

T-Spaces

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Abstract: In this paper, using generalized groups and their generalized actions, we define and study the notion of T -spaces. We study properties of the quotient space of a T -space and we present the conditions that imply the Hausdorff property for it. We also prove some essential results about topological generalized groups. As a main result, we show that for each positive integer n there is a topological generalized group T with n identity elements. Moreover, we study the maps between two T -spaces and we consider the notion of T -transitivity.

Key words: Generalized group, generalized action, transitivity, T -space, quotient space

1. Introduction

Groups appear in many branches of mathematics such as number theory, geometry, and the theory of Lie groups, and they can also find many applications even in physics and chemistry. Here we need to recall the notion of *topological groups*, which play a major role in the geometry and also group representation theory. They, along with their continuous actions, are used to study continuous symmetries, which have many applications in physics and chemistry. Topological groups have both algebraic and topological structures such that the multiplication of the group and inverse function are continuous functions with respect to the topology. More precisely, a topological group G is a group endowed with a topology such that the multiplication mapping $m : G \times G \rightarrow G$ defined by $(g, h) \rightarrow gh$ and the inversion mapping $i : G \rightarrow G$ defined by $g \rightarrow g^{-1}$ are continuous. For example, any group, endowed with discrete topology or indiscrete topology is a topological group. Many familiar examples of topological spaces are topological groups like the group $(\mathbb{R}, +)$ and $(\mathbb{R} \setminus \{0\}, \times)$ with the topology generated by the euclidean metric $(\mathbb{C}, +)$ and $(\mathbb{C} \setminus \{(0, 0)\}, \times)$. Moreover, every subgroup of a topological group endowed with the subspace topology is a topological group. Topological groups have many interesting properties; here we recall some important theorems.

Theorem 1.1 [11] *Let G be a topological group. Then:*

- every neighborhood U of the identity element e in G contains an open symmetric neighborhood V of e such that $VV \subseteq U$;
- the identity component is a closed normal subgroup;
- G is a homogeneous space;
- G is Hausdorff if and only if the trivial subgroup $\{e\}$ is closed in G .

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Theorem 1.2 [10] *Let G be a topological group and H be a subgroup of G . Then:*

- *if H is an open subset of G , then it is also closed;*
- *the mapping $p : G \rightarrow G/H$ is an open map;*
- *if H is compact, then $p : G \rightarrow G/H$ is a closed map;*
- *G/H is a Hausdorff space if and only if H is closed in G .*

We refer to [9, 10, 11, 20] for more theorems and more details about topological groups and their actions on topological spaces. A resource about the importance of topological groups is [3].

Generalized groups or completely simple semigroups [2, 13] are an interesting extension of groups. In a group, there is a unique identity element, but in a generalized group, each element has a special identity element. This kind of structure appears in genetic codes [16]. The role of identity as a mapping lead to many new important challenges. Generalized groups have been applied in genetic [1, 16], geometric [15], and dynamical systems [17]. The notion of generalized action [12] is an extension of the notion of group actions [4, 7]. This notion was studied first in 1999 [12, 14, 16]. Generalized action has been applied by other researchers [5, 6]. Let us recall the definition of a generalized group or completely simple semigroup [2]. A *generalized group* is a semigroup T with the following conditions:

- (i) For each $t \in T$ there exists a unique $e(t) \in T$ such that $t \cdot e(t) = e(t) \cdot t = t$;
- (ii) For each $t \in T$ there exists $s \in T$ such that $s \cdot t = t \cdot s = e(t)$.

One can easily prove that each t in T has a unique inverse in T . The inverse of t is denoted by t^{-1} . Moreover, for a given $t \in T$, $e(t) = e(t^{-1})$ and $e(e(t)) = e(t)$. Let T and S be two generalized groups. A map $f : T \rightarrow S$ is called a *homomorphism* if $f(st) = f(s) \cdot f(t)$ for every $s, t \in T$. We refer to [16] for more details. The geometrical viewpoint of completely simple semigroups is given in [18].

In Section 2, we study more properties of generalized groups and their generalized action on topological spaces. We present the notion of T -spaces. We consider the quotient space of a T -space and we deduce the conditions that imply the Hausdorff property for it. Moreover, we obtain some theorems about generalized actions of topological generalized groups. We also introduce a simple method for constructing topological generalized groups. As a corollary, we show that for each positive integer n , there exists a topological generalized group with n identity elements. In Section 3, the maps between two T -spaces and T -transitivity are considered.

2. T-Spaces

Let us begin this section by recalling the definition of a topological generalized group.

Definition 2.1 [1, 14] *A topological generalized group is a Hausdorff topological space T that is endowed with a generalized group structure such that the generalized group operations $m_1 : T \rightarrow T$ defined by $t \mapsto t^{-1}$ and $m_2 : T \times T \rightarrow T$ by $(s, t) \mapsto s \cdot t$ are continuous maps and*

$$e(s \cdot t) = e(s) \cdot e(t). \quad (1)$$

As shown in [16], every topological generalized group T is a disjoint union of topological groups. Moreover, the mapping $e : T \rightarrow T$ defined by $t \mapsto e(t)$ is a continuous map. Note that in a topological generalized group, property (1) implies that

$$e(t) \cdot e(s) \cdot e(t) = e(t), \quad \forall t, s \in T. \tag{2}$$

This property will be used frequently.

Theorem 2.2 *If T is a topological generalized group, then*

$$(s \cdot t)^{-1} = e(s) \cdot t^{-1} \cdot s^{-1} \cdot e(t).$$

Proof Since

$$\begin{aligned} & (s \cdot t)(e(s) \cdot t^{-1} \cdot s^{-1} \cdot e(t)) \\ &= s \cdot (t \cdot e(s) \cdot t^{-1} \cdot s^{-1} \cdot e(t)) \\ &= s \cdot (t \cdot e(t) \cdot e(s) \cdot e(t) \cdot t^{-1} \cdot s^{-1} \cdot e(t)) \quad (\text{because } t = t \cdot e(t) \text{ and } t^{-1} = e(t) \cdot t^{-1}) \\ &= s \cdot (t \cdot e(t) \cdot t^{-1} \cdot s^{-1} \cdot e(t)) \quad (\text{by equation (2)}) \\ &= s \cdot (e(t) \cdot s^{-1} \cdot e(t)) \quad (\text{because } t \cdot e(t) \cdot t^{-1} = e(t) \text{)} \\ &= (s \cdot e(t) \cdot s^{-1}) \cdot e(t) \\ &= (s \cdot e(s) \cdot e(t) \cdot e(s) \cdot s^{-1}) \cdot e(t) \quad (\text{because } s = s \cdot e(s) \text{ and } s^{-1} = e(s) \cdot s^{-1}) \\ &= (s \cdot e(s) \cdot s^{-1}) \cdot e(t) \quad (\text{by equation (2)}) \\ &= e(s) \cdot e(t) = e(s \cdot t), \end{aligned}$$

we can see that $(s \cdot t)(e(s) \cdot t^{-1} \cdot s^{-1} \cdot e(t)) = e(s \cdot t)$. Similarly, we can show that $(e(s) \cdot t^{-1} \cdot s^{-1} \cdot e(t)) \cdot (s \cdot t) = e(s \cdot t)$. Thus, $(s \cdot t)^{-1} = e(s) \cdot t^{-1} \cdot s^{-1} \cdot e(t)$. □

Example 2.3 *If T is the topological space*

$$\mathbb{R}^2 - \{(0,0)\} = \{re^{i\theta} \mid r > 0 \text{ and } 0 \leq \theta < 2\pi\}$$

with the Euclidean metric, then T with the multiplication

$$(r_1e^{i\theta_1}) \cdot (r_2e^{i\theta_2}) = r_1r_2e^{i\theta_2} \tag{3}$$

is a topological generalized group. We have $e(re^{i\theta}) = e^{i\theta}$ and $(re^{i\theta})^{-1} = \frac{1}{r}e^{i\theta}$. Thus, we can see that the identity set $e(T)$ is the unit circle S^1 , where $e(T)$ is the set all identity elements of T . However, T is not a topological group.

In Example 2.3, one can see that the identity set $e(T)$ is infinity. One of the main questions in generalized group theory is whether there is a generalized group T such that $e(T) = n$ for any positive integer number n . (See Corollary 2.26.)

Definition 2.4 Suppose that X is a topological space and T is a topological generalized group. A generalized action of T on X is a continuous map $\lambda : T \times X \rightarrow X$ with the following properties:

- (i) $\lambda(s, \lambda(t, x)) = \lambda(s \cdot t, x)$, for $s, t \in T$ and $x \in X$;
- (ii) If $x \in X$, then there is $e(t) \in T$ such that $\lambda(e(t), x) = x$.

Note that in the definition, if T is a group, then we have the definition of an action of a group that is defined in [9]. Henceforth, $\lambda(t, x)$ will be denoted by tx . For more details on the generalized action, one can see [6, 12]. Now let us define the notion of T -spaces.

Definition 2.5 A T -space is a triple (X, T, λ) where X is a Hausdorff topological space, T is a topological generalized group, and $\lambda : T \times X \rightarrow X$ is a generalized action of T on X . Moreover, for each $x \in X$:

- (i) $T_x = \{t \in T \mid tx = x\}$ is called the stabilizer of x in T ;
- (ii) $T(x) = \{tx \mid t \in T\}$ is called the T -orbit of x in X .

For $x \in X$, T_x is a generalized subgroup of T [6]. For $x \in X$, if the stabilizer T_x is trivial (i.e. $T_x = \{e(t)\}$ for some $t \in T$), we say that x is a regular point; otherwise, it is called a singular point. The singular set of X , denoted by \sum_X , is the set of singular points of X . We define two maps $\theta_t : X \rightarrow X$ and $\rho_x : T \rightarrow X$ by $\theta_t(x) = tx$ and $\rho_x(t) = tx$, respectively, where $t \in T$ and $x \in X$. We can see that θ_t and ρ_x are continuous maps. Clearly, $T(x) = \rho_x(T)$ and $T_x = (\rho_x)^{-1}(x)$. Thus, T_x is a closed subset of T and then we can say it is a closed generalized subgroup of T . Now for continuing we need the definition of top spaces that are introduced in [18]. The topological generalized group T is said to be a top space if T is a manifold and the generalized group operations m_1 and m_2 are smooth [18]. As shown in [6], if X is a manifold and $\text{Card}(e(T)) < \infty$, then we can show that for $x \in X$, T_x is a generalized subtop space of T and also a topological generalized subgroup of T .

By the action λ of a T -space (X, T, λ) , we can define the following equivalence relation on X :

$$x \sim y \text{ if and only if there is } t \in T \text{ such that } tx = y.$$

Now we consider the quotient space X/\sim , and by the projection map $\pi : X \rightarrow X/\sim$, we define a topology on X/\sim such that π is a continuous map. In fact, $U \subseteq X/\sim$ is open if $\pi^{-1}(U)$ is open in X . Henceforth, we will use of the notation X/T for the topological quotient space X/\sim .

Definition 2.6 Suppose that (X, T, λ) is a T -space. Then the generalized action λ is called:

- (i) effective if for any two distinct $s, t \in T$, there exists $x \in X$ such that $sx \neq tx$;
- (ii) transitive if for given $x, y \in X$, there exists $t \in T$ such that $tx = y$;
- (iii) free if for each $x \in X$, there exists precisely one $t \in T$ such that $tx = x$;
- (iv) regular if it is transitive and free.

λ is transitive if and only if $T(x) = X$ for each $x \in X$, and λ is free if and only if for each $x \in X$, $T_x = \{e(t)\}$ for some $t \in T$.

Example 2.7 Every topological generalized group T acts on itself by the multiplication of $T : st = s \cdot t$ for all $s, t \in T$. Note that this action need not be free; for instance, the action $xy = y$ as the multiplication of T implies $T_x = T$ for $x \in X$.

Example 2.8 Every topological generalized group T acts on itself by the multiplication $st = e(s) \cdot t$.

Example 2.9 Let T be $\mathbb{R} - \{0\}$ with the euclidean metric. T with the multiplication $x \cdot y = x$ is a topological generalized group such that if $x \in T$, then $e(x) = x^{-1} = x$. T acts on itself with this multiplication. This action is regular.

We recall that if T is a generalized group, X is a set, and $S = \{\varphi^t \mid \varphi^t : X \rightarrow X \text{ is a mapping and } t \in T\}$, then the triple (X, S, T) is called a *complete semidynamical system* if:

- (i) $\varphi^{t_1} \circ \varphi^{t_2} = \varphi^{t_1 \cdot t_2}$, for all $t_1, t_2 \in T$;
- (ii) For given $x \in X$, there is $\varphi^t \in D$ such that x is a fixed point of φ^t .

We see that each T -space (X, T, λ) generates a complete semidynamical system (X, S, T) where

$$S = \{\theta_t : X \rightarrow X \mid \theta_t(x) = tx, \text{ for } x \in X \text{ and } t \in T\}.$$

As shown in [6], if $e(T) \subseteq T_x$ for each $x \in X$, then S with the multiplication $\theta_s \circ \theta_t = \theta_{st}$ is a topological generalized group. In this case, for each $\theta_t \in S, e(\theta_t) = \theta_{e(t)}$ and $(\theta_t)^{-1} = \theta_{t^{-1}}$.

Theorem 2.10 If $e(T) \subseteq T_x$ for each $x \in X$, then each θ_t is a homeomorphism.

Proof Every θ_t is a continuous map on X . Now we claim that each θ_t is one to one and onto and has the inverse $\theta_{t^{-1}}$. If $\theta_t(x) = \theta_t(y)$, then $tx = ty$, and so $t^{-1}tx = e(t)x = x = y = e(t)y = t^{-1}ty$. Hence, θ_t is one to one. On the other hand, for $x \in X$, there exists $t^{-1}x \in X$ such that $\theta_t(t^{-1}x) = tt^{-1}x = e(t)x = x$. Thus, θ_t is also onto. If $t \in T$ and $x \in X$, then

$$\theta_t \circ \theta_{t^{-1}}(x) = \theta_{tt^{-1}}(x) = \theta_{e(t)}(x) = e(t)x = x.$$

The last equality follows from the fact $e(T) \subseteq T_x$. In the same way,

$$\theta_{t^{-1}} \circ \theta_t(x) = x.$$

Therefore, every θ_t is a homeomorphism. □

Theorem 2.11 Let (X, T, λ) be a T -space. If T is compact and $e(T) \subseteq T_x$ for each $x \in X$, then $\lambda : T \times X \rightarrow X$ is a closed map.

Proof Assume that C is a closed subset of $T \times X$ and $x \in X$ is a limit point of $\lambda(C)$. Then there exists a sequence $\{(t_i, x_i)\}$ in C such that $\lambda(t_i, x_i) = t_i x_i$ converges to x . As T is compact, then there is a subsequence of $\{t_i\}$ that converges to a t in T . We rename that subsequence $\{t_i\}$. T is a topological generalized group, so the map $m_1 : T \rightarrow T$, defined by $t \mapsto t^{-1}$, is continuous. This implies that $\{t_i^{-1}\}$ converges to t^{-1} . λ is also continuous, and $\lambda(t_i^{-1}, t_i x_i)$ converges to $\lambda(t^{-1}, x)$, so $\{e(t_i)x_i\}$ converges to $t^{-1}x$. However,

for each $x \in X$, $e(T) \subseteq T_x$, and thus $e(t_i)x_i = x_i$, and consequently x_i converges to $t^{-1}x$. Thus, the sequence $\{(t_i, x_i)\}$ in C converges to $(t, t^{-1}x)$. Since C is a closed subset of $T \times X$, then $(t, t^{-1}x) \in C$. Therefore, $\lambda(t, t^{-1}x) = e(t)x \in \lambda(C)$. According to the assumption, $e(t)x = x$. Thus, $x \in \lambda(C)$, which means that $\lambda(C)$ is closed; that is, λ is a closed map. \square

Let (X, T, λ) be a T -space. If Y is a subset of X , then

$$TY := \{ty \mid t \in T \text{ and } y \in Y\}.$$

Y is called *invariant* under T if $TY = Y$.

Theorem 2.12 *Let (X, T, λ) be a T -space. Moreover, let T be compact, $Y \subseteq X$, and for each $x \in X$, $e(T) \subseteq T_x$; then:*

- (i) TY is closed if Y is closed;
- (ii) TY is compact if Y is compact.

Theorem 2.13 *Let (X, T, λ) be a T -space and for each $x \in X$, $e(T) \subseteq T_x$. Then the projection map $\pi : X \rightarrow X/T$ is an open map.*

Proof To prove that λ is an open map, suppose $Y \subseteq X$ is an arbitrary open subset of X . We have

$$TY = \{ty \mid t \in T \text{ and } y \in Y\} = \pi^{-1}(\pi(Y)) = \bigcup_{t \in T} tY. \tag{4}$$

Since θ_t is a homeomorphism, then it is an open map. Thus, for each $t \in T$, $\theta_t(Y) = tY$ is an open set in X . Hence, $\pi^{-1}(\pi(Y))$ is an open set in X (from (4)). Therefore, $\pi(Y)$ is open in X/T , which means that π is an open map. \square

Definition 2.14 *Suppose that (X, T, λ) is a T -space. A generalized action λ is called *perfect* if $e(T) \subseteq T_x$ for each $x \in X$. In this sense, the T -space is called *perfect*. Moreover, λ is called *super perfect* if for each $x \in X$, $e(T) = T_x$.*

Suppose that X and Y are two topological spaces. A mapping $f : X \rightarrow Y$ is called *proper* if for each compact subset A of Y , $f^{-1}(A)$ is a compact subset of X [19]. We know that if the mapping $f : X \rightarrow Y$ is closed and for each $y \in Y$, $f^{-1}(y)$ is compact, then f is a proper map [19].

Theorem 2.15 *Let (X, T, λ) be a T -space, T be compact, and λ be perfect. Then:*

- (i) The projection map $\pi : X \rightarrow X/T$ is a closed map;
- (ii) The quotient space X/T is Hausdorff;
- (iii) The projection map $\pi : X \rightarrow X/T$ is a proper map;
- (iv) X is compact if and only if X/T is compact;
- (v) X is locally compact if and only if X/T is locally compact.

Proof

(i) Suppose that $Z \subseteq X$ is an arbitrary closed subset of X . As λ is perfect and T is compact, according to Theorem 2.13, $TZ = \pi^{-1}(\pi(Z))$ is closed in X . Since π is a surjective map, we can see that

$$\pi^{-1}(X/T - \pi(Z)) = X - \pi^{-1}(\pi(Z)). \tag{5}$$

Since $\pi^{-1}(\pi(Z))$ is closed in X , then $X - \pi^{-1}(\pi(Z))$ is open in X . $X/T - \pi(Z)$ is open in X/T . Thus, $\pi(Z)$ is closed in X/T , and hence π is a closed map.

(ii) Suppose that $[x]$ and $[y]$ are two distinct elements in X/T . Clearly, we can see that $T(x) \cap T(y) = \emptyset$. As mentioned, $T(x) = \rho_x(T)$; that is, the T -orbit $T(x)$ is the image of T . However, we know that ρ_x is continuous and T is compact, which implies that $T(x) = \pi^{-1}([x])$ is compact in X . In the same way, $T(y)$ is compact. Since X is Hausdorff and $T(x)$ is compact in X and $y \notin T(x)$, thus there exist disjoint open subsets U and V of X containing $T(x)$ and y , respectively [19]. It is especially easy to see that $T(x) \cap \bar{V} = \emptyset$. Thus, $[x] = \pi(x) \notin \pi(\bar{V})$. On the other hand, $\pi(V)$ is open in X/T . $\pi(V)$ contains $[y]$. Moreover, since \bar{V} is closed, $\pi(\bar{V})$ is closed in X/T . Thus, $X/T - \pi(\bar{V})$ is an open subset that contains $[x]$. Therefore, $X/T - \pi(\bar{V})$ and $\pi(V)$ are disjoint open sets of X/T containing $[x]$ and $[y]$, respectively.

(iii) In (i), we saw that π is a closed map, and also in (ii), we showed that $T(x) = \pi^{-1}([x])$ is compact for each $x \in X$. Thus, π is a proper map.

(iv) This follows easily from continuity of π and (iii).

(v) This follows from Theorem 2.13 and (iii).

□

Definition 2.16 Let (X, T, λ) be a T -space. The generalized action λ is called proper if the map $\hat{\lambda} : T \times X \rightarrow X \times X$ defined by $\hat{\lambda}(t, x) = (tx, x)$ is a proper map.

Theorem 2.17 Let (X, T, λ) be a T -space. The generalized action λ is proper if and only if for every compact subset Y of X , the set $Y_T = \{t \in T \mid tY \cap Y \neq \emptyset\}$ is compact.

Proof Suppose that the mapping $\hat{\lambda}$ is proper and $Y \subset X$ is compact. We see that $Y_T = P(\hat{\lambda}^{-1}(Y \times Y))$, where $P : T \times X \rightarrow T$ is the projection map. Thus, Y_T is compact. Conversely, suppose that Y_T is compact, where Y is a compact subset of X . If $Z \subset X \times X$ is compact, and $Y := \pi_1(Z) \cup \pi_2(Z) \subset X$, where $\pi_1, \pi_2 : X \times X \rightarrow X$ are the projections on the first and second components, respectively, then Y is compact. Moreover, $\hat{\lambda}^{-1}(Z) \subset \hat{\lambda}^{-1}(Y \times Y) \subset Y_T \times Y$. Since $X \times X$ is Hausdorff and Z is compact, then Z is a closed subset of $X \times X$. Thus, $\hat{\lambda}^{-1}(Z)$ is closed by continuity and it is also a closed subset of the compact set $Y_T \times Y$. Thus, Z is compact and λ is proper. □

Corollary 2.18 Let (X, T, λ) be a T -space. If T is compact, then the generalized action λ is proper.

Corollary 2.19 *Let (X, T, λ) be a T -space. If λ is proper, then:*

- (i) *The stabilizer T_x is compact, where $x \in X$;*
- (ii) *The orbit map ρ_x is proper, where $x \in X$.*

Corollary 2.20 *Let (X, T, λ) be a T -space. If λ is proper, then $T(x)$ is a closed subset of X and the quotient space X/T is Hausdorff.*

Example 2.21 *Let $X = \mathbb{R}^2$ and let T be the generalized group of Example 2.3 that acts on X by*

$$(r_1 e^{i\theta_1}).(r_2 e^{i\theta_2}) = r_1 r_2 e^{i\theta_2}.$$

The equivalence class $[x]$ is the set all $re^{i\theta_0}$ such that $r > 0$, where $x = r_0 e^{i\theta_0} \in \mathbb{R}^2 - \{(0, 0)\}$ and $[(0, 0)] = \{(0, 0)\}$. Moreover, for $x = r_1 e^{i\theta_1}$ and $y = r_2 e^{i\theta_2}$, $[x] = [y]$ if and only if $\theta_1 = \theta_2$. Thus, $Y = X/T \simeq S^1 \cup \{(0, 0)\}$, which is not a Hausdorff space with the quotient topology.

Now let X be a topological space that the group G acts on by the action λ . Then (X, G, λ) is called a G -space. We know that locally connectedness is preserved by the orbit space of a G -space; that is, if X is a locally connected space with a continuous action of a topological group G , then the orbit space $Y := X/G$, given with the quotient topology, is also locally connected. However, this property is not true for T -spaces, but we can show that if the generalized action of the generalized group T on X is perfect, then locally connectedness is preserved.

Theorem 2.22 *Suppose that (X, T, λ) is a T -space and λ is perfect. If X is locally connected, then $Y := X/T$ is also locally connected.*

Proof We know that X is locally connected if and only if every open subset $U \subseteq X$ can be decomposed into a disjoint union of open connected subspaces of X [19]. Now we consider the projection map $\pi : X \rightarrow X/T$. Let V be an open subset of $Y := X/T$ and consider the inverse image $\pi^{-1}(V)$ that is open in X . Since X is locally connected, $\pi^{-1}(V)$ is decomposed into a disjoint union of connected components of X . As λ is a perfect generalized action of T on X , we can define a natural generalized action of T on the set of connected components of X . The image of each component of X under π is the same within an orbit in Y . Thus, $\pi(\pi^{-1}(V)) = V$ is decomposed into a disjoint union of open connected subsets of Y . Hence, Y is locally connected. □

Let $C(X)$ be the set of all continuous maps on X . We define the mapping $\tilde{\lambda} : T \rightarrow C(X)$ by $t \mapsto \lambda_t$, where $\lambda_t(x) = tx$. Define $\text{Ker}\lambda = \{t \in T | \tilde{\lambda}(t) = id_X\}$. As shown in [6], if λ is perfect then $C(X)$ is a topological generalized group and so $\tilde{\lambda}$ is a generalized group homomorphism. Moreover, we can see that $\text{Ker}\lambda$ is a closed generalized subgroup of T .

Finding relations between groups and generalized groups and constructing new generalized groups with respect to a group are the most important problems in generalized groups theory. It is shown that if G is a generalized group then G is a disjoint union of groups, i.e. $G = \bigcup_{a \in G} G_a$ such that $G_a = \{g \in G | e(g) = e(a)\}$ [16]. Furthermore, Araujo and Konieczny characterized generalized groups by Rees matrix semigroups [2]. In the following, we introduce a new method for constructing a generalized group from a group.

Theorem 2.23 *Let G be a group. Then the set $T := G \times G$ by the multiplication*

$$(s_1, t_1)(s_2, t_2) = (s_1, t_1 s_2 t_2)$$

is a generalized group. Moreover, if G is a topological group, then T is also a topological generalized group.

Proof

$$(s, t).(s, s^{-1}) = (s, t s s^{-1}) = (s, te) = (s, t) = (s, et) = (s, s^{-1} st) = (s, s^{-1}).(s, t)$$

and thus $e((s, t)) = (s, s^{-1})$. Moreover, we have

$$\begin{aligned} (s, t).(s, s^{-1} t^{-1} s^{-1}) &= (s, t s s^{-1} t^{-1} s^{-1}) \\ &= (s, tet^{-1} s^{-1}) \\ &= (s, tt^{-1} s^{-1}) \\ &= (s, es^{-1}) \\ &= (s, s^{-1}) \\ &= e((s, t)). \end{aligned}$$

Also:

$$\begin{aligned} (s, s^{-1} t^{-1} s^{-1}).(s, t) &= (s, s^{-1} t^{-1} s^{-1} st) \\ &= (s, s^{-1} t^{-1} et) \\ &= (s, s^{-1} t^{-1} t) \\ &= (s, s^{-1} e) \\ &= (s, s^{-1}) \\ &= e((s, t)). \end{aligned}$$

Thus, $(s, t)^{-1} = (s, s^{-1} t^{-1} s^{-1})$ and $G \times G$ is a generalized group. If G is a topological group, then the generalized group operations m_1 and m_2 for $G \times G$ are also continuous. Therefore, $G \times G$ is a topological generalized group. □

The above topological generalized group T is called the *induced topological generalized group* of the topological group G . We can see that $e(T) = \{(t, t^{-1}) \mid t \in G\}$ (See the Figure). We are able to construct generalized groups from groups with this method.

Example 2.24 *Let G be the euclidean space \mathbb{R} . Suppose that $T := \mathbb{R}^2$. Using Theorem 2.23, we can say that T is a topological generalized group such that for each $(a, b), (c, d) \in T$,*

$$(a, b).(c, d) = (a, b + c + d),$$

$$e((a, b)) = (a, -a) \text{ and } (a, b)^{-1} = (a, -2a - b).$$

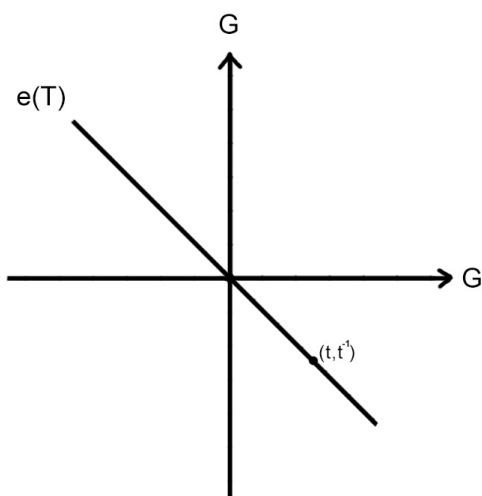


Figure. The induced topological generalized group.

Example 2.25 Let G be the unit circle S^1 . If we consider torus as the product space of two circles, then using Theorem 2.23, the torus $\mathbb{T}^2 = S^1 \times S^1$ is a topological generalized group.

Corollary 2.26 For any positive integer n , there is a topological generalized group T such that $\text{Card}(e(T)) = n$.

Proof In Theorem 2.23, if we consider $G := \mathbb{Z}_n$, then $T = G \times G$ is a generalized group such that $\text{Card}(e(T)) = n$. □

Example 2.27 Let G be the group $\{0, \theta, \theta^2, \theta^3\}$ where θ is the 90 degrees counterclockwise rotation of the xy -plane. The induced topological generalized group of G has 16 members and $\text{Card}(e(T)) = 4$.

Now suppose that the topological group G acts continuously on a topological space X . We can define a continuous generalized action of T on X . If the action of G on X is λ , then we define $\bar{\lambda}$ by

$$\bar{\lambda}((s, t), x) = \lambda(st, x).$$

We can easily show that $\bar{\lambda}$ is a continuous generalized action.

Theorem 2.28 Let G be a group that acts on X by λ and let T be the induced generalized group of G .

- (i) If λ is transitive, then $\bar{\lambda}$ is also transitive;
- (ii) $e(T) \subseteq T_x$ for each $x \in X$ (that is, $\bar{\lambda}$ is perfect);
- (iii) $G_x \times G_x \subseteq T_x$ for each $x \in X$.

Proof

- (i) Suppose x, y are two elements of X . Since λ is transitive, there is $s \in G$ such that $\lambda(s, x) = y$. Hence, $\bar{\lambda}((s, e), x) = \lambda(se, x) = \lambda(s, x) = y$, and so $\bar{\lambda}$ is transitive.

(ii) We can show easily that $(s, s^{-1}) \in T_x$ for each $s \in G$ and for each $x \in X$. Thus, $e(T) \subseteq T_x$ for each $x \in X$, and so $\bar{\lambda}$ is perfect.

(iii) It is obvious that if $(s, t) \in G_x \times G_x$ for some $x \in X$, then $(s, t) \in T_x$.

□

Note that if λ is effective (resp. free), then $\bar{\lambda}$ need not be effective (resp. free). However, we can see that $x \sim y$ under λ if and only if $x \sim y$ under $\bar{\lambda}$, and so $X/G = X/T$.

3. Maps on T-spaces

Now let us consider the maps of T -spaces. First we recall the notion of transitivity of maps on a topological space. Let X be a topological space and let $f : X \rightarrow X$ be a continuous mapping. Then the mapping f is called (topologically) transitive if for every pair of nonempty open subsets U and V of X , there is some $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. Moreover, if X is compact, then f is transitive if and only if f is onto and there is a point in X with dense orbit [8].

Definition 3.1 Let (X, T, λ) be a T -space. Suppose $f : X \rightarrow X$ is a continuous map. The map f is called T -transitive if for every pair of nonempty open subsets U and V in X , there is some positive integer n and $t \in T$ such that $tf^n(U) \cap V \neq \emptyset$.

We can see that if λ is not trivial and f is a transitive map, then f is T -transitive. However, the following example shows that each T -transitive map need not be transitive.

Example 3.2 Let X be the topological space $[-1, 1]$ with the topology generated by a euclidean metric. The topological generalized group $T = \{\pm 1\}$ with the multiplication $s \cdot t = s|t|$ acts on X by the generalized action $tx = t|x|$, where $t \in T$ and $x \in X$. Then $f : X \rightarrow X$ is defined by

$$f(x) = \begin{cases} \sqrt{x} & \text{if } 0 \leq x \leq 1 \\ -\sqrt{|x|} & \text{if } -1 \leq x \leq 0. \end{cases}$$

We can see that f is T -transitive but it is not transitive.

Definition 3.3 Let (X, T, λ) and (Y, T, μ) be two T -spaces. A continuous map $f : X \rightarrow Y$ is called:

(i) T -equivariant if $f(\lambda(t, x)) = \mu(t, f(x))$ for $t \in T$ and $x \in X$, or briefly, $f(tx) = tf(x)$;

(ii) T -pseudoequivariant if $f(T(x)) = T(f(x))$ for $x \in X$.

If $f : X \rightarrow Y$ is a T -equivariant between two T -spaces X and Y , then $T_x \subseteq T_{f(x)}$ for each $x \in X$. Moreover, an T -equivariant map is clearly T -pseudoequivariant but the converse is not true. The following example shows this.

Example 3.4 Let X and Y be two euclidean spaces \mathbb{R} and $T = \{\pm 1\}$. Then T with the multiplication $s \cdot t = s|t|$ is a topological generalized group. We define actions $\lambda, \theta : T \times \mathbb{R} \rightarrow \mathbb{R}$ by $\lambda(t, x) = t|x|$ and $\theta(t, x) = tx$, respectively. Now we see that the identity map on \mathbb{R} is a T -pseudoequivariant map that is not T -equivariant.

Theorem 3.5 *Let (X, d) be a compact metric space with no isolated point and let (X, T, λ) be a T -space for which λ is perfect. Suppose that $f : X \rightarrow X$ is a T -pseudoequivariant onto map. Then f is T -transitive if and only if there exists $x \in X$ such that $\{t \cdot f^n(x) \mid t \in T, n > 0\}$ is dense in X .*

Proof Suppose that f is T -transitive. By using the Baire category theorem and the fact that λ is perfect, we can see that there is some x such that $\{t \cdot f^n(x) \mid t \in T, n > 0\}$ is dense in X . Conversely, suppose that $\{tf^n(x) \mid t \in T, n > 0\}$ is dense in X for some $x \in X$. Let U and V be two nonempty open subsets of X . Then there are $s, t \in T$ and $m, n \in \mathbb{N}$ such that $sf^m(x) \in U$ and $tf^n(x) \in V$. We assume that $m < n$. Since λ is perfect, then $x \in f^{-m}(s^{-1}U)$. Thus, $f^n(x) \in f^{n-m}(s^{-1}U)$. Moreover, since f is T -pseudoequivariant, then there is $r \in T$ such that $f^{n-m}(s^{-1}U) = rf^{n-m}(U)$. Thus, $f^n(x) \in rf^{n-m}(U)$ and $tf^n(x) \in trf^{n-m}(U)$. Therefore, $tf^n(x) \in t'f^p(U) \cap V$, where $t' = tr \in T$ and $p = n - m \in \mathbb{N}$, and so f is T -transitive. \square

It may be that T -transitivity is not preserved by topological conjugacy. Now we introduce topological T -conjugacy that can preserve T -transitivity.

Definition 3.6 *Let (X, T, λ) and (Y, T, θ) be two T -spaces. Moreover, let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be two continuous maps. We say f is topologically T -conjugate to g if there is a T -pseudoequivariant homeomorphism $h : X \rightarrow Y$ such that $h \circ f = g \circ h$.*

Theorem 3.7 *Let (X, T, λ) and (Y, T, θ) be two perfect T -spaces. If the continuous mapping $f : X \rightarrow X$ is topologically T -conjugate to the continuous mapping $g : Y \rightarrow Y$, then f is T -transitive if and only if g is T -transitive.*

Proof Using Definitions 3.6 and 3.3, we can prove this theorem. \square

Theorem 3.8 *Let X and Y be smooth manifolds and let T be a top space. Suppose that $F : X \rightarrow Y$ is a smooth map that is equivariant with respect to a transitive perfect smooth generalized action λ of T on X and a perfect smooth generalized action θ of T on Y . Then F has constant rank.*

Proof Let $x \in X$. Since λ is transitive, then for any $y \in X$, there is some $t \in T$ such that $tx = y$. Since $\theta_t \circ F = F \circ \lambda_t$ for each $t \in T$, then we have $\theta_{t*} \circ F_* = F_* \circ \lambda_{t*}$. The generalized actions λ and θ are perfect, and then λ_{t*} and θ_{t*} are isomorphism. Thus, the rank F at y is the same as its rank at x . Thus, F has constant rank. \square

4. Conclusion

In this paper, we introduced a simple method for constructing new generalized groups from groups. According to this method, we could define a generalized group structure for the torus \mathbb{T}^2 . We also show that for each positive integer n , there is a topological generalized group T such that $\text{Card}(e(T)) = n$. We obtain some results about generalized actions and T -spaces such as group actions and G -spaces. For instance, locally connectedness is preserved by the orbit space of a T -space such that its generalized action is perfect.

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