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Special proper pointwise slant surfaces of a locally product Riemannian manifold

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Abstract: The structure of pointwise slant submanifolds in an almost product Riemannian manifold is investigated and the special proper pointwise slant surfaces of a locally product manifold are introduced. A relation involving the squared mean curvature and the Gauss curvature of pointwise slant surface of a locally product manifold is proved. Two examples of proper pointwise slant surfaces of a locally product manifold, one of which is special and the other one is not special, are given.

Key words: Almost product Riemannian manifold, special slant surface, curvature

1. Introduction

A slant surface M of a Kaehlerian manifold is called *special slant* if, with respect to some suitable adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$, the shape operator of the surface takes the following forms:

$$A_{e_3} = \begin{pmatrix} c\lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{and} \quad A_{e_4} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \tag{1.1}$$

where both c and λ are real numbers and $\{e_1, e_2\}$ is an orthonormal basis on T_pM. The special slant surfaces were studied in complex space forms by Chen in [9] and [10]. He proved the following relation involving the squared mean curvature $\|H(p)\|^2$ and the Gauss curvature K at a point p of proper slant surface M in a complex space form $\widetilde{M}(4c)$:

$$\|H(p)\|^2 \geq 2K(p) - 2(1 + \cos^2\theta)c, \tag{1.2}$$

where θ is the slant angle of the surface M. Furthermore, Chen showed that the equality case of (1.2) holds at all points $p \in M$ if and only if the surface is special slant with $c = 3$.

Later, the special slant surfaces were studied on complex projective spaces and complex hyperbolic spaces in [11], on Kaehlerian manifolds in [13], on Lorentz surfaces in [15], on nonflat complex space forms in [18], on Sasakian space forms in [24]. etc.

Besides these facts, geometric structures of the pointwise slant submanifolds in almost Hermitian manifolds were first introduced by Chen and Gray in [17] as a generalization of the slant submanifolds of almost

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Hermitian manifolds. Later, pointwise semislant submersions were studied in [23], warped product pointwise slant submanifolds were investigated in [30], pointwise H-slant submanifolds in almost Hermitian manifolds were studied in [28], and pointwise slant and pointwise semislant submanifolds in almost contact metric manifolds were studied in [29]. For more details, we can also refer to [16].

This paper is organized as follows: in Section 2, some basic facts on submanifold theory are given. In section 3, the basic concepts related to Riemannian product manifolds are mentioned. In Section 4, the proper pointwise slant submanifolds of a locally product manifold are introduced and a useful orthonormal basis for these submanifolds are given. The sectional curvature, the Ricci curvature and the scalar curvature are computed in the proper pointwise slant submanifold of an almost constant curvature manifold. In Section 5, an inequality involving the squared mean curvature and the Gauss curvature of a proper pointwise slant surface of an almost constant curvature manifold is established. The special proper pointwise slant surfaces on these submanifolds are introduced.

2. Preliminaries

Let $\widetilde{M}$ be an m-dimensional Riemannian manifold equipped with a Riemannian metric $\widetilde{g}$ and $\{e_1,\dots,e_m\}$ be any orthonormal basis for $T_p\widetilde{M}$. For a fixed $i\in\{1,\dots,m\}$, the *Ricci curvature* of e_i, denoted by $\widetilde{\mathrm{Ric}}\,(e_i)$, is defined by

$$\widetilde{\mathrm{Ric}}\,(e_i)=\sum_{j\neq i}^{m}\widetilde{K}_{ij} \tag{2.1}$$

and the *scalar curvature* at the point $p\in\widetilde{M}$, denoted by $\widetilde{\tau}(p)$, is defined by

$$\widetilde{\tau}\,(p)=\frac{1}{2}\sum_{i\neq j=1}^{m}\widetilde{K}_{ij}, \tag{2.2}$$

where $\widetilde{K}_{ij}$ denotes the sectional curvature of the plane section spanned by e_i and e_j.

Let (M,g) be an n-dimensional Riemannian submanifold of $(\widetilde{M},\widetilde{g})$ with the induced metric tensor g from the metric tensor $\widetilde{g}$. The Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_XY=\nabla_XY+\sigma(X,Y)\quad\text{and}\quad\widetilde{\nabla}_XN=-A_NX+\nabla_X^{\perp}N \tag{2.3}$$

for all $X,Y\in\Gamma(TM)$ and $N\in T^{\perp}M$, where $\widetilde{\nabla}$, ∇ are the Riemannian connections of $\widetilde{M}$ and $\widetilde{M}$, respectively, while $\nabla^{\perp}$ is the normal connection of M in $\widetilde{M}$ and $T^{\perp}M$ stands for the normal bundle of M. We denote the inner product of both the metrics g and $\widetilde{g}$ by $\langle,\rangle$. σ and A_N are related by

$$\langle\sigma(X,Y),N\rangle=\langle A_NX,Y\rangle\,. \tag{2.4}$$

The Gauss equation is given by

$$\begin{aligned} g\,(R(X,Y)Z,W) &= \widetilde{g}(\widetilde{R}(X,Y)Z,W)+\langle\sigma(X,W),\sigma(Y,Z)\rangle\\ &\quad-\langle\sigma(X,Z),\sigma(Y,W)\rangle \end{aligned} \tag{2.5}$$

for all $X,Y,Z,W\in\Gamma(TM)$, where $\widetilde{R}$ and R are the Riemann curvature tensors of $\widetilde{M}$ and M, respectively.

Now let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM, and e_r belongs to an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of the normal space $T_p^{\perp}M$. We put

$$\sigma_{ij}^r = \langle \sigma(e_i, e_j), e_r \rangle \quad \text{and} \quad \|\sigma\|^2 = \sum_{i,j=1}^{n} \langle \sigma(e_i, e_j), \sigma(e_i, e_j) \rangle . \tag{2.6}$$

In view of (2.5) and (2.6), we get

$$K_{ij} = \widetilde{K}_{ij} + \sum_{r=n+1}^{m} \left(\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2 \right), \tag{2.7}$$

where K_{ij} and $\widetilde{K}_{ij}$ denote the sectional curvature of the plane section spanned by e_i and e_j at p in the submanifold M and in the ambient manifold $\widetilde{M}$, respectively. Here, K_{ij} and $\widetilde{K}_{ij}$ are said to be the "intrinsic" and "extrinsic" sectional curvature of the plane $Span\{e_i, e_j\}$ at $p \in M$.

The mean curvature vector H is given by $H = \frac{1}{n}\text{trace}(\sigma)$. If $\sigma = 0$ then the submanifold is called *totally geodesic* in $\widetilde{M}$. If $H = 0$ then the submanifold is called *minimal*. If $\sigma(X, Y) = \langle X, Y \rangle H$ for all $X, Y \in \Gamma(TM)$, then the submanifold is called *totally umbilical* [8].

3. Locally product manifolds

Let $\widetilde{M}$ be an m-dimensional differentiable manifold and F be a $(1,1)$ type tensor on the tangent space of $\widetilde{M}$ such that

$$F^2 = I, \tag{3.1}$$

where I denotes the identity transformation. Then $\widetilde{M}$ is called an *almost product manifold*. If we put

$$P = \frac{1}{2}(I + F), \quad Q = \frac{1}{2}(I - F), \tag{3.2}$$

then we have

$$P + Q = I, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0, \quad F = P - Q. \tag{3.3}$$

If an almost product manifold admits a Riemannian metric $\widetilde{g}$ such that

$$\widetilde{g}(FX, FY) = \widetilde{g}(X, Y) \tag{3.4}$$

for any vector fields X, Y on $\widetilde{M}$, then $(\widetilde{M}, \widetilde{g})$ is called an *almost product Riemannian manifold* [33].

Let (M, g) be an n-dimensional Riemannian submanifold of an almost product Riemannian $(\widetilde{M}, \widetilde{g})$. Then we have

$$FX = fX + wX, \tag{3.5}$$

where fX is the tangential part of FX and wX is the normal part of FX for any vector field X tangent to M. Similarly, we can write

$$FV = tV + sV, \tag{3.6}$$

where tV is the tangential part of FV and sV is the normal part of FV for any vector field V normal to M.

From (3.4), it is easy to see that

$$\widetilde{g}(FX, Y) = \widetilde{g}(X, FY) \tag{3.7}$$

for any $X, Y \in \Gamma(TM)$.

Let us consider an m-dimensional manifold $\widetilde{M}$, which is covered by a system of coordinate neighborhoods $\left(x^i\right)$ such that in any intersection of two coordinate neighborhoods $\left(x^i\right)$ and $\left(x^{i'}\right)$, we have

$$x^{a'} = x^{a'}\left(x^a\right), \qquad x^{\alpha'} = x^{\alpha'}\left(x^\alpha\right),$$

with

$$\left|\frac{\partial x^{a'}}{\partial x^a}\right| \neq 0, \qquad \left|\frac{\partial x^{\alpha'}}{\partial x^\alpha}\right| \neq 0,$$

where the indices a, b, c, d run over the range $1, \ldots, m_1$, the indices $\alpha, \beta, \gamma, \nu$ run over $m_1+1, \ldots, m_1+m_2 = m$, and the indices i, j, k, h run over $1, \ldots, m$. Such a system of coordinate neighborhoods is called a *separating coordinate system*. We denote by $\widetilde{M}_1$ the system of subspaces defined by $x^\alpha = \text{constant}$, and by $\widetilde{M}_2$ the system of subspaces defined by $x^a = \text{constant}$. Then the manifold $\widetilde{M}$ is locally product $\widetilde{M}_1 \times \widetilde{M}_2$ of two manifolds. Thus, $\widetilde{M}$ is called a *locally product manifold*, which admits a *locally product structure* defined by the existence of a separating coordinate system. A locally product manifold always admits a natural tensor field F of type $(1,1)$ given by

$$F_j^i = \begin{pmatrix} \delta_b^a & 0 \\ 0 & -\delta_\beta^\alpha \end{pmatrix},$$

which satisfies $F^2 = I$.

In a locally product manifold $\widetilde{M}$, let a Riemannian metric

$$ds^2 = g_{ij}\left(x\right) dx^i dx^j,$$

which satisfies equality (3.4) for all vectors X and Y. Then $\widetilde{M}$ is called a *locally product Riemannian manifold*. If the metric of the manifold has the form

$$ds^2 = \widetilde{g}_{ab}\left(x^c\right) dx^a dx^b + \widetilde{g}_{\alpha\beta}\left(x^\gamma\right) dx^\alpha dx^\beta,$$

then $\widetilde{M}$ is called a *locally decomposable Riemannian manifold*. A locally product Riemannian manifold is a locally decomposable manifold if and only if $\widetilde{\nabla} F = 0$, where $\widetilde{\nabla}$ is the Riemannian connection of $\widetilde{M}$ [33].

Let $\widetilde{M} = \widetilde{M}_1 \times \widetilde{M}_2$ be a locally decomposable Riemannian manifold with $\dim(\widetilde{M}_\ell) = m_\ell > 2$, $\ell = 1, 2$. Then both $\widetilde{M}_1$ and $\widetilde{M}_2$ are manifolds of constant curvatures c_1 and c_2, respectively; that is,

$$\widetilde{R}^1_{abcd} = c_1 \left(\widetilde{g}_{ad}\widetilde{g}_{bc} - \widetilde{g}_{ac}\widetilde{g}_{bd}\right), \quad \widetilde{R}^2_{\alpha\beta\gamma\nu} = c_2 \left(\widetilde{g}_{\alpha\nu}\widetilde{g}_{\beta\gamma} - \widetilde{g}_{\alpha\gamma}\widetilde{g}_{\beta\nu}\right)$$

if and only if

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) \;=\;& a\left\{\left(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\right)\right.\\ &\left. + \left(\langle X,FW\rangle\langle Y,FZ\rangle - \langle X,FZ\rangle\langle Y,FW\rangle\right)\right\}\\ &+ b\left\{\left(\langle X,FW\rangle\langle Y,Z\rangle - \langle X,FZ\rangle\langle Y,W\rangle\right)\right.\\ &\left. + \left(\langle X,W\rangle\langle Y,FZ\rangle - \langle X,Z\rangle\langle Y,FW\rangle\right)\right\}\end{aligned} \tag{3.8}$$

for all $X,Y,Z,W \in T\widetilde{M}$, where $\widetilde{R}$, $\widetilde{R}^1$, and $\widetilde{R}^2$ are Riemannian curvatures of $\widetilde{M}$, $\widetilde{M}_1$, and $\widetilde{M}_2$, respectively, and

$$a = \frac{1}{4}\left(c_1 + c_2\right), \qquad b = \frac{1}{4}\left(c_1 - c_2\right).$$

A locally decomposable Riemannian manifold is called a *manifold of almost constant curvature* and denoted by $\widetilde{M}(a,b)$ if its curvature tensor $\widetilde{R}$ is given by (3.8). For more details, we refer to [32, 33, 34, 35].

4. Proper slant submanifolds of locally product manifolds

In [8], Chen introduced slant submanifolds as a generalization of invariant and antiinvariant submanifolds of almost Hermitian manifolds. On a submanifold M of an almost Hermitian manifold, for a vector $0 \neq X \in T_pM$, the angle $\theta(X)$ between JX and the tangent space T_pM is called the Wirtinger angle of X [14]. If the Wirtinger angle is independent of $p \in M$ and $X \in T_pM$, then M is called a slant submanifold. On the other hand, slant submanifolds of an almost product manifold were introduced by Sahin in [31] following Chen's definition for a Hermitian manifold.

Now we shall state the notions of the Wirtinger angle and the pointwise slant submanifold in an almost product manifold following Chen's [17] definition of pointwise slant submanifolds for a Hermitian manifold.

Definition 4.1 *Let (M,g) be a submanifold of an almost product manifold. For any nonzero vector $X \in T_pM$, the angle $\theta(X)$ between FX and T_pM is called the Wirtinger angle of X. The Wirtinger angle gives rise to a real-valued function $\theta : TM - \{0\} \to R$, called the Wirtinger function or slant function. The submanifold (M,g) is called pointwise slant if, at each given point $p \in M$, the angle θ is independent of the choice of the nonzero tangent vector $X \in T_pM$.*

We note that a pointwise slant submanifold of an almost product manifold is called *slant*, in the sense of [7] and [8], if its Wirtinger function θ is globally constant. Moreover, F-invariant and F-antiinvariant submanifolds introduced in [35] are pointwise slant submanifold with slant angle 0 and $\frac{\pi}{2}$, respectively. A pointwise slant submanifold of an almost product manifold is called a *proper pointwise slant submanifold* if it is neither F-invariant nor F-antiinvariant.

In view of Lemma 3.1 in [3] and [31], the following lemma is given:

Lemma 4.2 *Let* (M,g) *be an* n*-dimensional pointwise slant submanifold of an* m*-dimensional almost product Riemannian manifold* $(\widetilde{M},\widetilde{g})$*. Then we have*

$$\langle fX, fY\rangle = \cos^2\theta\langle X,Y\rangle \tag{4.1}$$

and

$$\langle wX, wY\rangle = \sin^2\theta\langle X,Y\rangle \tag{4.2}$$

for any $X,Y \in \Gamma(TM)$*.*

Now we need the following existence theorem:

Theorem 4.3 *Let* (M,g) *be an* n*-dimensional proper pointwise slant submanifold of an* m*-dimensional almost product Riemannian manifold* $(\widetilde{M},\widetilde{g})$*. Then there exists an orthonormal basis* $\{e_1,...,e_n\}$ *of* T_pM*,* $p\in M$ *such that* $\langle fe_a, e_b\rangle = \cos\theta$*, and* $\langle fe_a, e_c\rangle = 0$ *for* $a,b,c \in \{1,2,...,n\}$*,* $b\neq c$ *and* $\theta\in(0,\frac{\pi}{2})$*.*

Proof For $n=2$, let $\{e_1,e_2\}$ be an orthonormal basis of T_pM, $p\in M$. For a unit vector Y in T_pM, we can write $Y = a_1e_1 + a_2e_2$ such that $a_1^2 + a_2^2 = 1$, $a_1,a_2\in\mathbb{R}$. Since M is a proper pointwise slant submanifold of $\widetilde{M}$, we have

$$\langle fe_1, Y\rangle = a_1\langle fe_1, e_1\rangle + a_2\langle fe_1,e_2\rangle = \cos\theta, \tag{4.3}$$

$$\langle fe_2, Y\rangle = a_1\langle fe_2, e_1\rangle + a_2\langle fe_2,e_2\rangle = \cos\theta \tag{4.4}$$

and

$$\begin{aligned}\langle fY, Y\rangle &= \langle a_1fe_1 + a_2fe_2, a_1e_1 + a_2e_2\rangle \\ &= a_1^2\langle fe_1,e_1\rangle + a_2^2\langle fe_2,e_2\rangle + 2a_1a_2\langle fe_1,e_2\rangle = \cos\theta.\end{aligned} \tag{4.5}$$

By using (4.3) and (4.4), respectively, we get

$$a_1^2\langle fe_1,e_1\rangle = a_1\cos\theta - a_1a_2\langle fe_1,e_2\rangle \tag{4.6}$$

and

$$a_2^2\langle fe_2,e_2\rangle = a_2\cos\theta - a_1a_2\langle fe_1,e_2\rangle. \tag{4.7}$$

If we put (4.6) and (4.7) in (4.5), we have $a_1 + a_2 = 1$. Since $a_1^2 + a_2^2 = 1$, we obtain $a_1a_2 = 0$. In other words, $a_1 = 0$ or $a_2 = 0$. This shows that we can choose $\langle fe_a, e_1\rangle = \cos\theta$ for a fixed e_a vector, $a=1$ or $a=2$. Thus, we can write

$$fe_a = \langle fe_a,e_1\rangle e_1 + \langle fe_a,e_2\rangle e_2.$$

Then we have

$$\langle fe_a, fe_a\rangle = \cos^2\theta + \langle fe_a,e_2\rangle^2.$$

From Lemma 4.2, we obtain

$$\cos^2\theta = \cos^2\theta + \langle fe_a,e_2\rangle^2. \tag{4.8}$$

By using (4.8), it is clear that $\langle fe_a, e_2\rangle = 0$. If we choose $\langle fe_a, e_2\rangle = \cos\theta$ then in a similar way, we find $\langle fe_a, e_1\rangle = 0$. Therefore, assertion of the theorem is true for $n = 2$.

For $n = k$, let $\{e_1, e_2, ..., e_k\}$ be an orthonormal basis of T_pM, $p \in M$. For a unit vector Y in T_pM, we can write $Y = \sum_{i=1}^{k} a_i e_i$ such that $\sum_{i=1}^{k} a_i^2 = 1$, $a_i \in R$, $i \in \{1, ..., k\}$. Since M is a proper pointwise slant submanifold of $\widetilde{M}$, we have

$$\langle fe_i, Y\rangle = \sum_{i,j}^{k} a_i \langle fe_i, e_j\rangle = \cos\theta, \quad 1 \leq i < j \leq k \tag{4.9}$$

and

$$\langle fY, Y\rangle = \cos\theta. \tag{4.10}$$

By using (4.9) and (4.10), we get

$$a_i^2 \langle fe_i, e_i\rangle = a_i \cos\theta - \sum_{i,j}^{k} a_i a_j \langle fe_i, e_j\rangle, 1 \leq i < j \leq k. \tag{4.11}$$

If we put (4.11) in (4.10), we have $a_1 + a_2 + ... + a_k = 1$. Since $a_1^2 + a_2^2 + ... + a_k^2 = 1$, we obtain $\sum_{i,j}^{k} a_i a_j = 0$ for $1 \leq i < j \leq k$. Thus, we can choose $\langle fe_a, e_b\rangle = \cos\theta$ for fixed e_a and e_b vectors, $a, b = \{1, ..., k\}$. Thus, we write

$$fe_a = \langle fe_a, e_1\rangle e_1 + ... + \langle fe_a, e_b\rangle e_b + ... + \langle fe_a, e_k\rangle e_k.$$

Then we have

$$\langle fe_a, fe_a\rangle = \langle fe_a, e_1\rangle^2 + ... + \langle fe_a, e_b\rangle^2 + ... + \langle fe_a, e_k\rangle^2.$$

From Lemma 4.2, it is clear that

$$\cos^2\theta = \langle fe_a, e_1\rangle^2 + ... + \cos^2\theta + ... + \langle fe_a, e_k\rangle^2, \tag{4.12}$$

which shows that $\langle fe_a, e_c\rangle = 0$ for $b \neq c$. Hence, the proof is complete. □

In view of Theorem 3.5 in [31] and Theorem 4.3, we get the following:

Corollary 4.4 *Let (M, g) be a proper pointwise θ-slant surface of an almost product Riemannian manifold $(\widetilde{M}, \widetilde{g})$ such that $\{e_1, e_2\}$ is an orthonormal basis of T_pM. Then we have one of the following two statements for all $i \neq j \in \{1, 2\}$:*

i) If $\widetilde{g}(Fe_i, e_i) = \cos\theta$ then M is a product pointwise slant surface with $\widetilde{g}(Fe_i, e_j) = 0$.

ii) If $\widetilde{g}(Fe_i, e_j) = \cos\theta$ then M is a product pointwise slant surface with $\widetilde{g}(Fe_i, e_i) = 0$ if and only if each of the e_i is parallel.

Lemma 4.5 *Let (M,g) be a θ-slant submanifold of an m-dimensional locally product manifold $(\widetilde{M},\widetilde{g})$. Then the following statements occur:*

i) For any vectors X,Y tangent to M, we have

$$f^2 = (\cos^2\theta)I \quad \text{and} \quad \langle fX,Y\rangle - \langle X,fY\rangle = 0, \tag{4.13}$$

where I denotes the identity transformation.

ii) For any vectors X,Y tangent to M, we have

$$(\nabla_X f)Y = t\sigma(X,Y) + A_{wY}X. \tag{4.14}$$

Hence, $\nabla f = 0$ if and only if $A_{wX}Y = A_{wY}X$.

iii) For any vectors X, Y tangent to M, we have

$$\nabla^{\perp}_X wY - w\nabla_X Y = s\sigma(X,Y) - \sigma(X,fY). \tag{4.15}$$

Hence, $\nabla w = 0$ if and only if $A_{s\xi}X = A_\xi fX$.

Proof Proof of (i) is clear from (3.7). Now we shall prove statements (ii) and (iii).

$$\begin{aligned}(\nabla_X f)Y &= \nabla_X fY - f\nabla_X Y\\ &= \widetilde{\nabla}_X fY - \sigma(X,fY) - f\nabla_X Y\\ &= \widetilde{\nabla}_X FY - \widetilde{\nabla}_X wY - \sigma(X,fY) - f\nabla_X Y\end{aligned}$$

Since $\widetilde{M}$ is a locally product manifold, we obtain

$$\begin{aligned}(\nabla_X f)Y &= F\widetilde{\nabla}_X Y + A_{wY}X - \nabla^{\perp}_X wY - \sigma(X,fY) - f\nabla_X Y\\ &= w\nabla_X Y + t\sigma(X,Y) + s\sigma(X,Y) + A_{wY}X - \nabla^{\perp}_X wY - \sigma(X,fY).\end{aligned}$$

Since $(\nabla_X f)Y \in \Gamma(TM)$, we have (4.14) and (4.15).

Considering (4.14) and (4.15), it is clear that $\nabla f = 0$ if and only if $A_{wX}Y = A_{wY}X$ and $\nabla w = 0$ if and only if $A_{s\xi}X = A_\xi fX$, respectively. □

Corollary 4.6 *Let (M,g) be a product submanifold of an m-dimensional locally product manifold $(\widetilde{M},\widetilde{g})$. Then M is pointwise θ-slant if and only if $A_{wX}Y = A_{wY}X$.*

For simplicity, let us put

$$X^* = (csc\theta)wX$$

for any $X \in \Gamma(TM)$ and define a symmetric bilinear TM-valued form on M by

$$\alpha(X,Y) = t\sigma(X,Y).$$

Thus, we can write

$$F\alpha(X,Y) = f\alpha(X,Y) + \sin\theta\alpha^*(X,Y) \tag{4.16}$$

and

$$F\sigma(X,Y) = \alpha(X,Y) + \beta^*(X,Y), \tag{4.17}$$

where β is also a symmetric bilinear TM-valued form on M. From (4.16) and (4.17), we have

$$\sigma(X,Y) = f\alpha(X,Y) + \sin\theta\alpha^*(X,Y) + \sin\theta\beta(X,Y) + f\beta(X,Y)^*. \tag{4.18}$$

Therefore, we have

$$\beta(X,Y) = -(\csc\theta)f\alpha(X,Y) \quad \text{and} \quad \sigma(X,Y) = (\csc\theta)\alpha^*(X,Y). \tag{4.19}$$

Consequently, we obtain

$$\sigma(X,Y) = (\csc^2\theta)(F\alpha(X,Y) - f\alpha(X,Y)) \tag{4.20}$$

and

$$\begin{aligned} \widetilde{R}(X,Y,Z,W) \;=\; & (\csc^2\theta)\{\langle\alpha(X,W),\alpha(Y,Z)\rangle - \langle\alpha(X,Z),\alpha(Y,W)\rangle\} \\ & + a\{(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle) \\ & + (\langle X,fW\rangle\langle Y,fZ\rangle - \langle X,fZ\rangle\langle Y,fW\rangle)\} \\ & + b\{(\langle X,fW\rangle\langle Y,Z\rangle - \langle X,fZ\rangle\langle Y,W\rangle) \\ & + (\langle X,W\rangle\langle Y,fZ\rangle - \langle X,Z\rangle\langle Y,fW\rangle)\} \end{aligned} \tag{4.21}$$

for any $X,Y,W,Z \in \Gamma(TM)$.

We shall now need the following remarks for later use:

Remark 4.7 *Suppose (M,g) is an n-dimensional submanifold of any Riemannian manifold $(\widetilde{M},\widetilde{g})$ endowed with any tensor field E of type $(1,1)$. Let T_pM (resp. $T_p^\perp M$) denote the tangent space (resp. the normal space) to M at $p \in M$. For any $X \in T_pM$, we decompose EX into tangential and normal parts given by*

$$EX = f_EX + w_EX, \qquad f_EX \in T_pM,\; w_EX \in T_p^\perp M. \tag{4.22}$$

*The manifold M is said to be E-*invariant *[34](resp. E-*antiinvariant *[34]) if $N_E = 0$ (resp. $P_E = 0$). The squared norm of f_E at $p \in M$ is given by*

$$\|f_E\|^2 = \sum_{i,j=1}^{n} \langle f_Ee_i, e_j\rangle^2,$$

where $\{e_1,\ldots,e_n\}$ is any orthonormal basis of the tangent space T_pM.

Remark 4.8 *Since every linear form defines a pair of linear maps from the vector space to dual space of the vector space, we can define a linear map $\alpha_1 : TM \to (TM^\perp)^*$ by*

$$\alpha_1(X) = \alpha(X,.)$$

for any $X \in \Gamma(TM)$. *The squared norms of* α *and* α_1 *on* $X \in TM$ *are given by*

$$\|\alpha\|^2 = \sum_{i,j=1}^{n} \langle \alpha(e_i, e_j), \alpha(e_i, e_j)\rangle \quad \textit{and} \quad \|\alpha_1(X)\|^2 = \sum_{j=1}^{n} \langle \alpha(X, e_j), \alpha(X, e_j)\rangle, \tag{4.23}$$

respectively.

Taking into consideration (4.21), Remark 4.7, and Remark 4.8, we give the following lemma:

Lemma 4.9 *Let* (M, g) *be an* n*-dimensional proper pointwise* θ*-slant submanifold of an* m*-dimensional manifold of almost constant curvature* $\widetilde{M}(a, b)$. *Let* $\{e_1, \dots, e_n\}$ *be an orthonormal basis of the tangent space* T_pM. *Then the sectional curvature* K_{ij} *on a plane section of* T_pM *spanned by* e_i *and* e_j *is given by*

$$\begin{aligned} K_{ij} &= (\csc^2\theta)\{\langle \alpha(e_i, e_i), \alpha(e_j, e_j)\rangle - \langle \alpha(e_i, e_j), \alpha(e_i, e_j)\rangle\} \\ &\quad + a\left\{1 + \langle e_i, f_F e_i\rangle \langle e_j, f_F e_j\rangle - \langle e_i, f_F e_j\rangle^2\right\} \\ &\quad + b\{\langle e_i, f_F e_i\rangle + \langle e_j, f_F e_j\rangle\}. \end{aligned} \tag{4.24}$$

The Ricci curvature $\mathrm{Ric}(e_i)$ *of* e_i *is given by*

$$\begin{aligned} \mathrm{Ric}(e_i) &= (\csc^2\theta)\{n\langle \alpha(e_i, e_i), tH(p)\rangle - \|\alpha_1(e_i)\|^2\} \\ &\quad + a\left\{(n-1) + \langle e_i, f_F e_i\rangle \,\mathrm{trace}\,(f_F) - \|f_F e_i\|^2\right\} \\ &\quad + b\{(n-2)\langle e_i, f_F e_i\rangle + \mathrm{trace}\,(f_F)\}, \end{aligned} \tag{4.25}$$

The scalar curvature $\tau(p)$ *at* $p \in M$ *is given by*

$$\begin{aligned} \tau(p) &= (\csc^2\theta)\{n^2\|tH(p)\|^2 - \|\alpha\|^2\} \\ &\quad + \frac{a}{2}\left\{(n-1)n + (\mathrm{trace}\,(f_F))^2 - \|f_F\|^2\right\} \\ &\quad + b(n-1)\mathrm{trace}\,(f_F), \end{aligned} \tag{4.26}$$

where $tH(p)$ *is the tangential part of the vector* $FH(p)$.

Theorem 4.10 *Let* (M, g) *be an* n*-dimensional* $(n > 2)$ *proper pointwise* θ*-slant submanifold of an* m*-dimensional manifold of almost constant curvature* $\widetilde{M}(a, b)$. *Then we have the following inequalities:*
i) For any unit vector $X \in T_pM$,

$$\begin{aligned} \mathrm{Ric}(X) &\leq n(\csc^2\theta)\langle \alpha(X, X), tH(p)\rangle + a\left\{(n-1) + \langle X, f_F X\rangle \,\mathrm{trace}\,(f_F) - \|f_F X\|^2\right\} \\ &\quad + b\{(n-2)\langle X, f_F X\rangle + \mathrm{trace}\,(f_F)\}. \end{aligned} \tag{4.27}$$

The equality case of (4.27) *holds for a unit vector* $X \in T_pM$ *if and only if*

$$\sigma(X, Y) = 0 \tag{4.28}$$

for any $Y \in T_pM$*.*

ii)

$$\tau(p) \leq n^2(\csc^2\theta)\|tH(p)\|^2 + \frac{a}{2}\left\{(n-1)n + (\mathrm{trace}\,(f_F))^2 - \|f_F\|^2\right\} + b(n-1)\mathrm{trace}\,(f_F)\,. \tag{4.29}$$

The equality case of (4.27) *holds for the point* $p \in M$ *if and only if* p *is a totally geodesic point.*

Corollary 4.11 *Let* M *be a proper pointwise* θ*-slant surface of a manifold of an* m*-dimensional almost constant curvature* $\widetilde{M}\,(a,b)$ *and* $\{e_1, e_2\}$ *be an orthonormal basis of* T_pM*. Then:*

i) If each e_i *is not parallel for* $i \in \{1,2\}$*, we have*

$$K \leq 4(\csc^2\theta)\|tH(p)\|^2 + a(1+\cos^2\theta) + 2b\cos\theta, \tag{4.30}$$

where K *is the Gaussian curvature of* M*, with the equality if and only if* M *is totally geodesic.*

ii) If each e_i *is parallel for* $i \in \{1,2\}$*, we have*

$$K \leq 4(\csc^2\theta)\|tH(p)\|^2 - a\sin^2\theta, \tag{4.31}$$

with the equality if and only if M *is totally geodesic.*

5. Special slant surfaces on almost constant curvature manifolds

First, we give the following theorem:

Theorem 5.1 *Every pointwise* θ*-slant surface of an almost product Riemannian manifold* $\widetilde{M}$ *with dim*$(\widetilde{M}) = 3$ *is* F*-invariant.*

Proof Suppose (M,g) is a pointwise slant surface of a 3-dimensional almost product Riemannian manifold. From Theorem 4.3, there exists an orthonormal basis $\{e_1, e_2\}$ of T_pM such that

$$Fe_i = \cos\theta e_j + \sin\theta e_3, \text{ for all } i,j \in \{1,2\}, \tag{5.1}$$

where $\{e_1, e_2, e_3\}$ is orthonormal basis of $T_p\widetilde{M}$. If we put (5.1) in (3.4) then we have

$$\sin^2\theta = 0, \tag{5.2}$$

which shows that M is F-invariant. □

Theorem 5.2 *Let* (M,g) *be a product pointwise slant surface of a* 4*-dimensional almost constant curvature manifold* $\widetilde{M}(a,b)$ *with the adapted orthonormal basis* $\{e_1, e_2, e_3, e_4\}$*. Then there are the following relations involving the squared mean curvature and the Gaussian curvature of* M*.*

i) If each e_i *is not parallel for any* $i \in \{1,2\}$ *then*

$$\|H(p)\|^2 \geq 2K(p) - 2a(1+\cos^2\theta) - 4b\cos\theta. \tag{5.3}$$

ii) If each e_i is parallel for all $i \in \{1,2\}$ then

$$\|H(p)\|^2 \geq 2K(p) - 2a\sin^2\theta. \tag{5.4}$$

iii) The equality sign of (5.3) or (5.4) is satisfied at a point $p \in M$ if and only if the shape operators of M at p take the following form:

$$A_{e_3} = \begin{pmatrix} 3\lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \textit{and} \quad A_{e_4} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \tag{5.5}$$

where λ is a real number.

Proof Since M is a pointwise proper slant product surface of $\widetilde{M}(a,b)$, according to Corollary 4.6, we have

$$\langle A_{wX}Y, Z\rangle = \langle A_{wY}X, Z\rangle \tag{5.6}$$

for any vectors X, Y, and Z tangent to M.

Furthermore, from Theorem 4.3, we can choose an orthonormal basis $\{e_1, e_2\}$ of T_pM such that

$$e_i = \sec\theta\ fe_j, \quad e_k = \csc\theta\ fe_\ell \tag{5.7}$$

for any $i, j \in \{1,2\}$ and $k, \ell \in \{3,4\}$. From (5.6) and (5.7) we obtain

$$A_{e_3} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 \end{pmatrix} \quad \text{and} \quad A_{e_4} = \begin{pmatrix} \lambda_2 & \lambda_3 \\ \lambda_3 & \lambda_4 \end{pmatrix}, \tag{5.8}$$

where $\lambda_1, \lambda_2, \lambda_3$, and λ_4 are real numbers. Thus,

$$4\|H(p)\|^2 = (\lambda_1 + \lambda_3)^2 + (\lambda_2 + \lambda_4)^2. \tag{5.9}$$

If each e_i is not parallel for any $i \in \{1,2\}$ then

$$K(p) = \lambda_1\lambda_3 - \lambda_2^2 + \lambda_2\lambda_4 - \lambda_3^2 + a(1+\cos^2\theta) + 2b\cos\theta. \tag{5.10}$$

From (5.9) and (5.10), we get

$$4\|H(p)\|^2 - 8K(p) + 8a(1+\cos^2\theta) + 16b\cos\theta = (\lambda_1 - 3\lambda_3)^2 + (\lambda_2 - \lambda_4)^2 \geq 0, \tag{5.11}$$

which shows the proof of statement (i).

If each e_i is parallel for any $i \in \{1,2\}$ then we have

$$K(p) = \lambda_1\lambda_3 - \lambda_2^2 + \lambda_2\lambda_4 - \lambda_3^2 + a\sin^2\theta. \tag{5.12}$$

Considering (5.9) and (5.12) we get

$$4\|H(p)\|^2 - 8K(p) + 8a\sin^2\theta = (\lambda_1 - 3\lambda_3)^2 + (\lambda_2 - \lambda_4)^2 \geq 0, \tag{5.13}$$

which shows the proof of statement (ii). □

Remark 5.3 *Suppose each of the orthonormal vectors e_i of T_pM is not parallel and the equality case of (5.3) holds at the point $p \in M$ in Theorem 5.2. If we choose another orthonormal basis $\{e_1', e_2'\}$ of T_pM for which each vector is parallel then it is obvious from the proof of Theorem 5.2 that the equality case of (5.4) also holds for the point $p \in M$.*

For these reasons, we can state the following definition:

Definition 5.4 *A proper product pointwise slant surface of a 4-dimensional almost constant curvature manifold is called special pointwise slant if the shape operator of the surface takes the following form with respect to some adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$,*

$$A_{e_3} = \begin{pmatrix} c\lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{and} \quad A_{e_4} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \tag{5.14}$$

for any c and λ that are real numbers.

Theorem 5.5 *Let M be a product pointwise slant surface of a 4-dimensional almost constant curvature manifold $\widetilde{M}(a,b)$. Then the following statements occur:*

a. *If M is totally geodesic then it is also special slant.*

b. *The surface M is minimal if and only if $c = -1$.*

c. *If $c = 1$ then M preserves the Riemannian curvature tensor.*

d. *Let M is minimal then M preserves the Riemannian curvature tensor if and only if it is totally geodesic.*

e. *The surface M has the best living way; that is, its energy density receives optimum value. (The notion of "best living way was introduced by Chen in [12, 16], etc.)*

Remark 5.6 *The surface in Euclidean 4-space satisfying (5.14) for $c = -1$ caught the special attention of geometers (see [19, 20], etc.) in many ways. Such a surface was named a minimal superconformal surface by Milousheva in [27].*

Let $\mathbb{E}^4 = \mathbb{E}^2 \times \mathbb{E}^2$ be the 4-dimensional Euclidean space endowed with the Euclidean metric. Let F be a product structure on $\mathbb{E}^4$ by

$$F(X) = F(x_1, x_2, x_3, x_4) = (x_2, x_1, x_4, x_3) \tag{5.15}$$

for any $X \in \mathbb{E}^4$.

Now we give the following proper pointwise slant surface examples with respect to the almost product structure F given in (5.15).

Example 5.7 *Consider in $\mathbb{E}^4$ the submanifold given by*

$$X(u,v) = (\cosh v \cos u, \sinh v \cos u, \cosh v \sin u, \sinh v \sin u) \tag{5.16}$$

for $v \in R - \{0\}$ and $0 < u < \frac{\pi}{2}$. Then we have

$$\frac{\partial X}{\partial u} = (-\cosh v \sin u, -\sinh v \sin u, \cosh v \cos u, \sinh v \cos u),$$

$$\frac{\partial X}{\partial v} = (\sinh v \cos u, \cosh v \cos u, \sinh v \sin u, \cosh v \sin u). \tag{5.17}$$

Thus, we have the following orthonormal basis of $\mathbb{E}^4$*:*

$$\{e_1 = \frac{\frac{\partial X}{\partial u}}{\|\frac{\partial X}{\partial u}\|} = \frac{1}{\sinh^2 v + \cosh^2 v}(-\cosh v \sin u, -\sinh v \sin u, \cosh v \cos u, \sinh v \cos u),$$

$$e_2 = \frac{\frac{\partial X}{\partial v}}{\|\frac{\partial X}{\partial v}\|} = \frac{1}{\sinh^2 v + \cosh^2 v}(\sinh v \cos u, \cosh v \cos u, \sinh v \sin u, \cosh v \sin u),$$

$$e_3 = \frac{1}{\sinh^2 v + \cosh^2 v}(-\cosh v \cos u, \sinh v \cos u, -\cosh v \sin u, \sinh v \sin u),$$

$$e_4 = \frac{1}{\sinh^2 v + \cosh^2 v}(\sinh v \sin u, \cosh v \sin u, -\sinh v \cos u, -\cosh v \cos u)\},$$

where the tangent space of the surface $X(u,v)$ *is spanned by* e_1 *and* e_2*. Then the surface is a proper pointwise slant surface with the Wirtinger function* $\theta = \arccos \frac{\cosh v \sinh v}{\sinh^2 v + \cosh^2 v}$.

Furthermore, the second derivatives of the surface $X(u,v)$ *are expressed as follows:*

$$X_{uu}(u,v) = (-\cosh v \cos u, -\sinh v \cos u, -\cosh v \sin u, -\sinh v \sin u),$$
$$X_{uv}(u,v) = (-\sinh v \cos u, -\cosh v \sin u, \sinh v \cos u, \cosh v \cos u),$$
$$X_{vv}(u,v) = (\cosh v \cos u, \sinh v \cos u, \cosh v \sin u, \sinh v \sin u). \tag{5.18}$$

Thus, the components of the second fundamental form are

$$\langle \sigma(e_1,e_1), e_3\rangle = \langle X_{uu}, e_3\rangle = \frac{1}{\sinh^2 v + \cosh^2 v},$$
$$\langle \sigma(e_1,e_2), e_3\rangle = \langle X_{uv}, e_3\rangle = 0,$$
$$\langle \sigma(e_2,e_2), e_3\rangle = \langle X_{vv}, e_3\rangle = -\frac{1}{\sinh^2 v + \cosh^2 v}$$
$$\langle \sigma(e_1,e_1), e_4\rangle = \langle X_{uu}, e_3\rangle = 0,$$
$$\langle \sigma(e_1,e_2), e_4\rangle = \langle X_{uv}, e_3\rangle = -\frac{1}{\sinh^2 v + \cosh^2 v},$$
$$\langle \sigma(e_2,e_2), e_3\rangle = \langle X_{vv}, e_3\rangle = 0,$$

which imply that the surface is a special pointwise slant surface.

The surface given in Example 5.7 is a tensor product surface of $\mathbb{E}^4$, which was defined in [25] and studied in [1, 2, 6, 4, 5, 26], etc.

Example 5.8 *Consider in* $\mathbb{E}^4 = \mathbb{E}^2 \times \mathbb{E}^2$ *the surface given by*

$$X(u,v) = (\cos u, \cos v, \sin u, \sin v), \;\; u, v \in (0, \frac{\pi}{2}). \tag{5.19}$$

The surface is a proper pointwise slant surface with the Wirtinger function $\theta = |u - v|$*, but it is not special.*

We also note that the surfaces given in Example 5.7 and Example 5.8 are members of a larger family of surfaces studied by Dursun and Turgay in [21, 22].

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