 [(x_n x_{n-1}^{k-1} - x_{n-1}^k) + (x_n^2 x_{n-1}^{k-2} - x_{n-1}^k) + \dots + (x_n^{k-1} x_{n-1} - x_{n-1}^k)]}{\left(Ax_n^k + B \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + Cx_{n-1}^k \right) (A + (k-1)B + C)}.
 \end{aligned}$$

After some computations, we get

$$\begin{aligned}
 x_{n+1} - \bar{x} &= \frac{r_1 (x_n - x_{n-1}) \left[\sum_{j=0}^{k-2} x_n^{j+1} x_{n-1}^{k-j-2} + \sum_{j=0}^{k-3} x_n^{j+2} x_{n-1}^{k-j-3} + \dots + \sum_{j=0}^0 x_n^{j+k-1} x_{n-1}^{-j} \right]}{\left(Ax_n^k + B \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + Cx_{n-1}^k \right) (A + (k-1)B + C)} \\
 &+ \frac{r_2 (x_n - x_{n-1}) \sum_{j=0}^{k-1} x_n^j x_{n-1}^{k-j-1}}{\left(Ax_n^k + B \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + Cx_{n-1}^k \right) (A + (k-1)B + C)} \\
 &+ \frac{r_3 (x_n - x_{n-1}) \left[\sum_{j=0}^0 x_n^{-j} x_{n-1}^{j+k-1} + \sum_{j=0}^1 x_n^{1-j} x_{n-1}^{j+k-2} + \dots + \sum_{j=0}^{k-2} x_n^{k-j-2} x_{n-1}^{j+1} \right]}{\left(Ax_n^k + B \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + Cx_{n-1}^k \right) (A + (k-1)B + C)}
 \end{aligned}$$

or

$$x_{n+1} - \bar{x} = (r_1 S_1 + r_2 S_2 + r_3 S_3) (x_n - x_{n-1}),$$

where

$$S_1 = \frac{\sum_{j=0}^{k-2} x_n^{j+1} x_{n-1}^{k-j-2} + \sum_{j=0}^{k-3} x_n^{j+2} x_{n-1}^{k-j-3} + \dots + \sum_{j=0}^0 x_n^{j+k-1} x_{n-1}^{-j}}{\left(Ax_n^k + B \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + Cx_{n-1}^k \right) (A + (k-1)B + C)},$$

$$S_2 = \frac{\sum_{j=0}^{k-1} x_n^j x_{n-1}^{k-j-1}}{\left(Ax_n^k + B \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + Cx_{n-1}^k \right) (A + (k-1)B + C)}$$

and

$$S_3 = \frac{\sum_{j=0}^0 x_n^{-j} x_{n-1}^{j+k-1} + \sum_{j=0}^1 x_n^{1-j} x_{n-1}^{j+k-2} + \dots + \sum_{j=0}^{k-2} x_n^{k-j-2} x_{n-1}^{j+1}}{\left(Ax_n^k + B \sum_{j=1}^{k-1} x_n^j x_{n-1}^{k-j} + Cx_{n-1}^k \right) (A + (k-1)B + C)}.$$

Hence, we get the equation

$$x_{n+1} - \bar{x} = (r_1 S_1 + r_2 S_2 + r_3 S_3) (x_n - \bar{x}) - (r_1 S_1 + r_2 S_2 + r_3 S_3) (x_{n-1} - \bar{x}). \tag{15}$$

Note that

$$\lim_{n \rightarrow \infty} S_1 = \lim_{n \rightarrow \infty} S_3 = \frac{k(k-1)}{2(a + (k-1)b + c)(A + (k-1)B + C)},$$

$$\lim_{n \rightarrow \infty} S_2 = \frac{k}{(a + (k-1)b + c)(A + (k-1)B + C)},$$

since $x_n \rightarrow \bar{x}$ as $n \rightarrow \infty$. Let

$$e_n = x_n - \bar{x}.$$

Then Eq. (15) becomes

$$e_{n+1} = (p + \epsilon_1(n))e_n + (q + \epsilon_2(n))e_{n-1} \tag{16}$$

where $\epsilon_1(n), \epsilon_2(n) \rightarrow 0$ as $n \rightarrow \infty$. Clearly Eq. (16) can be written in the matrix form

$$\begin{pmatrix} e_n \\ e_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix} \begin{pmatrix} e_{n-1} \\ e_n \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \epsilon_2(n) & \epsilon_1(n) \end{pmatrix} \begin{pmatrix} e_{n-1} \\ e_n \end{pmatrix}$$

and the characteristic equation of the matrix $\begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}$ is the same as the characteristic equation of the linearized equation at $\bar{x} = \frac{a+(k-1)b+c}{A+(k-1)B+C}$. Using Perron's Theorems, we have the following result.

Theorem 11 *Let \bar{x} be the positive equilibrium point and the sequence $(x_n)_{n \geq 1}$ be a positive solution of Eq.*

(4). *Then the error vector $E_n = \begin{pmatrix} e_n \\ e_{n-1} \end{pmatrix}$ of every solution of Eq. (4) satisfies both of the asymptotic relations*

$$\rho = \lim_{n \rightarrow \infty} \frac{\|E_{n+1}\|}{\|E_n\|}, \quad \rho = \lim_{n \rightarrow \infty} (\|E_n\|)^{1/n},$$

where $e_n = x_n - \bar{x}$ and ρ is equal to the modulus of one of the roots of the characteristic equation.

For confirming the results of this paper, we consider the following numerical examples.

Example 1 If we take $k = 3$, $a = 2$, $b = 1$, $c = 2$, $A = 5$, $B = 2$, $C = 3$ we obtain the equation

$$x_{n+1} = \frac{2x_n^3 + x_n^2x_{n-1} + x_nx_{n-1}^2 + 2x_{n-1}^3}{5x_n^3 + 2x_n^2x_{n-1} + 2x_nx_{n-1}^2 + 3x_{n-1}^3}. \tag{17}$$

Let $x_{-1} = 0.45$, $x_0 = 0.55$, and we have $\bar{x} = 0.5$. All conditions of Theorem (9) are satisfied and $\lim_{n \rightarrow +\infty} x_n = \bar{x}$. (See Figure 1).

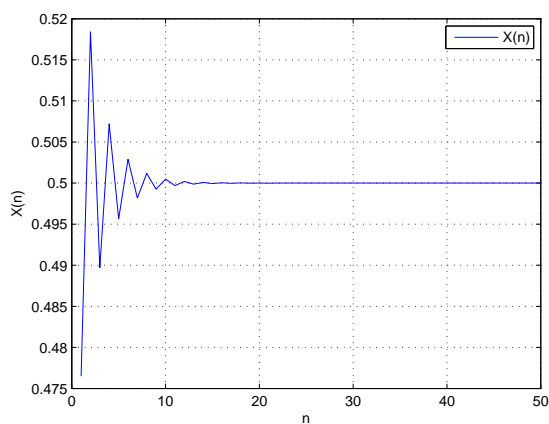


Figure 1. This figure shows the global attractivity of the equilibrium point ($\bar{x} = 0.5$) of Eq. (17).

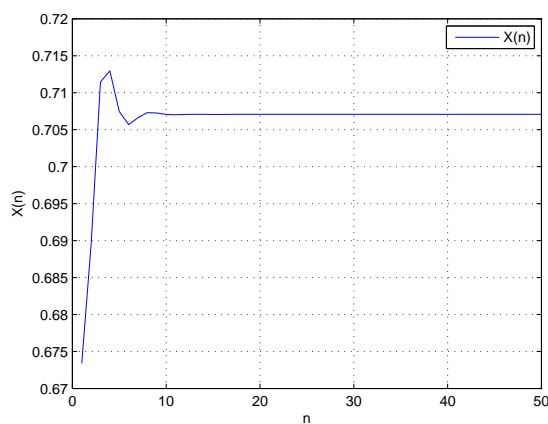


Figure 2. This figure shows the global attractivity of the equilibrium point ($\bar{x} = 0.707\dots$) of Eq. (18).

Example 2 If we take $k = 4$, $a = 2$, $b = 2$, $c = 6$, $A = 1.3$, $B = 3$, $C = 9.5$ we obtain the equation

$$x_{n+1} = \frac{2x_n^4 + 2x_n^3x_{n-1} + 2x_n^2x_{n-1}^2 + 2x_nx_{n-1}^3 + 6x_{n-1}^4}{1.3x_n^4 + 3x_n^3x_{n-1} + 3x_n^2x_{n-1}^2 + 3x_nx_{n-1}^3 + 9.5x_{n-1}^4}. \tag{18}$$

Let $x_{-1} = 0.95$, $x_0 = 0.75$, and we have $\bar{x} = 0.707\dots$. All conditions of Theorem (10) are satisfied and $\lim_{n \rightarrow +\infty} x_n = \bar{x}$. (See Figure 2).

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