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Notes on cotorsion dimension of Hopf–Galois extensions

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Abstract: Let H be a finite dimensional Hopf algebra over a field k and A/B be a right H-Galois extension. In this note the relationship of cotorsion dimensions between A and B will be studied. We prove that $\mathrm{r.cot.D}(A) \leq \mathrm{r.cot.D}(B) + \mathrm{l.D}(H)$. Moreover, we give some sufficient conditions for which $\mathrm{r.cot.D}(A) = \mathrm{r.cot.D}(B)$. As applications, we obtain some results about cotorsion dimension of the smash product.

Key words: Hopf–Galois extension, cotorsion dimension, smash product

1. Introduction and preliminaries

Throughout this paper, k denotes a fixed field, and we will always work over k. The tensor product $\otimes = \otimes_k$ and Hom is always assumed to be over k. For an algebra A, denote by Mod-A the category of right A-modules. We write M_A to indicate a right A-module. For an A-module M, let $\mathrm{pd}(M)$ and $\mathrm{id}(M)$ denote the projective dimension and the injective dimension of M, respectively. We refer the reader to [12] for details about Hopf algebras.

The definition of Hopf–Galois extension has its roots in the Chase–Harrison–Rosenberg approach to Galois theory for groups acting on commutative rings (see [2]). In 1969 Chase and Sweedler extended these ideas to coaction of a Hopf algebra H acting on a commutative k-algebra, for k a commutative ring (see [3]); the general definition appeared in [8] in 1981. Hopf–Galois extensions also generalize strongly graded algebras (here H is a group algebra) and certain inseparable field extensions (here the Hopf algebra is the restricted enveloping algebra of a restricted Lie algebra), twisted group rings $R*G$ of a group G acting on a ring R, and so on.

Let H be a Hopf algebra over a field k and A be a right H-comodule algebra, i.e. A is a k-algebra together with an H-comodule structure $\rho_A : A \to A\otimes H$ (with notation $a \mapsto a_0 \otimes a_1$) such that ρ_A is an algebra map. Let B be the subalgebra of the H-coinvariant elements, $B := A^{coH} := \{a \in A |\ \rho_A(a) = a \otimes 1\}$. Then the extension A/B is called right H-Galois if the map $\beta : A \otimes_B A \to A \otimes H$, given by $a \otimes_B b \mapsto (a \otimes 1)\rho(b)$, is bijective. For more details and unexplained concepts we refer the reader to [12].

Let R be a ring. For any right R-module M, the cotorsion dimension $\mathrm{cd}(M)$ of M is defined to be the smallest integer $n \geqslant 0$ such that $\mathrm{Ext}_R^{n+1}(F, M) = 0$ for any flat right R-module F. If there is no such

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n, set $\mathrm{cd}(M) = \infty$. The right global cotorsion dimension $\mathrm{r.cot.D}(R)$ of R is defined as the supremum of the cotorsion dimensions of right R-modules (see [11]).

Recall from [6] that M is called cotorsion if $\mathrm{Ext}^1_R(F, M) = 0$ for any flat right R-module F, i.e. $\mathrm{cd}(M) = 0$. So the cotorsion dimension of M measures how far away a module is from being cotorsion. The class of cotorsion modules contains all pure-injective (hence injective) modules. Using it, some new characterizations of right perfect rings and von Neumann regular rings can be given (see [10]).

The aim of this paper is to study the relationship of cotorsion dimensions of Hopf–Galois extensions. We will prove the following two main results:

1. Let A/B be a right H-Galois extension for a finite dimensional Hopf algebra H. Then

$$\mathrm{r.cot.D}(A) \leq \mathrm{r.cot.D}(B) + \mathrm{l.D}(H).$$

2. Let H be a finite dimensional Hopf algebra that is semisimple as well as its dual H^* (here $H^* = \mathrm{Hom}(H, k)$), and A/B be a right faithfully flat H-Galois extension. Then

$$\mathrm{r.cot.D}(A) = \mathrm{r.cot.D}(B).$$

2. The main results, and their proof and corollaries

Let A/B be a right H-Galois extension. Consider the following two functors:

$$\begin{aligned} - \otimes_B A : \text{Mod-}B \to \text{Mod-}A, \qquad & M \mapsto M \otimes_B A, \\ (-)_B : \text{Mod-}A \to \text{Mod-}B, \qquad & M_A \mapsto M_B, \end{aligned}$$

where $(-)_B$ is the restriction functor.

Lemma 2.1 *Let A/B be a right H-Galois extension for a finite dimensional Hopf algebra H. Then $(- \otimes_B A, (-)_B)$ and $((-)_B, - \otimes_B A)$ are both adjoint pairs.*

Proof By adjoint isomorphism theorem, $(-\otimes_B A, (-)_B)$ is an adjoint pair. By Theorem 5 in [5], $((-)_B, -\otimes_B A)$ is also an adjoint pair. $\square$

Remark 2.2 *Let (F, G) be an adjoint pair of functors of abelian categories. Then F is right exact and G is left exact. If G is exact, then F preserves projective objects; if F is exact, then G preserves injective objects. Thus, by Lemma 2.1, the above functors $- \otimes_B A$ and $(-)_B$ are both exact, and so they preserve projective objects and injective objects.*

By Lemma 2.1 and the Remark, we immediately get the following lemma.

Lemma 2.3 *Let A/B be a right H-Galois extension for a finite dimensional Hopf algebra H and P be a right A-module. Then:*

(1) P_A being projective implies P_B and $P \otimes_B A$ are both projective;

(2) P_A being injective implies P_B and $P \otimes_B A$ are both injective;

(3) P_A being flat implies P_B and $P \otimes_B A$ are both flat.

Lemma 2.4 *(Lemma 3.1 of [9]) Let A/B be a right H-Galois extension for a semisimple Hopf algebra H. Then for any right A-module M, M is an A-direct summand of $M \otimes_B A$.*

The following lemma gives another equivalent definition of the right global cotorsion dimension of a ring R proved in Theorem 7.2.5 of [11].

Lemma 2.5 *Let R be a ring. Then*

$$r.cot.D(R) = sup\{pd(F) | F \text{ is a flat right} R\text{-module}\}.$$

Lemma 2.6 *Let A/B be a right H-Galois extension for a semisimple Hopf algebra H. Then for any flat right A-module F, $pd(F_A) = pd(F_B)$.*

Proof First, by Lemma 2.2 and the Remark, any projective resolution of F_A is also a projective resolution of F_B. It follows that $\mathrm{pd}(F_B) \leq \mathrm{pd}(F_A)$.

Conversely, we may assume that $\mathrm{pd}(F_B) = n < \infty$, and let $\mathcal{P}$ be a projective resolution of F_B of length n. Then by Lemma 2.2 and the Remark, $\mathcal{P} \otimes_B A$ is a projective resolution of $F \otimes_B A$ as a right A-module. This implies $\mathrm{pd}((F \otimes_B A)_A) \leq \mathrm{pd}(F_B)$. Also by Lemma 2.3, F is an A-direct summand of $F \otimes_B A$, and it follows that $\mathrm{pd}(F_A) \leq \mathrm{pd}((F \otimes_B A)_A)$. Thus, $\mathrm{pd}(F_A) \leq \mathrm{pd}(F_B)$. The proof is completed. □

Combining Lemma 2.4 and Lemma 2.5, we immediately obtain the following result.

Proposition 2.7 *Let A/B be a right H-Galois extension for a semisimple Hopf algebra H. Then*

$$r.cot.D(A) \leq r.cot.D(B).$$

Now we want to discuss when the right global cotorsion dimension of A is equal to that of B.

First we introduce the definitions of smash products. Let H be a Hopf algebra and A be a left H-module algebra, i.e. A is a k-algebra together with an H-module structure $\cdot : H \otimes A \to A$ (with notation $h \otimes a \mapsto h \cdot a$) such that $h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b)$ and $h \cdot 1 = \varepsilon(h)1$, for all $a, b \in A$ and $h \in H$. Then the smash product algebra $A\#H$ is the set $A \otimes H$ as a vector space, with multiplication

$$(a\#h)(b\#k) = a(h_1 \cdot b)\#h_2k_2$$

for $a, b \in A, h, k \in H$. Here we write $a\#h$ for the element $a \otimes h$ (see [12]).

In [4], the authors discussed the cotorsion dimension of the smash product $A\#H$. Let H be a finite dimensional Hopf algebra and A be a left H-module algebra. They proved that

$$l.cot.D(A\#H) \leq l.cot.D(A) + r.D(H),$$

where $l.cot.D(A)$ is the left global cotorsion dimension of A and $r.D(H)$ is the right global dimension of H.

Let $A\#H$ be a smash product. It is well known that $A\#H/A$ is a right H-Galois extension (see [12]). In the following, we prove that the above result is also true for the Hopf–Galois extension and we give the right version.

Theorem 2.8 *Let A/B be a right H-Galois extension for a finite dimensional Hopf algebra H. Then*

$$r.cot.D(A) \leq r.cot.D(B) + l.D(H).$$

Proof Compared to Proposition 2.6, we mainly discuss the left global dimension of H : $\mathrm{l.D}(H)$. Since H is finite dimensional, by Theorem 2.1.3 of [12] H is a Frobenius algebra. It follows that the projective modules of H and injective modules of H coincide. So for any H-module M, $\mathrm{pd}(M) = 0 \,\mathrm{or}\, \infty$. Indeed, let $\mathcal{P}$ be a projective resolution of M of length n, denoting

$$\mathcal{P} : 0 \longrightarrow P_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0.$$

Set $K_i = \mathrm{Ker}\, d_i$ (the kernel of d_i), $i = 0, \ldots, n-1$. Consider the short exact sequence

$$0 \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow K_{n-1} \longrightarrow 0.$$

Since P_n is injective, this sequence is split; that is, $P_{n-1} \cong P_n \oplus K_{n-1}$, and P_{n-1} is also injective, so one can get that K_{n-1} is injective (the category of injective modules is closed under the direct summands). Similarly, we get that $K_i, i = 0, \ldots, n-1$ are all injective (hence projective). Consequently, one can obtain the following short exact sequence:

$$0 \longrightarrow K_0 \longrightarrow P_0 \xrightarrow{d_0} M \longrightarrow 0,$$

with K_0, P_0 projective. Thus, $\mathrm{pd}(M) = 0$.

From all of the above, $\mathrm{l.D}(H) = 0 \,\mathrm{or}\, \infty$. If $\mathrm{l.D}(H) = 0$, then H is semisimple and this theorem is just Proposition 2.6. If $\mathrm{l.D}(H) = \infty$, then this theorem is obviously satisfied. The proof is completed. □

Now we give a duality theorem of Hopf–Galois extensions. Let H be a finite dimensional Hopf algebra. Then a right H-comodule algebra A corresponds to a left H^*-module algebra A via $f \rightharpoonup a = a_0 < f, a_1 >$ (see [12]). Thus, A and H^* form a smash product algebra $A\#H^*$. Let A/B be a right H-Galois extension for a finite dimensional Hopf algebra H. From Theorem 8.3.3 of [12], there is a canonical isomorphism between the smash product algebra $A\#H^*$ and the endomorphism algebra $\mathrm{End}A_B$; that is, $A\#H^* \cong \mathrm{End}A_B$, where the right B-module action on A is the multiplication.

Lemma 2.9 *Let A/B be a right H-Galois extension for a finite dimensional Hopf algebra H. If A/B is faithfully flat, then $A\#H^*$ is Morita equivalent to B.*

Proof By the above, $A\#H^* \cong \mathrm{End}A_B$. Since A/B is right faithfully flat, by [12] or the right version of Theorem 2.6 in [1], we obtain that A is a right B-progenerator. Hence, $A\#H^*$ is Morita equivalent to B. □

Note that there are many examples of faithfully flat Hopf–Galois extensions (cf. [14]). For example, the smash product extension $A\#H/A$ is a right faithfully flat H-Galois extension. In [14], the author studied the representation theory of the faithfully flat Hopf–Galois extension.

Now we obtain the main result as follows.

Theorem 2.10 *Let H be a finite dimensional Hopf algebra that is semisimple as well as its dual H^*, and A/B be a right faithfully flat H-Galois extension. Then:*

(1) $r.cot.D(A) = r.cot.D(B)$.

(2) A is right perfect if and only if so is B.

Proof

(1) First, by Proposition 2.6, $\text{r.cot.D}(A) \leq \text{r.cot.D}(B)$.

Next we consider the smash product algebra $A\#H^*$. Since $A\#H^*/A$ is a right H^*-Galois extension, combining the semisimplicity of H^*, we have $\text{r.cot.D}(A\#H^*) \leq \text{r.cot.D}(A)$. Since A/B is faithfully flat, by Lemma 2.8, $A\#H^*$ is Morita equivalent to B. It follows that $\text{r.cot.D}(B) = \text{r.cot.D}(A\#H^*)$. Then

$$\text{r.cot.D}(B) = \text{r.cot.D}(A\#H^*) \leq \text{r.cot.D}(A) \leq \text{r.cot.D}(B).$$

Therefore, $\text{r.cot.D}(A) = \text{r.cot.D}(B)$.

(2) It immediately follows from (1) since A is right perfect if and only if $\text{r.cot.D}(A) = 0$ by Corollary 7.2.7 of [11].

□

Let $A\#H$ be a smash product. Then $A\#H/A$ is a right faithfully flat H-Galois extension, and so we have the following corollary.

Corollary 2.11 *Let H be a finite dimensional Hopf algebra that is semisimple as well as its dual H^*, and $A\#H$ be a smash product. Then*

$$\textit{r.cot.D}(A\#H) = \textit{r.cot.D}(A).$$

Note that the result of the above corollary is also true for the crossed product $A\#_\sigma H$, which are generalizations of the smash products (for the definition of the crossed product, see Definition 7.1.1 of [12]), since the crossed product extension $A\#_\sigma H/A$ is also a right faithfully flat H-Galois extension (see [14]).

Let A/B be a right H-Galois extension. We now give another sufficient condition for which $\text{r.cot.D}(A) = \text{r.cot.D}(B)$ using separable functor.

Now we recall the definition of a separable functor. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor. F induces a natural transformation

$$\mathcal{F} : \text{Hom}_{\mathcal{C}}(\cdot,\cdot) \to \text{Hom}_{\mathcal{D}}(F(\cdot), F(\cdot)); \;\; \mathcal{F}_{C,C'}(f) = F(f).$$

We say that F is a separable functor if $\mathcal{F}$ splits, i.e. we have a natural transformation

$$\mathcal{P} : \text{Hom}_{\mathcal{D}}(F(\cdot), F(\cdot)) \to \text{Hom}_{\mathcal{C}}(\cdot,\cdot)$$

such that

$$\mathcal{P} \circ \mathcal{F} = 1_{\text{Hom}_{\mathcal{C}}(\cdot,\cdot)},$$

the identity natural transformation on $\text{Hom}_{\mathcal{C}}(\cdot,\cdot)$. The more explicit form of the definition can be found in [13] in which separable functors were first introduced.

The terminology comes from the fact that, for a ring extension $R \to S$, the restriction functor $(-)_R$ is separable if and only if the extension S/R is separable.

Lemma 2.12 *Let A/B be a right H-Galois extension for a finite dimensional Hopf algebra H. If $- \otimes_B A$ is separable, then for any right B-module M, M is a B-direct summand of $M \otimes_B A$.*

Proof Consider the adjoint pair $(-\otimes_B A, (-)_B)$. If the functor $-\otimes_B A$ is separable, then we obtain by Proposition 5 of [7] that the natural map $\eta_M : M_B \to (M \otimes_B A)_B$ is a split monomorphism for every $M \in$ Mod-B. □

Proposition 2.13 *Let A/B be a right H-Galois extension for a semisimple Hopf algebra H. If $-\otimes_B A$ is separable, then*

$$r.cot.D(A) = r.cot.D(B).$$

Proof First, by Proposition 2.6, $\text{r.cot.D}(A) \le \text{r.cot.D}(B)$.

Next we prove that $\text{r.cot.D}(B) \le \text{r.cot.D}(A)$. For this, by Lemma 2.2 and Lemma 2.4 we only need to show that for any flat right B-module F, $\text{pd}(F_B) = \text{pd}((F \otimes_B A)_A)$. It is clear that $\text{pd}((F \otimes_B A)_B) \le \text{pd}((F \otimes_B A)_A)$ and $\text{pd}((F \otimes_B A)_A) \le \text{pd}(F_B)$ by Lemma 2.2 and the Remark. Also by Lemma 2.11, F is a B-direct summand of $F \otimes_B A$, and it follows that $\text{pd}(F_B) \le \text{pd}((F \otimes_B A)_B)$. The proof is completed. □

Finally, we remark here that the left global cotorsion dimension and the right global cotorsion dimension of a finite dimensional Hopf algebra H are both equal to 0. Indeed, since H is finite dimensional, it follows that H is left and right Noetherian and $\text{id}({}_H H) = \text{id}(H_H) = 0$ (note that H is a Frobenius algebra, and so the projective modules of H and injective modules of H coincide). Hence, H is a 0-Gorenstein algebra. By Proposition 7.2.12 of [11], $\text{l.cot.D}(H) = \text{r.cot.D}(H) = 0$.

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References

[1] Caenepeel S, De Groot E. Galois theory for weak Hopf algebras. Rev Roumaine Math Pures Appl 2007; 52: 151–176.

[2] Chase SU, Harrison DK, Rosenberg A. Galois theory and cohomology of commutative rings. Mem Amer Math Soc 1965; 52: 15–33.

[3] Chase SU, Sweedler M. Hopf Algebras and Galois Theory. Lecture Notes in Mathematics, Volume 97. Berlin, Germany: Springer-Verlag, 1969.

[4] Chen XL, Zhu HY, Li F. Cotorsion dimensions and Hopf algebra actions. Math Notes 2013; 93: 616–623.

[5] Doi Y. Hopf extensions of algebras and Maschke type theorems. Israel J Math 1990; 72: 99–108.

[6] Enochs EE. Flat covers and flat cotorsion modules. P Am Math Soc 1984; 92: 179–184.

[7] García Rozas JR, Torrecillas B. Preserving and reflecting covers by functors. Applications to graded modules. J Pure Appl Algebra 1996; 112: 91–107.

[8] Kreimer HF, Takeuchi M. Hopf algebras and Galois extensions of an algebra. Indiana Univ Math J 1981; 30: 675–692.

[9] Liu L, Guo QL. Global and finitistic dimension of Hopf-Galois extensions. Turk J Math 2013; 37: 210–217.

[10] Mao LX, Ding NQ. Notes on cotorsion modules. Comm Algebra 2005; 33: 349–360.

[11] Mao LX, Ding NQ. The cotorsion dimension of modules and rings. Lect Notes Pure Appl Math 2006; 249: 217–233.

[12] Montgomery S. Hopf Algebras and Their Actions on Rings. CBMS Regional Conference Series in Mathematics, Vol 82. Providence, RI, USA: American Mathematical Society, 1993.

[13] Năstăsescu C, Van den Bergh M, Van Oystaeyen F. Separable functors applied to graded rings. J Algebra 1989; 123: 397–413.

[14] Schneider HJ. Representation theory of Hopf-Galois extensions. Israel J Math 1990; 72: 196–231.