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On the block sequence space $l_p(E)$ and related matrix transformations

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Abstract: The purpose of the present study is to introduce the sequence space

$$l_p(E) = \left\{ x = (x_n)_{n=1}^{\infty} \ : \ \sum_{n=1}^{\infty} \left| \sum_{j \in E_n} x_j \right|^p < \infty \right\},$$

where $E = (E_n)$ is a partition of finite subsets of the positive integers and $1 \le p < \infty$. We investigate some topological properties of this space and also give some inclusion relations concerning it. Furthermore, we compute α- and β-duals of this space and characterize the matrix transformations from the space $l_p(E)$ to the space X, where $X \in \{l_\infty, c, c_0\}$.

Key words: Sequence spaces, matrix domains, α- and β-duals, matrix transformations

1. Introduction

Let ω denote the space of all real-valued sequences. Any vector subspace of ω is called a sequence space. For $1 \le p < \infty$, denote by l_p the space of all real sequences $x = (x_n) \in \omega$ such that

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < \infty.$$

For $p = \infty$, $\left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$ is interpreted as $\sup_{n \ge 1} |x_n|$. We write c and c_0 for the spaces of all convergent and null sequences, respectively. By cs, we denote the space of all convergent series.

Let X, Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N} = \{1, 2, \cdots\}$. We say that A defines a matrix mapping from X into Y, and we denote it by $A : X \to Y$, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}_{n=1}^{\infty}$ exists and is in Y, where $(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k$ for $n = 1, 2, \cdots$. By (X, Y), we denote the class of all infinite matrices A such that $A : X \to Y$.

For a sequence space X, the matrix domain X_A of an infinite matrix A is defined by

$$X_A = \{x = (x_n) \in \omega \ : \ Ax \in X\}, \tag{1.1}$$

which is a sequence space. The new sequence space X_A generated by the limitation matrix A from a sequence space X can be the expansion or the contraction and/or the overlap of the original space X. A matrix

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$A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. If A is triangle, then one can easily observe that the sequence spaces X_A and X are linearly isomorphic, i.e. $X_A \cong X$.

In the past, several authors studied matrix transformations on sequence spaces that are the matrix domains of triangle matrices in classical spaces l_p, l_∞, c, and c_0. For instance, some matrix domains of the difference operator were studied in [4, 8, 9, 13], of the Riesz matrices in [1, 3], of the Euler matrices in [2, 6, 12], of the Cesàro matrices in [5, 14, 15], and of the Nörlund matrices in [16, 17]. In these studies the matrix domains were obtained by triangle matrices, and hence these spaces are normed sequence spaces. For more details on the domain of triangle matrices in some sequence spaces, the reader may refer to Chapter 4 of [7]. The matrix domains given in this paper specify a certain nontriangle matrix, so we should not expect that related spaces are normed sequence spaces.

In this study, the normed sequence space l_p is extended to seminormed space $l_p(E)$. We consider some topological properties of this space and derive inclusion relations concerning it. Moreover, we determine the α- and β-duals for the space $l_p(E)$, and we also obtain the necessary and sufficient conditions on an infinite matrix belonging to the classes $(l_p(E), l_\infty)$, $(l_p(E), c)$, and $(l_p(E), c_0)$. The results are generalizations of some results of Malkowsky and Rakocevic [11]. In a similar way, Erfanmanesh and Foroutannia introduced the sequence spaces $l_\infty(E)$, $c(E)$ and $c_0(E)$ in [10].

2. The sequence space $l_p(E)$ of nonabsolute type

Let $E = (E_n)$ be a partition of finite subsets of the positive integers such that

$$\max E_n < \min E_{n+1}, \tag{2.1}$$

for $n = 1, 2, \cdots$. We define the sequence spaces $l_p(E)$ by

$$l_p(E) = \left\{ x = (x_n) \in \omega \ : \ \sum_{n=1}^{\infty} \left| \sum_{j \in E_n} x_j \right|^p < \infty \right\}, \qquad (1 \leq p < \infty),$$

with the seminorm $|\!|\!|.|\!|\!|_{p,E}$, which is defined in the following way:

$$|\!|\!|x|\!|\!|_{p,E} = \left(\sum_{n=1}^{\infty} \left| \sum_{j \in E_n} x_j \right|^p \right)^{1/p}. \tag{2.2}$$

For example, if $E_n = \{2n-1, 2n\}$ for all n, then $x = (x_n) \in l_p(E)$ if and only if $\sum_{n=1}^{\infty} |x_{2n-1} + x_{2n}|^p < \infty$. It should be noted that the function $|\!|\!|.|\!|\!|_{p,E}$ cannot be the norm, since if $x = (1, -1, 0, 0, \cdots)$ and $E_n = \{2n-1, 2n\}$ for all n, then $|\!|\!|x|\!|\!|_{p,E} = 0$ while $x \neq \theta$. It is also significant that in the special case $E_n = \{n\}$ for $n = 1, 2, \cdots$, we have $l_p(E) = l_p$ and $|\!|\!|x|\!|\!|_{p,E} = \|x\|_p$.

Suppose that $E = (E_n)$ is a partition of finite subsets of the positive integers such that it satisfies the condition (2.1). If the infinite matrix $A = (a_{nk})$ is defined by

$$a_{nk} = \begin{cases} 1 & if\ k \in E_n \\ 0 & otherwise, \end{cases} \tag{2.3}$$

with the notation of (1.1), we can redefine the space $l_p(E)$ as follows:

$$l_p(E) = (l_p)_A.$$

Now we may begin with the following theorem, which is essential in this study.

Theorem 2.1 *The set $l_p(E)$ becomes a vector space with coordinate-wise addition and scalar multiplication, which is the complete seminormed space by $\|\!|.\|\!|_{p,E}$ defined by (2.2).*

Proof This is a routine verification and so we omit the detail. □

It can easily be checked that the absolute property does not hold on the space $l_p(E)$; that is, $\|\!|x\|\!|_{p,E} \neq \|\!||x|\|\!|_{p,E}$ for at least one sequence in the space $l_p(E)$, and this says that $l_p(E)$ is a sequence space of nonabsolute type, where $|x| = (|x_k|)$. Throughout this article, the cardinal number of the set E_k is denoted by $|E_k|$.

Theorem 2.2 *Let $M = \left\{x = (x_n) \ : \ \sum_{j \in E_n} x_j = 0, \forall n\right\}$. The quotient space $l_p(E)/M$ is linearly isomorphic to the space l_p, i.e. $l_p(E)/M \simeq l_p$.*

Proof Consider the map $T : l_p(E) \longrightarrow l_p$ defined by

$$Tx = \left(\sum_{j \in E_n} x_j\right)_{n=1}^{\infty},$$

for all $x \in l_p(E)$. The linearity of T is trivial. Letting $y \in l_p$ and $\alpha_n = |E_n|$ for all n, we define the sequence $x = (x_k)$ by $x_k = y_n/\alpha_n$ for all $k \in E_n$. It is clear that $x \in l_p(E)$ and $Tx = y$, and so the map T is surjective. By applying the first isomorphism theorem we have $l_p(E)/M \simeq l_p$, because $kerT = M$. □

Note that the mapping defined in Theorem 2.2, T is not injective, while $\|Tx\|_p = \|\!|x\|\!|_{p,E}$ for all $x \in l_p(E)$.

One may expect a similar result for the space $l_p(E)$ as was observed for the space l_p and ask the following natural question: Is the space $l_p(E)$ a semiinner product space for $p = 2$? The answer is positive and is given by the following theorem:

Theorem 2.3 *Except for the case $p = 2$, the space $l_p(E)$ is not a semiinner product space.*

Proof If we define $< x, y >= \sum_{n=1}^{\infty} \sum_{i,j \in E_n} x_i y_j$, then it is a semiinner product on the space $l_2(E)$ and $\|\!|x\|\!|_{2,E}^2 =< x, x >$. Now considering the sequences x and y such that

$$\sum_{j \in E_1} x_j = 1, \ \sum_{j \in E_2} x_j = 2, \ \sum_{j \in E_3} x_j = \sum_{j \in E_4} x_j = \cdots = 0, \ \sum_{j \in E_1} y_j = 3, \ \sum_{j \in E_2} y_j = \sum_{j \in E_3} y_j = \cdots = 0,$$

we see that

$$\|\!|x + y\|\!|_{p,E}^2 + \|\!|x - y\|\!|_{p,E}^2 \neq 2\left(\|\!|x\|\!|_{p,E}^2 + \|\!|y\|\!|_{p,E}^2\right) \quad (p \neq 2).$$

Since the equation $(2^p + 1)^{2/p} + 2^{(2/p)+1} = 9$ has only one root, $p = 2$, the seminorm of the space $l_p(E)$ does not satisfy the parallelogram equality, which means that the seminorm cannot be obtained from the semiinner product. Hence, the space $l_p(E)$ with $p \neq 2$ is not a semiinner product space. □

Let X be a seminormed space with a seminorm g. A sequence (b_n) of the elements of seminormed space X is called a Schauder basis (or briefly a basis) for X iff for each $x \in X$ there exists a unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} g\left(x - \sum_{k=1}^{n} \alpha_k b_k\right) = 0.$$

The series $\sum_{k=1}^{\infty} \alpha_k b_k$ that has the sum x is then called the expansion of x with respect to (b_n) and is written as $x = \sum_{k=1}^{\infty} \alpha_k b_k$.

In the following, we will give a sequence of the points of the space $l_p(E)$, which forms a basis for the space $l_p(E)$.

Theorem 2.4 *If the sequence $b^{(k)} = \{b_j^{(k)}\}_{j=1}^{\infty}$ is defined such that $\sum_{j\in E_k} b_j^{(k)} = 1$ and the remaining elements are zero, for $k = 1, 2, \cdots$, then the sequence $\{b^{(k)}\}_{k=1}^{\infty}$ is a basis for the space $l_p(E)$, and any $x \in l_p(E)$ has a unique representation of the form*

$$x = \sum_{k=1}^{\infty} \alpha_k b^{(k)},$$

where $\alpha_k = \sum_{j\in E_k} x_j$ for $k = 1, 2, \cdots$.

Proof This is a routine verification and so we omit the detail. □

Definition 2.5 *Let $E = (E_n)$ be a partition of finite subsets of the positive integers that satisfies the condition (2.1), and let $s = (s_n)$ be a strictly increasing sequence of the positive integers. The generated partition $H = (H_n)$ is defined by E and s as follows*

$$H_n = \cup_{j=s_{n-1}+1}^{s_n} E_j,$$

for $n = 1, 2, \cdots$.

Here and in the sequel, we shall use the convention that any term with a zero subscript is equal to zero. Note that any arbitrary partition $H = (H_n)$ that satisfies the condition (2.1) is generated by the partition $E = (E_n)$ and the sequence $s = (s_n)$, where $E_n = \{n\}$ and $s_n = \max H_n$ for all n. It is also important to know $s_n - s_{n-1} = |H_n|$.

In the following, the inclusion relation between the spaces $l_p(E)$ and $l_p(H)$ is examined. Obviously, if $s_n - s_{n-1} > 1$ only for a finite number of n, then

$$l_p(E) = l_p(H).$$

Especially if $|H_n| > 1$ only for a finite number of n, then $l_p = l_p(H)$.

Theorem 2.6 *Let E, s, and H be as in Definition 2.5. We have $l_1(E) \subset l_1(H)$, and $l_p(E) \subset l_p(H)$ when $1 < p < \infty$ and*

$$\sup_n (s_n - s_{n-1})^{p-1} < \infty. \tag{2.4}$$

Moreover, if $s_n - s_{n-1} > 1$ for an infinite number of n, then these inclusions are strict.

Proof Let $1 \leq p < \infty$ and $\zeta = \sup_n (s_n - s_{n-1})^{p-1}$. To prove the validity of the inclusion $l_p(E) \subset l_p(H)$, it suffices to show

$$\|x\|_{p,H} \leq \zeta^{\frac{1}{p}} \|x\|_{p,E}, \tag{2.5}$$

for each $x \in l_p(E)$. Note that $\zeta = 1$ when $p = 1$. Suppose that $x = (x_n) \in l_p(E)$ is an arbitrary sequence. By applying Hölder's inequality, we have

$$\left| \sum_{j \in H_n} x_j \right|^p = \left| \sum_{k=s_{n-1}+1}^{s_n} \sum_{j \in E_k} x_j \right|^p \leq (s_n - s_{n-1})^{p-1} \sum_{k=s_{n-1}+1}^{s_n} \left| \sum_{j \in E_k} x_j \right|^p,$$

so

$$\|x\|_{p,H}^p \leq \zeta \sum_{n=1}^{\infty} \sum_{k=s_{n-1}+1}^{s_n} \left| \sum_{j \in E_k} x_j \right|^p = \zeta \|x\|_{p,E}^p.$$

Moreover, letting $s_n - s_{n-1} > 1$ for an infinite number of n, one can choose a subsequence (n_j) in $\mathbb{N}$ with $s_{n_j} - s_{n_{j-1}} > 1$ for $j = 1, 2, \cdots$. We define the sequence $x = (x_k)$ such that

$$\sum_{i \in E_k} x_i = \begin{cases} j & if\ k = s_{n_{j-1}} + 1 \\ -j & if\ k = s_{n_{j-1}} + 2 \\ 0 & otherwise, \end{cases} \tag{2.6}$$

for $k = 1, 2, \cdots$. It is obvious that $\sum_{i \in H_k} x_i = 0$, so $x \in l_p(H)$ while $x \notin l_p(E)$. Hence, $x \in l_p(H) - l_p(E)$, and the inclusion $l_p(E) \subset l_p(H)$ strictly holds. □

Corollary 2.7 *Let $H = (H_n)$ be a partition of finite subsets of the positive integers that satisfies the condition (2.1). We have $l_1 \subset l_1(H)$, and $l_p \subset l_p(H)$ when $1 < p < \infty$ and $\sup_n |H_n|^{p-1} < \infty$. Moreover, if $|H_n| > 1$ for an infinite number of n, then these inclusions are strict.*

Proof If $E_n = \{n\}$ and $s_n = \max H_n$ for all n, then the partition $H = (H_n)$ is generated by $E = (E_n)$ and $s = (s_n)$. The desired result follows from Theorem 2.6. □

Corollary 2.8 *Let $1 \leq p < \infty$, and let M and N be two positive integers. If we put $E_i = \{Mi - M + 1, Mi - M + 2, \cdots, Mi\}$ and $H_i = \{MNi - MN + 1, MNi - MN + 2, \cdots, MNi\}$ for all i, then $l_p(E) \subset l_p(H)$. Moreover, if $N > 1$, then this inclusion strictly holds.*

Proof If $s_i = Ni$ for all i, then the partition $H = (H_n)$ is generated by E and s. The desired result follows from Theorem 2.6. □

In the following, we consider the necessity of condition (2.4) in Theorem (2.6).

Theorem 2.9 *Let E, s, and H be as in Definition 2.5. If $1 < p < \infty$ and*

$$\sup_n (s_n - s_{n-1})^{p-1} = \infty, \tag{2.7}$$

then neither of the spaces $l_p(E)$ nor $l_p(H)$ includes the other one.

Proof There exists a subsequence $\{n_k\}$ such that $s_{n_k} - s_{n_{k-1}} \geq k \geq 2$, by (2.7). Consider the sequence $y = (y_i)$ such that

$$\sum_{j \in E_i} y_j = \frac{1}{\left(s_{n_k} - s_{n_{k-1}}\right) k^{1/p}},$$

for $s_{n_{k-1}} + 1 \leq i \leq s_{n_k}$. We conclude that $y \in l_p(E) - l_p(H)$. Also, if the sequence $x = (x_n)$ is defined as in (2.6), then $x \in l_p(H) - l_p(E)$. This completes the proof. □

Corollary 2.10 *Let $H = (H_n)$ be a sequence of finite subsets of the positive integers that satisfies the condition (2.1). If $1 < p < \infty$ and*

$$\sup_n |H_n|^{p-1} = \infty,$$

then neither of the spaces $l_p(H)$ nor l_p includes the other one.

Proof If $E_n = \{n\}$ and $a_n = \max H_n$ for all n, the desired result follows from Theorem 2.9. □

Theorem 2.11 *If $1 \leq p < s$, then $l_p(H) \subset l_s(H)$.*

Proof This is a routine verification and so we omit the details. □

3. The α- and β-duals of sequence space $l_p(E)$

In this section, we compute the α- and β-duals for the sequence space $l_p(E)$. For the sequence spaces X and Y, the set $M(X,Y)$ defined by

$$M(X,Y) = \{a = (a_k) \in \omega \ : \ (a_k x_k)_{k=1}^{\infty} \in Y \quad \forall x = (x_k) \in X\}$$

is called the multiplier space of X and Y. With the above notation, the α- and β-duals of a sequence space X, which are respectively denoted by X^α and X^β, are defined by

$$X^\alpha = M(X, l_1), \qquad X^\beta = M(X, cs).$$

Lemma 3.1 *([11], pp. 156–157) Let $X, Y, Z \subset \omega$ and $Y \subset Z$. We have $M_{E,F}(X,Y) \subset M_{E,F}(X,Z)$, and in particular $X^\alpha \subset X^\beta$.*

Theorem 3.2 *If $1 < p < \infty$ and $q = p/(p-1)$ and the set d_q is defined as follows:*

$$d_q = \left\{ a = (a_k) \in \omega \ : \ \sum_{k=1}^{\infty} \left(\sup_{i \in E_k} |a_i|^q \right) < \infty \right\},$$

then $(l_p(E))^\beta = d_q$. Moreover, for $p = 1$, we have $(l_1(E))^\beta = l_\infty$.

Proof We only prove the statement for the case $1 < p < \infty$; the case $p = 1$ is proved similarly. Let $a \in d_q$ be given. Since $E = (E_n)$ is a partition of the positive integers, by Hölder's inequality we have

$$\begin{aligned} \left|\sum_{k=1}^{\infty} a_k x_k\right| &= \left|\sum_{k=1}^{\infty}\sum_{i\in E_k} a_i x_i\right| \\ &\le \sum_{k=1}^{\infty}\left(\sup_{i\in E_k}|a_i|\right)\left|\sum_{i\in E_k} x_i\right| \\ &\le \left(\sum_{k=1}^{\infty}\sup_{i\in E_k}|a_i|^q\right)^{1/q}\left(\sum_{k=1}^{\infty}\left|\sum_{i\in E_k} x_i\right|^p\right)^{1/p} < \infty, \end{aligned} \tag{3.1}$$

for all $x \in l_p(E)$. This shows $ax \in cs$; thus $a \in (l_p(E))^\beta$ and hence $d_q \subset (l_p(E))^\beta$.

Now, let $a \in (l_p(E))^\beta$ be given. We consider the linear functional $f_n : l_p(E) \to \mathbb{R}$ defined by

$$f_n(x) = \sum_{k=1}^{n}\sum_{i\in E_k} a_i x_i \qquad (\, x = (x_k) \in l_p(E)),$$

for $n = 1, 2, \cdots$. Similar to (3.1), we obtain

$$|f_n(x)| \le \left(\sum_{k=1}^{n}\sup_{i\in E_k}|a_i|^q\right)^{1/q}\left(\sum_{k=1}^{n}\left|\sum_{i\in E_k} x_i\right|^p\right)^{1/p},$$

for every $x \in l_p(E)$. So the linear functional f_n is bounded and

$$\|f_n\| \le \left(\sum_{k=1}^{n}\sup_{i\in E_k}|a_i|^q\right)^{1/q},$$

for all n. We now prove the reverse of the above inequality. Without loss of generality we assume there is an index i such that $1 \le i \le \max E_n$ and $a_i \ne 0$, since the case $a_i = 0$ for all $1 \le i \le \max E_n$ is trivial. We define the sequence $x = (x_i)$ by

$$x_i = (sgn\ a_i)\,|a_i|^{q-1},$$

where $i \in E_k$ is the first index of E_k such that $|a_i| = \sup_{j\in E_k}|a_j|$, for $1 \le k \le n$, and we put the remaining elements as zero. Obviously $\left|\sum_{j\in E_k} x_j\right| = \sup_{j\in E_k}|a_j|^{q-1}$ and $x \in l_p(E)$, so

$$\|f_n\| \ge \frac{|f_n(x)|}{\|x\|_{p,E}} = \frac{\sum_{k=1}^{n}\sup_{j\in E_k}|a_j|^q}{\left(\sum_{k=1}^{n}\sup_{j\in E_k}|a_j|^q\right)^{1/p}} = \left(\sum_{k=1}^{n}\sup_{i\in E_k}|a_i|^q\right)^{1/q},$$

for $n = 1, 2, \cdots$. Since $a \in (l_p(E))^\beta$, the map $f_a : l_p(E) \to \mathbb{R}$ defined by

$$f_a(x) = \sum_{k=1}^{\infty}\sum_{i\in E_k} a_i x_i \qquad (\, x = (x_k) \in l_p(E)),$$

is well defined and linear, and also the sequence (f_n) is pointwise convergent to f_a. By using the Banach–Steinhaus theorem, it can be shown that $\|f_a\| \le \sup_n \|f_n\| < \infty$, so $\left(\sum_{k=1}^{\infty} \sup_{i\in E_k} |a_i|^q\right)^{1/q} < \infty$ and $a \in d_q$. This establishes the proof. □

Note that $d_q = l_q$ when $\sup_k |E_k| < \infty$ and $1 < q < \infty$, since

$$\sum_{k=1}^{\infty} \sup_{i\in E_k} |a_i|^q \le \sum_{k=1}^{\infty} |a_k|^q,$$

and

$$\sum_{k=1}^{\infty} |a_k|^q = \sum_{k=1}^{\infty} \sum_{i\in E_k} |a_i|^q \le \left(\sup_k |E_k|\right) \sum_{k=1}^{\infty} \sup_{i\in E_k} |a_i|^q.$$

It is noteworthy that the inclusion $l_q \subset d_q$ strictly holds when $\sup_k |E_k| = \infty$, because if $E_1 = \{1\}$, $E_2 = \{2,3\}$, $E_3 = \{4,5,6\}, \cdots$ and $x_i = \frac{1}{k}$ for $i \in E_k$, we deduce that $x \in d_q - l_q$, when $1 < q < 2$.

Corollary 3.3 *Let* $\sup_k |E_k| < \infty$. *We have* $(l_p(E))^\beta = l_q$, *when* $1 < p < \infty$ *and* $q = p/(p-1)$.

Corollary 3.4 *([11], Theorem 1.29) We have* $l_1^\beta = l_\infty$, *and* $l_p^\beta = l_q$ *when* $1 < p < \infty$ *and* $q = p/(p-1)$.

Proof If $E_k = \{k\}$ for all k, by applying Theorem 3.2, we obtain the desired result. □

Theorem 3.5 *We have* $(l_1(E))^\alpha \subset l_\infty$, *and* $(l_p(E))^\alpha \subset d_q$ *when* $1 < p < \infty$ *and* $q = p/(p-1)$. *Moreover, if* $|E_n| > 1$ *for an infinite number of* n, *then these inclusions are strict.*

Proof We only prove the statement for the case $1 < p < \infty$; the other case is proved in the same way. One can conclude from Theorem 3.2 and Lemma 3.1 that

$$(l_p(E))^\alpha \subset (l_p(E))^\beta = d_q.$$

Moreover, if the sequence $a = (a_i)$ is defined by $a_i = 1/k$ whenever $i \in E_k$, then we have $a \in d_q$. Because $|E_n| > 1$ for an infinite number of n, we may choose an index subsequence (n_j) of the positive integers with $|E_{n_j}| > 1$ for $j = 1, 2, \cdots$. Letting $\alpha_j = \min E_{n_j}$, we define the sequence $x = (x_i)$ as follows:

$$x_i = \begin{cases} n_j & if \;\; i = \alpha_j \\ -n_j & if \; i = \alpha_j + 1 \\ 0 & otherwise, \end{cases}$$

for $i = 1, 2, \cdots$. Thus, $\sum_{i\in E_k} x_i = 0$ and $x \in l_p(E)$, while

$$\sum_{k=1}^{\infty} |a_k x_k| = \sum_{j=1}^{\infty} \sum_{i\in E_{n_j}} |a_i x_i| = \sum_{j=1}^{\infty} 2 = \infty,$$

and this shows $a \notin (l_p(E))^\alpha$. Therefore, this inclusion strictly holds. □

Corollary 3.6 *We have* $(l_p(E))^\alpha \subset l_q$ *when* $\sup_k |E_k| < \infty$, $1 < p < \infty$ *and* $q = p/(p-1)$. *Moreover, if* $|E_n| > 1$ *for an infinite number of* n, *then the inclusion strictly holds.*

4. Matrix transformations on sequence spaces $l_p(E)$

In the present section, some classes of infinite matrices related with new sequence space are characterized. Let $A = (a_{nk})$ be an infinite matrix of real numbers and X and Y be two sequence spaces. We write $A_n = (a_{n,k})_{k=1}^{\infty}$ for the sequence in the nth row of A. It is clear that $A \in (X, Y)$ if and only if $A_n \in X^{\beta}$ for all n and $Ax \in Y$ for all $x \in X$.

We start with the following lemma, which is needed to prove our main result.

Lemma 4.1 *Suppose that $a = (a_k) \in \omega$, and the linear functional $f : l_p(E) \to \mathbb{R}$ is defined by*

$$f(x) = \sum_{k=1}^{\infty} a_k x_k \qquad (\, x = (x_k) \in l_p(E))\,.$$

(i) If $p = 1$ and $a \in l_\infty$, then f is bounded and $\|f\| = \sup_k |a_k|$.
(ii) If $1 < p < \infty$, $q = p/(p-1)$, and $a \in d_q$, then f is bounded and

$$\|f\| = \left(\sum_{k=1}^{\infty} \sup_{i \in E_k} |a_i|^q \right)^{1/q}.$$

Proof Since $f(x) = \sum_{k=1}^{\infty} \sum_{i \in E_k} a_i x_i$, the proof is obtained by the proof of Theorem 3.2. □

Let $A = (a_{nk})$ be an infinite matrix. We consider the following conditions:

$$\sup_n \left(\sum_{k=1}^{\infty} \sup_{i \in E_k} |a_{ni}|^q \right) < \infty \qquad (1 < q < \infty), \tag{4.1}$$

$$\sup_{n,k} |a_{nk}| < \infty, \tag{4.2}$$

$$\lim_{n\to\infty} a_{nk} = 0 \quad (k = 1, 2, \cdots), \tag{4.3}$$

$$\lim_{n\to\infty} a_{nk} = l_k \;\; for\ some\ l_k \in \mathbb{R} \;\; (k = 1, 2, \cdots). \tag{4.4}$$

Theorem 4.2 *We have:*
(i) $A \in (l_p(E), l_\infty)$ if and only if condition (4.1) holds, where $1 < p < \infty$;
(ii) $A \in (l_1(E), l_\infty)$ if and only if condition (4.2) holds;
(iii) $A \in (l_p(E), c_0)$ if and only if conditions (4.1) and (4.3) hold, where $1 < p < \infty$;
(iv) $A \in (l_1(E), c_0)$ if and only if conditions (4.2) and (4.3) hold;
(v) $A \in (l_p(E), c)$ if and only if conditions (4.1) and (4.4) hold, where $1 < p < \infty$;
(vi) $A \in (l_1(E), c)$ if and only if conditions (4.2) and (4.4) hold.

Proof (*i*) Let $1 < p < \infty$ and $A \in (l_p(E), l_\infty)$. We have $A_n \in d_q$ for all n, by Theorem 3.2. Thus, due to Lemma 4.1, the linear functional $f_n : l_p(E) \to \mathbb{R}$ defined by

$$f_n(x) = \sum_{k=1}^{\infty} a_{nk} x_k \qquad (\, x = (x_k) \in l_p(E))\,,$$

is bounded and $\|f_n\|^q = \sum_{k=1}^{\infty} \sup_{i\in E_k} |a_{ni}|^q$, for all n. Since $Ax \in l_\infty$ for every $x \in l_p(E)$, by applying the Banach–Steinhaus theorem it follows that $\sup_n \|f_n\| < \infty$. Thus, the condition (4.1) must hold.

Conversely, suppose that condition (4.1) holds. By using Theorem 3.2, $A_n \in (l_p(E))^\beta$ for $n = 1, 2, \cdots$ and letting $x \in l_p(E)$, similar to the proof of (3.1), we deduce that

$$|A_n(x)| \leq \left(\sum_{k=1}^{\infty} \sup_{i\in E_k} |a_{ni}|^q \right) \|x\|_{p,E},$$

for all n. Condition (4.1) implies $Ax \in l_\infty$ for each $x \in l_p(E)$, so $A \in (l_p(E), l_\infty)$.

The proof of part (ii) is similar to part (i).

(iii) Suppose that $A \in (l_p(E), c_0)$. We define the sequence $e^k = (e_i^k)_{i=1}^{\infty}$ by $e_i^k = 1$ for $i = k$, and we put the remaining elements to zero. Obviously, $e^k \in l_p(E)$ and $Ae^k \in c_0$, and this proves the necessity of condition (4.3). The proof of the necessity of condition (4.1) is similar to the previous part.

Conversely, suppose that conditions (4.1) and (4.3) hold. By Hölder's inequality we have

$$\begin{aligned} |A_n x| &= \left| \sum_{k=1}^{\infty} \sum_{i\in E_k} a_{ni} x_i \right| \\ &\leq \sum_{k=1}^{m} \sup_{i\in E_k} |a_{ni}| \left| \sum_{i\in E_k} x_i \right| + \sum_{k=m+1}^{\infty} \sup_{i\in E_k} |a_{ni}| \left| \sum_{i\in E_k} x_i \right| \\ &\leq \|x\|_{p,E} \left(\sum_{k=1}^{m} \sup_{i\in E_k} |a_{ni}|^q \right)^{1/q} + \left(\sum_{k=m+1}^{\infty} \left| \sum_{i\in E_k} x_i \right|^p \right)^{1/p} \left(\sum_{k=m+1}^{\infty} \sup_{i\in E_k} |a_{ni}|^q \right)^{1/q}. \end{aligned}$$

Now taking m so large that

$$\left(\sum_{k=m+1}^{\infty} \left| \sum_{i\in E_k} x_i \right|^p \right)^{1/p} < \epsilon,$$

and then taking n so large that

$$\left(\sum_{k=1}^{m} \sup_{i\in E_k} |a_{ni}|^q \right)^{1/q} < \epsilon$$

(possible since $\lim_{n\to\infty} \sup_{i\in E_k} |a_{ni}| = 0$), we conclude that $Ax \in c_0$.

The proof of part (iv) is similar to part (iii).

(v) Let $A \in (l_p(E), c)$. Using the sequences e^k, the necessity of condition (4.4) is immediate. The proof of the necessity of condition (4.1) is similar to part (i).

Conversely, suppose that conditions (4.1) and (4.4) hold. If $B = (b_{nk})$ is a matrix such that $b_{nk} = a_{nk} - l_k$ for every $n, k \in \mathbb{N}$, we first prove that $B \in (l_p(E), c_0)$. Let $M > 0$ and $\epsilon > 0$ be given. Due to (4.4), $\lim_{n\to\infty} \sup_{i\in E_k} |a_{ni} - l_i| = 0$, and hence there is a positive number N_k such that $\sup_{i\in E_k} |a_{ni} - l_i| \leq \frac{\epsilon}{M^{1/q}}$, for all $n \geq N_k$. Taking $N = \max_{1 \leq k \leq M} N_k$, by Minkowski's inequality we obtain

$$\begin{aligned}\left(\sum_{k=1}^{M} \sup_{i \in E_k} |l_i|^q\right)^{1/q} &\leq \left(\sum_{k=1}^{M} \sup_{i \in E_k} |a_{ni} - l_i|^q\right)^{1/q} + \left(\sum_{k=1}^{M} \sup_{i \in E_k} |a_{ni}|^q\right)^{1/q} \\ &\leq \epsilon + \left(\sup_n \sum_{k=1}^{M} \sup_{i \in E_k} |a_{ni}|^q\right)^{1/q},\end{aligned}$$

for every $n \geq N$. If $M \to \infty$, due to condition (4.1), one can conclude that $(l_k)_{k=1}^{\infty} \in d_q$, and hence $B \in (l_p(E), c_0)$ by part (iii). This implies that $Bx \in c_0$ for all $x \in l_p(E)$, so

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} x_k = \sum_{k=1}^{\infty} l_k x_k. \tag{4.5}$$

Since $(l_k)_{k=1}^{\infty} \in (l_p(E))^{\beta}$, by Theorem 3.2, we have $\sum_{k=1}^{\infty} l_k x_k < \infty$ for all $x \in l_p(E)$. This result and relation (4.5) show that $A \in (l_p(E), c)$. The proof of part (vi) is similar to part (v). □

It should be noted that when $\sup_k |E_k| < \infty$, condition (4.1) is equivalent to the following condition:

$$\sup_n \left(\sum_{k=1}^{\infty} |a_{nk}|^q\right) < \infty, \qquad (1 < p < \infty). \tag{4.6}$$

Corollary 4.3 *([11], Theorem 1.37) We have:*
(i) $A \in (l_p, l_\infty)$ *if and only if condition (4.6) holds, where* $1 < p < \infty$*;*
(ii) $A \in (l_1, l_\infty)$ *if and only if condition (4.2) holds;*
(iii) $A \in (l_p, c_0)$ *if and only if conditions (4.6) and (4.3) hold, where* $1 < p < \infty$*;*
(iv) $A \in (l_1, c_0)$ *if and only if conditions (4.2) and (4.3) hold;*
(v) $A \in (l_p, c)$ *if and only if conditions (4.6) and (4.4) hold, where* $1 < p < \infty$*;*
(vi) $A \in (l_1, c)$ *if and only if conditions (4.2) and (4.4) hold.*

Proof If $E_n = \{n\}$ for all n, by applying Theorem 4.2, we obtain the desired result. □

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